System Reliability Theory: Models and Statistical Methods
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Counting Processes

7.1 INTRODUCTION

In this chapter and in Chapter 8 we will study the reliability of a repairable system as a function of time. We are interested in finding system reliability measures like the availability of the system, the mean number of failures during a specified time interval, the mean time to the first system failure, and the mean time between system failures. For this purpose, we study the system by using stochastic processes. A *stochastic process* $\{X(t), t \in \Theta\}$ is a collection of random variables. The set Θ is called the *index set* of the process. For each *index* t in Θ, $X(t)$ is a random variable. The index t is often interpreted as time, and $X(t)$ is called the *state* of the process at time t. When the index set Θ is countable, we say that the process is a discrete-time stochastic process. When Θ is a continuum, we say that it is a continuous-time stochastic process. In this chapter and in Chapter 8 we only look at continuous-time stochastic processes. The presentation of the various processes in this book is very brief and limited, as we have focused on results that can be applied in practice instead of mathematical rigor. The reader should therefore consult a textbook on stochastic processes for more details. An excellent introduction to stochastic processes may be found in, for example, Ross (1996) and Cocozza-Thivent (1997).

In this chapter we consider a repairable system that is put into operation at time $t = 0$. When the system fails, it will be repaired to a functioning state. The repair time is assumed to be negligible. When the second failure occurs, the system will again be repaired, and so on. We thus get a sequence of failure times. We will primarily be interested in the random variable $N(t)$, the number of failures in the time interval $(0, t]$. This particular stochastic process $\{N(t), t \geq 0\}$ is called a *counting process*.

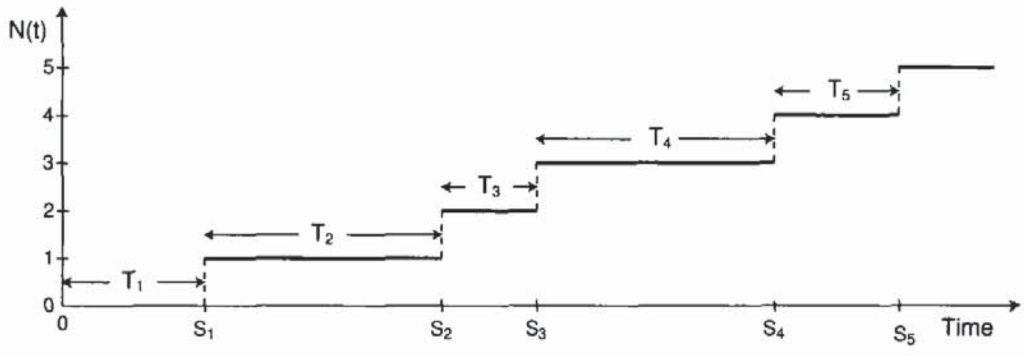

Fig. 7.1 Relation between the number of events $N(t)$, the interoccurrence times (T_i), and the calendar times (S_i).

In Chapter 8 we study the various *states* of a repairable system. A multicomponent, repairable system will have a number of possible states, depending on how many of its components are in operation. The state of the system at time t is denoted $X(t)$, and we are interested in finding the probability that the system is in a specific state at time t. We also find the steady-state probabilities, or the average proportion of time the system is in the various states. The presentation will be limited to a special class of stochastic processes $\{X(t), t \geq 0\}$ having the *Markov property*. Such a stochastic process is called a *Markov process* and is characterized by its lack of memory. If a Markov process is in state j at time t, we will get no more knowledge about its future states by knowing the history of the process up to time t.

7.1.1 Counting Processes

Consider a repairable system that is put into operation at time $t = 0$. The first failure (event) of the system will occur at time S_1. When the system has failed, it will be replaced or restored to a functioning state. The repair time is assumed to be so short that it may be neglected. The second failure will occur at time S_2 and so on. We thus get a sequence of failure times $S_1, S_2, \ldots$. Let T_i be the time between failure $i-1$ and failure i for $i = 1, 2, \ldots$, where S_0 is taken to be 0. T_i will be called the *interoccurrence time i* for $i = 1, 2, \ldots$. T_i may also be called the *time between failures*, and the *interarrival time*. In general, counting processes are used to model sequences of *events*. In this book, most of the events considered are failures, but the results presented will apply for more general events.

Throughout this chapter t denotes a specified point of time, irrespective whether t is *calendar time* (a realization of S_i) or *local time* (a realization of an interoccurrence time T_i. We hope that this convention will not confuse the reader. The time concepts are illustrated in Fig. 7.1.

The sequence of interoccurrence times, $T_1, T_2, \ldots$ will generally not be independent and identically distributed—unless the system is replaced upon failure or restored to an "as good as new" condition, and the environmental and operational conditions remain constant throughout the whole period.

A precise definition of a counting process is given below (from Ross 1996, p. 59).

Definition 7.1 A stochastic process $\{N(t), t \geq 0\}$ is said to be a *counting process* if $N(t)$ satisfies:

1. $N(t) \geq 0$.
2. $N(t)$ is integer valued.
3. If $s < t$, then $N(s) \leq N(t)$.
4. For $s < t$, $[N(t) - N(s)]$ represents the number of failures that have occurred in the interval $(s, t]$. □

A counting process $\{N(t), t \geq 0\}$ may alternatively be represented by the sequence of failure (calendar) times $S_1, S_2, \ldots$, or by the sequence of interoccurrence times $T_1, T_2, \ldots$. The three representations contain the same information about the counting process.

Example 7.1
The following failure times (calendar time in days) are presented by Ascher and Feingold (1984, p. 79). The data set is recorded from time $t = 0$ until 7 failures have been recorded during a total time of 410 (days). The data come from a single system, and the repair times are assumed to be negligible. This means that the system is assumed to be functioning again almost immediately after a failure is encountered.

Number of failures $N(t)$	Calendar time S_j	Interoccurrence time T_j
0	0	0
1	177	177
2	242	65
3	293	51
4	336	43
5	368	32
6	395	27
7	410	15

The data are illustrated in Fig. 7.2. The interoccurrence times are seen to become shorter with time. The system seems to be deteriorating, and failures tend to become more frequent. A system with this property is called a *sad* system by Ascher and Feingold (1984), for obvious reasons. A system with the opposite property, where failures become less frequent with operating time, is called a *happy* system.

The number of failures $N(t)$ may also be illustrated as a function of (calendar) time t as illustrated in Fig. 7.3. Note that $N(t)$ by definition is constant between failures and jumps (a height of 1 unit) at the failure times S_i for $i = 1, 2, \ldots$. It is thus sufficient to plot the jumping points $(S_i, N(S_i))$ for $i = 1, 2, \ldots$. The plot is called an $N(t)$ plot, or a Nelson-Aalen plot (see Section 7.4.3).

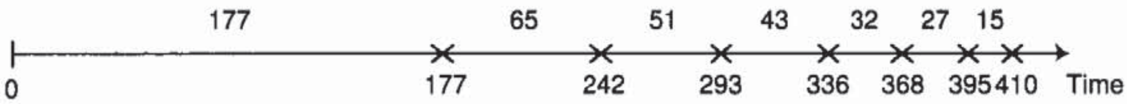

Fig. 7.2 The data set in Example 7.1.

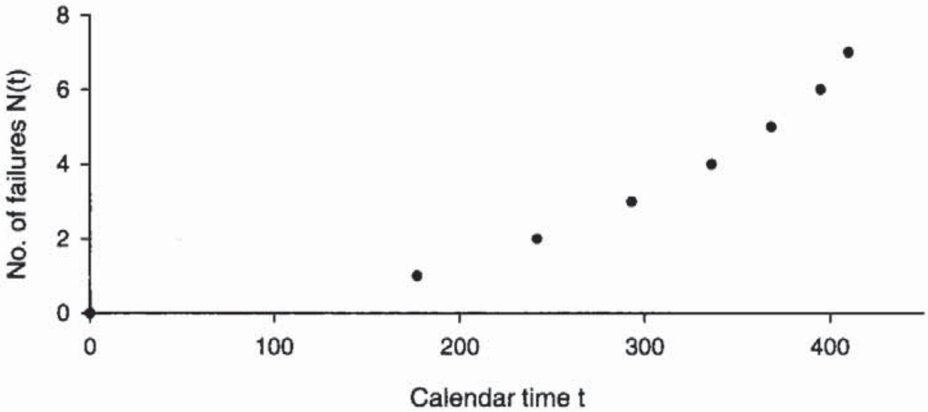

Fig. 7.3 Number of failures $N(t)$ as a function of time for the data in Example 7.1.

Note that $N(t)$ as a function of t will tend to be convex when the system is *sad*. In the same way, $N(t)$ will tend to be concave when the system is *happy*.[1] If $N(t)$ is (approximately) linear, the system is steady, that is, the interoccurrence times will have the same expected length. In Fig. 7.3 $N(t)$ is clearly seen to be convex. Thus the system is *sad*. □

Example 7.2 Compressor Failure Data

Failure time data for a specific compressor at a Norwegian process plant have been collected as part of a student thesis at the Norwegian University of Science and Technology. All compressor failures in the time period from 1968 until 1989 have been recorded. In this period a total of 321 failures occurred, 90 of which were critical failures and 231 were noncritical. In this context, a critical failure is defined to be a failure causing compressor downtime. Noncritical failures may be corrected without having to close down the compressor. The majority of the noncritical failures were instrument failures and failures of the seal oil system and the lubrication oil system.

As above, let $N(t)$ denote the number of compressor failures in the time interval $(0, t]$. From a production regularity point of view, the critical failures are the most important, since these failures are causing process shutdown. The operating times (in days) at which the 90 critical failures occurred are listed in Table 7.1. Here the time t denotes the *operating* time, which means that the downtimes caused by compressor

[1] Notice that we are using the terms *convex* and *concave* in a rather inaccurate way here. What we mean is that the observed points $\{t_i, N(t_i)\}$ for $i = 1, 2, \ldots$ approximately follow a convex/concave curve.

Table 7.1 Failure Times (Operating Days) in Chronological Order.

1.0	4.0	4.5	92.0	252.0	277.0
277.5	284.5	374.0	440.0	444.0	475.0
536.0	568.0	744.0	884.0	904.0	1017.5
1288.0	1337.0	1338.0	1351.0	1393.0	1412.0
1413.0	1414.0	1546.0	1546.5	1575.0	1576.0
1666.0	1752.0	1884.0	1884.2	1884.4	1884.6
1884.8	1887.0	1894.0	1907.0	1939.0	1998.0
2178.0	2179.0	2188.5	2195.5	2826.0	2847.0
2914.0	3156.0	3156.5	3159.0	3211.0	3268.0
3276.0	3277.0	3321.0	3566.5	3573.0	3594.0
3640.0	3663.0	3740.0	3806.0	3806.5	3809.0
3886.0	3886.5	3892.0	3962.0	4004.0	4187.0
4191.0	4719.0	4843.0	4942.0	4946.0	5084.0
5084.5	5355.0	5503.0	5545.0	5545.2	5545.5
5671.0	5939.0	6077.0	6206.0	6206.5	6305.0

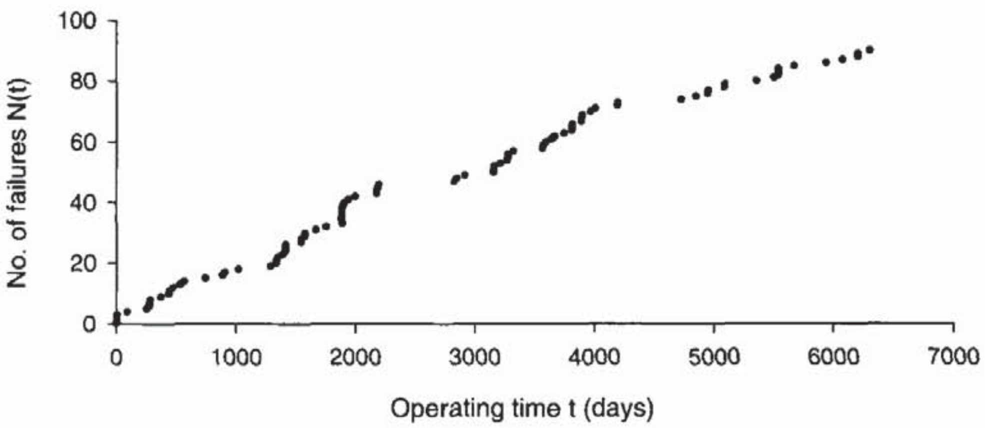

Fig. 7.4 Number of critical compressor failures $N(t)$ as a function of time (days), (totaling 90 failures).

failures and process shutdowns are not included. An $N(t)$ plot with respect to the 90 critical failures is presented in Fig. 7.4. In this case the $N(t)$ plot is slightly concave, which indicates a *happy* system. The time between critical failures hence seems to increase with the time in operation. Also note that several failures have occurred within short intervals. This indicates that the failures may be dependent, or that the maintenance crew has not been able to correct the failures properly at the first attempt. □

An analysis of life data from a repairable system should always be started by establishing an $N(t)$ plot. If $N(t)$ as a function of the time t is nonlinear, methods based on the assumption of independent and identically distributed times between failures are obviously not appropriate. It is, however, not certain that such methods are appropriate even if the $N(t)$ plot is very close to a straight line. The interoccurrence times may be strongly correlated. Methods to check whether the interoccurrence times are correlated or not are discussed, for example, by Ascher and Feingold (1984) and Bendell and Walls (1985). The $N(t)$ plot is further discussed in Section 7.4.

7.1.2 Some Basic Concepts

A number of concepts associated with counting processes are defined in the following. Throughout this section we assume that the events that are counted are *failures*. In some of the applications later in this chapter we also study other types of events, like repairs. Some of the concepts must be reformulated to be meaningful in these applications. We hope that this will not confuse the reader.

- *Independent increments.* A counting process $\{N(t), t \geq 0\}$ is said to have independent increments if for $0 < t_1 < \cdots < t_k$, $k = 2, 3, \ldots$ $[N(t_1) - N(0)], [N(t_2) - N(t_1)], \ldots, [N(t_k) - N(t_{k-1})]$ are all independent random variables. In that case the number of failures in an interval is not influenced by the number of failures in any strictly earlier interval (i.e., with no overlap). This means that even if the system has experienced an unusual high number of failures in a certain time interval, this will not influence the distribution of future failures.

- *Stationary increments.* A counting process is said to have stationary increments if for any two disjoint time points $t > s \geq 0$ and any constant $c > 0$, the random variables $[N(t) - N(s)]$ and $[N(t+c) - N(s+c)]$ are identically distributed. This means that the distribution of the number of failures in a time interval depends only on the length of the interval and not on the interval's distance from the origin.

- *Stationary process.* A counting process is said to be stationary (or homogeneous) if it has stationary increments.

- *Nonstationary process.* A counting process is said to be nonstationary (or nonhomogeneous) if it is neither stationary nor eventually becomes stationary.

- *Regular process.* A counting process is said to be regular (or orderly) if

$$\Pr(N(t + \Delta t) - N(t) \geq 2) = o(\Delta t) \tag{7.1}$$

  when Δt is small, and $o(\Delta t)$ denotes a function of Δt with the property that $\lim_{\Delta t \to 0} o(\Delta t)/\Delta t = 0$. In practice this means that the system will not experience two or more failures simultaneously.

- *Rate of the process.* The rate of the counting process at time t is defined as:

$$w(t) = W'(t) = \frac{d}{dt} E(N(t)) \tag{7.2}$$

where $W(t) = E(N(t))$ denotes the mean number of failures (events) in the interval $(0, t]$. Thus

$$w(t) = W'(t) = \lim_{\Delta t \to 0} \frac{E(N(t + \Delta t) - N(t))}{\Delta t} \tag{7.3}$$

and when Δt is small,

$$\begin{aligned} w(t) &\approx \frac{E(N(t + \Delta t) - N(t))}{\Delta t} \\ &= \frac{\text{Mean no. of failures in } (t, t + \Delta t]}{\Delta t} \end{aligned}$$

Thus a natural estimator of $w(t)$ is

$$\hat{w}(t) = \frac{\text{Number of failures in } (t, t + \Delta t]}{\Delta t} \tag{7.4}$$

for some suitable Δt. It follows that the rate $w(t)$ of the counting process may be regarded as the mean number of failures (events) per time unit at time t.

When we are dealing with a *regular* process, the probability of two or more failures in $(t, t + \Delta t]$ is negligible when Δt is small. Thus for small Δt we may assume that

$$N(t + \Delta t) - N(t) = 0 \text{ or } 1$$

Thus the mean number of failures in $(t, t + \Delta t]$ is approximately equal to the probability of failure in $(t, t + \Delta t]$, and

$$w(t) \approx \frac{\text{Probability of failure in } (t, t + \Delta t]}{\Delta t} \tag{7.5}$$

Hence $w(t)\, \Delta t$ can be interpreted as the probability of failure in the time interval $(t, t + \Delta t]$.

Some authors use (7.5) written as

$$w(t) = \lim_{\Delta t \to 0} \frac{\Pr(N(t + \Delta t) - N(t) = 1)}{\Delta t}$$

as definition of the rate of the process. Observe also that

$$E(N(t_0)) = W(t_0) = \int_0^{t_0} w(t)\, dt \tag{7.6}$$

- ROCOF. When the events of a counting process are failures, the rate $w(t)$ of the process is often called the *rate of occurrence of failures* (ROCOF).

- *Time between failures.* We have denoted the time T_i between failure $i - 1$ and failure i, for $i = 1, 2, \ldots$, the interoccurrence times. For a general counting

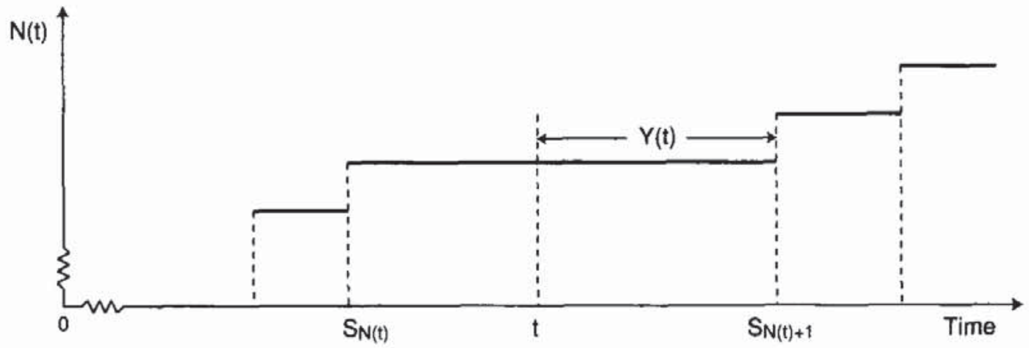

Fig. 7.5 The forward recurrence time $Y(t)$.

process the interoccurrence times will neither be identically distributed nor independent. Hence the mean time between failures, $\text{MTBF}_i = E(T_i)$, will in general be a function of i and $T_1, T_2, \ldots, T_{i-1}$.

- *Forward recurrence time.* The forward recurrence time $Y(t)$ is the time to the next failure measured from an arbitrary point of time t. Thus $Y(t) = S_{N(t)+1} - t$. The forward recurrence time is also called the *residual lifetime*, the *remaining lifetime* or the *excess life*. The forward recurrence time is illustrated in Fig. 7.5.

Many of the results in this chapter are only valid for *nonlattice* distributions. The definition of a lattice distribution follows.

Definition 7.2 A nonnegative random variable is said to have a *lattice* (or periodic) distribution if there exists a number $d \geq 0$ such that

$$\sum_{n=0}^{\infty} \Pr(X = nd) = 1$$

In words, X has a lattice distribution if X can only take on values that are integral multiples of some nonnegative number d. □

7.1.3 Martingale Theory

Martingale theory can be applied to counting processes to make a record of the *history* of the process. Let $\mathcal{H}_t$ denote the history of the process up to, but not including, time t. Usually we think of $\mathcal{H}_t$ as $\{N(s), 0 \leq s < t\}$ which keeps records of all failures before time t. It could, however, contain more specific information about each failure.

We may define a conditional rate of failures as

$$w_C(t \mid \mathcal{H}_t) = \lim_{\Delta t \to \infty} \frac{\Pr(N(t + \Delta t) - N(t) = 1 \mid \mathcal{H}_t)}{\Delta t} \tag{7.7}$$

Thus, $w_C(t \mid \mathcal{H}_t) \cdot \Delta t$ is approximately the probability of failure in the interval $[t, t + \Delta t)$ conditional on the failure history up to, but not including time t. Note that the rate of the process (ROCOF) defined in (7.2) is the corresponding unconditional rate of failures.

Usually the process depends on the history through random variables and $w_C(t \mid \mathcal{H}_t)$ will consequently be *stochastic*. It should, however, be noted that $w_C(t \mid \mathcal{H}_t)$ is stochastic only through the history: for a fixed history (i.e., for a given state just before time t), $w_C(t \mid \mathcal{H}_t)$ is not stochastic. To simplify the notation, we will in the following omit the explicit reference to the history $\mathcal{H}_t$ and let $w_C(t)$ denote the conditional ROCOF.

The martingale approach for modeling counting processes requires rather sophisticated mathematics. We will therefore avoid using this approach during the main part of the chapter but will touch upon martingales on page 254 and in Section 7.5 where we discuss imperfect repair models.

A brief, but clear, introduction to martingales used in counting processes is given by Hokstad (1997). A more thorough description is given by Andersen et al. (1993).

7.1.4 Four Types of Counting Processes

In this chapter four types of counting processes are discussed:

1. Homogeneous Poisson processes (HPP)
2. Renewal processes
3. Nonhomogeneous Poisson processes (NHPP)
4. Imperfect repair processes

The Poisson process got its name after the French mathematician Siméon Denis Poisson (1781–1840).

The HPP was introduced in Section 2.10. In the HPP model all the interoccurrence times are independent and exponentially distributed with the same parameter (failure rate) λ.

The renewal process as well as the NHPP are generalizations of the HPP, both having the HPP as a special case. A renewal process is a counting process where the interoccurrence times are independent and identically distributed with an arbitrary life distribution. Upon failure the component is thus replaced or restored to an "as good as new" condition. This is often called a *perfect repair*. Statistical analysis of observed interoccurrence times from a renewal process is discussed in detail in Chapter 11.

The NHPP differs from the HPP in that the rate of occurrences of failures varies with time rather than being a constant. This implies that for an NHPP model the interoccurrence times are neither independent nor identically distributed. The NHPP is often used to model repairable systems that are subject to a *minimal repair* strategy, with negligible repair times. Minimal repair means that a failed system is restored just back to functioning state. After a minimal repair the system continues as if nothing had happened. The likelihood of system failure is the same immediately before and after a failure. A minimal repair thus restores the system to an "as bad as

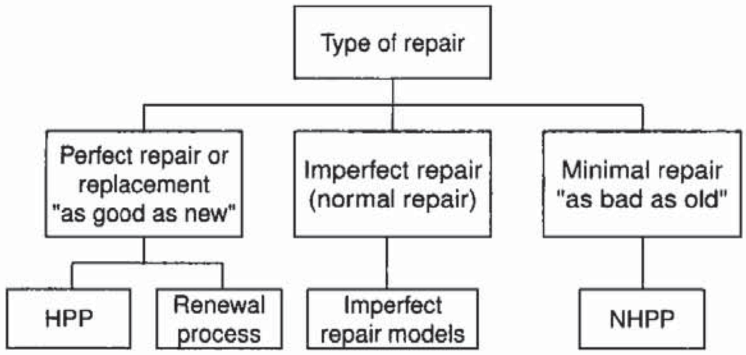

Fig. 7.6 Types of repair and stochastic point processes covered in this book.

old" condition. The minimal repair strategy is discussed, for example, by Ascher and Feingold (1984) and Akersten (1991) who gives a detailed list of relevant references on this subject.

The renewal process and the NHPP represent two extreme types of repair: replacement to an "as good as new" condition and replacement to "as bad as old" (minimal repair), respectively. Most repair actions are, however, somewhere between these extremes and are often called *imperfect repair* or *normal repair*. A number of different models have been proposed for imperfect repair. A survey of some of these models is given in Section 7.5.

The various types of repair and the models covered in this book are illustrated in Fig. 7.6.

7.2 HOMOGENEOUS POISSON PROCESSES

The homogeneous Poisson process was introduced in Section 2.10. The HPP may be defined in a number of different ways. Three alternative definitions of the HPP are presented in the following to illustrate different features of the HPP. The first two definitions are from Ross (1996, pp. 59–60).

Definition 7.3 The counting process $\{N(t), t \geq 0\}$ is said to be an HPP having rate λ, for $\lambda > 0$, if

1. $N(0) = 0$.
2. The process has independent increments.
3. The number of events in any interval of length t is Poisson distributed with mean λt. That is, for all $s, t > 0$,

$$\Pr(N(t+s) - N(s) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \quad \text{for } n = 0, 1, 2, \ldots \tag{7.8}$$

□

Note that it follows from property 3 that an HPP has stationary increments and also that $E(N(t)) = \lambda t$, which explains why λ is called the rate of the process.

Definition 7.4 The counting process $\{N(t), t \geq 0\}$ is said to be an HPP having rate λ, for $\lambda > 0$, if

1. $N(0) = 0$.
2. The process has stationary and independent increments.
3. $\Pr(N(\Delta t) = 1) = \lambda \Delta t + o(\Delta t)$.
4. $\Pr(N(\Delta t) \geq 2) = o(\Delta t)$.

□

These two alternative definitions of the HPP are presented to clarify the analogy to the definition of the NHPP which is presented in Section 7.4.

A third definition of the HPP is given, for example, by Cocozza-Thivent (1997, p. 24):

Definition 7.5 The counting process $\{N(t), t \geq 0\}$ is said to be an HPP having rate λ, for $\lambda > 0$, if $N(0) = 0$, and the interoccurrence times $T_1, T_2, \ldots$ are independent and exponentially distributed with parameter λ. □

7.2.1 Main Features of the HPP

The main features of the HPP can be easily deduced from the three alternative definitions:

1. The HPP is a regular (orderly) counting process with independent and stationary increments.
2. The ROCOFof the HPP is constant and independent of time,
$$w(t) = \lambda \quad \text{for all } t \geq 0 \tag{7.9}$$
3. The number of failures in the interval $(t, t+v]$ is Poisson distributed with mean λv,
$$\Pr(N(t+v) - N(t) = n) = \frac{(\lambda v)^n}{n!} e^{-\lambda v} \quad \text{for all } t \geq 0,\ v > 0 \tag{7.10}$$
4. The mean number of failures in the time interval $(t, t+v]$ is
$$W(t+v) - W(t) = E(N(t+v) - N(t)) = \lambda v \tag{7.11}$$
Especially note that $E(N(t)) = \lambda t$, and $\text{var}(N(t)) = \lambda t$.

5. The interoccurrence times $T_1, T_2, \ldots$ are independent and identically distributed exponential random variables having mean $1/\lambda$.

6. The time of the nth failure $S_n = \sum_{i=1}^{n} T_i$ has a gamma distribution with parameters (n, λ). Its probability density function is

$$f_{S_n}(t) = \frac{\lambda}{(n-1)!}(\lambda t)^{n-1} e^{-\lambda t} \quad \text{for } t \geq 0 \tag{7.12}$$

Further features of the HPP are presented and discussed, for example, by Ross (1996), Thompson (1988), and Ascher and Feingold (1984).

***Remark*:** Consider a counting process $\{N(t), t \geq 0\}$ where the interoccurrence times $T_1, T_2, \ldots$ are independent and exponentially distributed with parameter λ (i.e., Definition 7.5). The arrival time S_n has, according to (7.12), a gamma distribution with parameters (n, λ).

Since $N(t) = n$ if and only if $S_n \leq t < S_{n+1}$, and the interoccurrence time $T_{n+1} = S_{n+1} - S_n$, we can use the law of total probability to write

$$\begin{aligned} \Pr(N(t) = n) &= \Pr(S_n \leq t < S_{n+1}) \\ &= \int_0^t \Pr(T_{n+1} > t - s \mid S_n = s)\, f_{S_n}(s)\, ds \\ &= \int_0^t e^{-\lambda(t-s)} \frac{\lambda}{(n-1)!}(\lambda s)^{n-1} e^{-\lambda s}\, ds \\ &= \frac{(\lambda t)^n}{n!} e^{-\lambda t} \end{aligned} \tag{7.13}$$

We have thus shown that $N(t)$ has a Poisson distribution with mean λt, in accordance with Definition 7.3. □

7.2.2 Asymptotic Properties

The following asymptotic results apply:

$$\frac{N(t)}{t} \to \lambda \quad \text{with probability 1, when } t \to \infty$$

and

$$\frac{N(t) - \lambda t}{\sqrt{\lambda t}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1)$$

such that

$$\Pr\left(\frac{N(t) - \lambda t}{\sqrt{\lambda t}} \leq t\right) \approx \Phi(t) \quad \text{when } t \to \infty \tag{7.14}$$

where $\Phi(t)$ denotes the distribution function of the standard normal (Gaussian) distribution $\mathcal{N}(0, 1)$.

7.2.3 Estimate and Confidence Interval

An obvious estimator for λ is

$$\hat{\lambda} = \frac{N(t)}{t} \tag{7.15}$$

The estimator is unbiased, $E(\hat{\lambda}) = \lambda$, with variance, $\text{var}(\hat{\lambda}) = \lambda/t$.

A $1 - \varepsilon$ confidence interval for λ, when $N(t) = n$ events (failures) are observed during a time interval of length t, is given by (e.g., see Cocozza-Thivent 1997, p. 63)

$$\left(\frac{1}{2t} z_{1-\varepsilon/2,\,2n},\ \frac{1}{2t} z_{\varepsilon/2,\,2(n+1)}\right) \tag{7.16}$$

where $z_{\varepsilon,\nu}$ denotes the upper $100\varepsilon\%$ percentile of the chi-square (χ^2) distribution with ν degrees of freedom. A table of $z_{\varepsilon,\nu}$ for some values of ε and ν is given in Appendix F.

In some situations it is of interest to give an upper $(1 - \varepsilon)$ confidence limit for λ. Such a limit is obtained through the one-sided confidence interval given by

$$\left(0,\ \frac{1}{2t} z_{\varepsilon,\,2(n+1)}\right) \tag{7.17}$$

Note that this interval is applicable even if no failures ($N(t) = 0$) are observed during the interval $(0, t)$.

7.2.4 Sum and Decomposition of HPPs

Let $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ be two independent HPPs with rates λ_1 and λ_2, respectively. Further, let $N(t) = N_1(t) + N_2(t)$. It is then easy to verify that $\{N(t), t \geq 0\}$ is an HPP with rate $\lambda = \lambda_1 + \lambda_2$.

Suppose that in an HPP $\{N(t), t \geq 0\}$ we can classify each event as type 1 and type 2 that are occurring with probability p and $(1 - p)$, respectively. This is, for example, the case when we have a sequence of failures with two different failure modes (1 and 2), and p equals the relative number of failure mode 1. Then the number of events, $N_1(t)$ of type 1, and $N_2(t)$ of type 2, in the interval $(0, t]$ also give rise to HPPs, $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ with rates $p\lambda$ and $(1 - p)\lambda$, respectively. Furthermore, the two processes are independent. For a formal proof, see, for example, Ross (1996, p. 69). These results can be easily generalized to more than two cases.

Example 7.3

Consider an HPP $\{N(t), t \geq 0\}$ with rate λ. Some failures develop into a consequence C, others do not. The failures developing into a consequence C are denoted a C-failure. The consequence C may, for example, be a specific failure mode. The probability that a failure develops into consequence C is denoted p and is constant for each failure. The failure consequences are further assumed to be independent of each other. Let $N_C(t)$ denote the number of C-failures in the time interval $(0, t]$. When $N(t)$ is equal to n, $N_C(t)$ will have a binomial distribution:

$$\Pr(N_C(t) = y \mid N(t) = n) = \binom{n}{y} p^y (1-p)^{n-y} \quad \text{for } y = 0, 1, 2, \ldots, n$$

The marginal distribution of $N_C(t)$ is

$$\begin{aligned}
\Pr(N_C(t) = y) &= \sum_{n=y}^{\infty} \binom{n}{y} p^y (1-p)^{n-y} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\
&= \frac{p^y e^{-\lambda t}}{y!} (\lambda t)^y \sum_{n=y}^{\infty} \frac{[\lambda t(1-p)]^{n-y}}{(n-y)!} \\
&= \frac{(p\lambda t)^y e^{-\lambda t}}{y!} \sum_{x=0}^{\infty} \frac{[\lambda t(1-p)]^x}{x!} \\
&= \frac{(p\lambda t)^y e^{-\lambda t}}{y!} e^{\lambda t(1-p)} \\
&= \frac{(p\lambda t)^y}{y!} e^{-p\lambda t} \qquad (7.18)
\end{aligned}$$

Thus $\{N_C(t), t \geq 0\}$ is an HPP with rate $p\lambda$, and the mean number of C-failures in the time interval $(0, t]$ is

$$E(N_C(t)) = p\lambda t$$

□

7.2.5 Conditional Distribution of Failure Time

Suppose that exactly one failure of an HPP with rate λ is known to have occurred some time in the interval $(0, t_0]$. We want to determine the distribution of the time T_1 at which the failure occurred:

$$\begin{aligned}
\Pr(T_1 \leq t \mid N(t_0) = 1) &= \frac{\Pr(T_1 \leq t \cap N(t_0) = 1)}{\Pr(N(t_0) = 1)} \\
&= \frac{\Pr(1 \text{ failure in } (0, t] \cap 0 \text{ failures in } (t, t_0])}{\Pr(N(t_0) = 1)} \\
&= \frac{\Pr(N(t) = 1) \cdot \Pr(N(t_0) - N(t) = 0)}{\Pr(N(t_0) = 1)} \\
&= \frac{\lambda t e^{-\lambda t}\, e^{-\lambda(t_0 - t)}}{\lambda t_0 e^{-\lambda t_0}} \\
&= \frac{t}{t_0} \quad \text{for } 0 < t \leq t_0 \qquad (7.19)
\end{aligned}$$

When we know that exactly one failure (event) takes place in the time interval $(0, t_0]$, the time at which the failure occurs is *uniformly* distributed over $(0, t_0]$. Thus each interval of equal length in $(0, t_0]$ has the same probability of containing the failure. The expected time at which the failure occurs is

$$E(T_1 \mid N(t_0) = 1) = \frac{t_0}{2} \tag{7.20}$$

7.2.6 Compound HPPs

Consider an HPP $\{N(t), t \geq 0\}$ with rate λ. A random variable V_i is associated to failure event i, for $i = 1, 2, \ldots$. The variable V_i may, for example, be the consequence (economic loss) associated to failure i. The variables $V_1, V_2, \ldots$ are assumed to be independent with common distribution function

$$F_V(v) = \Pr(V \leq v)$$

The variables $V_1, V_2, \ldots$ are further assumed to be independent of $N(t)$. The cumulative consequence at time t is

$$Z(t) = \sum_{i=1}^{N(t)} V_i \quad \text{for } t \geq 0 \tag{7.21}$$

The process $\{Z(t), t \geq 0\}$ is called a *compound Poisson process*. Compound Poisson processes are discussed, for example, by Ross (1996, p. 82) and Taylor and Karlin (1984, p. 200). The same model is called a *cumulative damage model* by Barlow and Proschan (1975, p. 91). To determine the mean value of $Z(t)$, we need the following important theorem:

Theorem 7.1 ***(Wald's Equation)*** Let $X_1, X_2, X_3, \ldots$ be independent and identically distributed random variables with finite mean μ. Further let N be a stochastic integer variable so that the event $(N = n)$ is independent of $X_{n+1}, X_{n+2}, \ldots$ for all $n = 1, 2, \ldots$. Then

$$E\left(\sum_{i=1}^{N} X_i\right) = E(N) \cdot \mu \tag{7.22}$$

□

A proof of Wald's equation may be found, for example, in Ross (1996, p. 105). The variance of $\sum_{i=1}^{N} X_i$ is (see Ross 1996, pp. 22–23):

$$\text{var}\left(\sum_{i=1}^{N} X_i\right) = E(N) \cdot \text{var}(X_i) + [E(X_i)]^2 \cdot \text{var}(N) \tag{7.23}$$

Let $E(V_i) = \nu$ and $\text{var}(V_i) = \tau^2$. From (7.22) and (7.23) we get

$$E(Z(t)) = \nu\lambda t \quad \text{and} \quad \text{var}(Z(t)) = \lambda(\nu^2 + \tau^2)t$$

Assume now that the consequences V_i are all positive, that is, $\Pr(V_i > 0) = 1$ for all i, and that a total system failure occurs as soon as $Z(t) > c$ for some specified critical value c. Let T_c denote the time to system failure. Note that $T_c > t$ if and only if $Z(t) \leq c$.

Let $V_0 = 0$, then

$$\begin{aligned}
\Pr(T_c > t) &= \Pr(Z(t) \leq c) = \Pr\left(\sum_{i=0}^{N(t)} V_i \leq c\right) \\
&= \sum_{n=0}^{\infty} \Pr\left(\sum_{i=0}^{n} V_i \leq c \mid N(t) = n\right) \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\
&= \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} F_V^{(n)}(c) \qquad (7.24)
\end{aligned}$$

where $F_V^{(n)}(v)$ denotes the distribution function of $\sum_{i=0}^{n} V_i$, and the last equality is due to the fact that $N(t)$ is independent of $V_1, V_2, \ldots$.

The mean time to total system failure is thus

$$\begin{aligned}
E(T_c) &= \int_0^{\infty} \Pr(T_c > t)\, dt \\
&= \sum_{n=0}^{\infty} \left(\int_0^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t}\, dt\right) F_V^{(n)}(c) \\
&= \frac{1}{\lambda} \sum_{n=0}^{\infty} F_V^{(n)}(c) \qquad (7.25)
\end{aligned}$$

Example 7.4

Consider a sequence of failure events that can be described as an HPP $\{N(t), t \geq t\}$ with rate λ. Failure i has consequence V_i, where $V_1, V_2, \ldots$ are independent and exponentially distributed with parameter ρ. The sum $\sum_{i=1}^{n} V_i$ therefore has a gamma distribution with parameters (n, ρ) [see Section 2.11 (2.45)]:

$$F_V^{(n)}(v) = 1 - \sum_{k=0}^{n-1} \frac{(\rho v)^k}{k!} e^{-\rho v} = \sum_{k=n}^{\infty} \frac{(\rho v)^k}{k!} e^{-\rho v}$$

Total system failure occurs as soon as $Z(t) = \sum_{i=1}^{N(t)} V_i > c$. The mean time to total system failure is given by (7.25) where

$$\begin{aligned}
\sum_{n=0}^{\infty} F_V^{(n)}(c) &= \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{(\rho c)^k}{k!} e^{-\rho c} = \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{(\rho c)^k}{k!} e^{-\rho c} \\
&= \sum_{k=0}^{\infty} (1 + k) \frac{(\rho c)^k}{k!} e^{-\rho c} = 1 + \rho c
\end{aligned}$$

Hence when the consequences $V_1, V_2, \ldots$ are exponentially distributed with parameter ρ, the mean time to total system failure is

$$E(T_c) = \frac{1+\rho c}{\lambda} \tag{7.26}$$

□

The distribution of the time T_c to total system failure is by Barlow and Proschan (1975, p. 94) shown to be an increasing failure rate average (IFRA) distribution for *any* distribution $F_V(v)$. (IFRA distributions are discussed in Section 2.19).

7.3 RENEWAL PROCESSES

Renewal theory had its origin in the study of strategies for replacement of technical components, but later it was developed as a general theory within stochastic processes. As the name of the process indicates, it is used to model *renewals*, or replacement of equipment. This section gives a summary of some main aspects of renewal theory which are of particular interest in reliability analysis. This includes formulas for calculation of exact availability and mean number of failures within a given time interval. The latter can, for example, be used to determine optimal allocation of spare parts.

Example 7.5
A component is put into operation and is functioning at time $t = 0$. When the component fails at time T_1, it is replaced by a new component of the same type, or restored to an "as good as new" condition. When this component fails at time $T_1 + T_2$, it is again replaced, and so on. The replacement time is assumed to be negligible. The life lengths $T_1, T_2, \ldots$ are assumed to be independent and identically distributed. The number of failures, and *renewals*, in a time interval $(0, t]$ is denoted $N(t)$. □

7.3.1 Basic Concepts

A *renewal process* is a counting process $\{N(t), t \geq 0\}$ with interoccurrence times $T_1, T_2, \ldots$ that are independent and identically distributed with distribution function

$$F_T(t) = \Pr(T_i \leq t) \quad \text{for } t \geq 0,\ i = 1, 2, \ldots$$

The events that are observed (mainly failures) are called *renewals*, and $F_T(t)$ is called the underlying distribution of the renewal process. We will assume that $E(T_i) = \mu$ and $\text{var}(T_i) = \sigma^2 < \infty$ for $i = 1, 2, 3, \ldots$. Note that the HPP discussed in Section 7.2 is a renewal process where the underlying distribution is exponential with parameter λ. A renewal process may thus be considered as a generalization of the HPP.

The concepts that were introduced for a general counting process in Section 7.1.2 are also relevant for a renewal process. The theory of renewal processes has, however, been developed as a specific theory, and many of the concepts have therefore been given specific names. We will therefore list the main concepts of renewal processes and introduce the necessary terminology.

1. The time until the nth renewal (the nth arrival time), S_n:

$$S_n = T_1 + T_2 + \cdots + T_n = \sum_{i=1}^{n} T_i \tag{7.27}$$

2. The number of renewals in the time interval $(0, t]$:

$$N(t) = \max\{n : \ S_n \le t\} \tag{7.28}$$

3. The renewal function:

$$W(t) = E(N(t)) \tag{7.29}$$

   Thus $W(t)$ is the mean number of renewals in the time interval $(0, t]$.

4. The renewal density:

$$w(t) = \frac{d}{dt} W(t) \tag{7.30}$$

   Note that the renewal density coincides with the rate of the process defined in (7.2), which is called the rate of occurrence of failures when the renewals are failures. The mean number of renewals in the time interval $(t_1, t_2]$ is

$$W(t_2) - W(t_1) = \int_{t_1}^{t_2} w(t)\, dt \tag{7.31}$$

The relation between the renewal periods T_i and the number of renewals $N(t)$, the renewal process is illustrated in Fig. 7.1. The properties of renewal processes are discussed in detail by Cox (1962), Ross (1996), and Cocozza-Thivent (1997).

7.3.2 The Distribution of S_n

To find the exact distribution of the time to the nth renewal S_n is often very complicated. We will outline an approach that may be used, at least in some cases. Let $F^{(n)}(t)$ denote the distribution function of $S_n = \sum_{i=1}^{n} T_i$.

Since S_n may be written as $S_n = S_{n-1} + T_n$, and S_{n-1} and T_n are independent, the distribution function of S_n is the *convolution* of the distribution functions of S_{n-1} and T_n, respectively,

$$F^{(n)}(t) = \int_0^t F^{(n-1)}(t - x)\, dF_T(x) \tag{7.32}$$

The convolution of two (life) distributions F and G is often denoted $F * G$, meaning that $F * G(t) = \int_0^t G(t-x)\,dF(x)$. Equation (7.32) can therefore be written $F^{(n)} = F_T * F^{(n-1)}$.

When $F_T(t)$ is absolutely continuous with probability density function $f_T(t)$, the probability density function $f^{(n)}(t)$ of S_n may be found from

$$f^{(n)}(t) = \int_0^t f^{(n-1)}(t-x) f_T(x)\,dx \tag{7.33}$$

By successive integration of (7.32) for $n = 2, 3, 4, \ldots$, the probability distribution of S_n for a specified value of n can, in principle, be found.

It may also sometimes be relevant to use Laplace transforms to find the distribution of S_n. The Laplace transform of Equation (7.33) is (see Appendix B)

$$f^{*(n)}(s) = (f_T^*(s))^n \tag{7.34}$$

The probability density function of S_n can now, at least in principle, be determined from the inverse Laplace transform of (7.34).

In practice it is often very time-consuming and complicated to find the exact distribution of S_n from formulas (7.32) and (7.34). Often an approximation to the distribution of S_n is sufficient.

From the strong law of large numbers, that is, with probability 1,

$$\frac{S_n}{n} \to \mu \quad \text{as} \quad n \to \infty \tag{7.35}$$

According to the central limit theorem, $S_n = \sum_{i=1}^{n} T_i$ is asymptotically normally distributed:

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$$

and

$$F^{(n)}(t) = \Pr(S_n \le t) \approx \Phi\left(\frac{t - n\mu}{\sigma\sqrt{n}}\right) \tag{7.36}$$

where $\Phi(\cdot)$ denotes the distribution function of the standard normal distribution $\mathcal{N}(0,1)$.

Example 7.6 IFR Interoccurrence Times

Consider a renewal process where the interoccurrence times have an increasing failure rate (IFR) distribution $F_T(t)$ (see Section 2.19) with mean time to failure μ. In this case, Barlow and Proschan (1965, p. 27) have shown that the survivor function, $R_T(t) = 1 - F_T(t)$, satisfies

$$R_T(t) \ge e^{-t/\mu} \quad \text{when} \quad t < \mu \tag{7.37}$$

The right-hand side of (7.37) is the survivor function of a random variable U_j with exponential distribution with failure rate $1/\mu$. Let us assume that we have n independent random variable $U_1, U_2, \ldots, U_n$ with the same distribution. The distribution of $\sum_{j=1}^{n} U_j$ has then a gamma distribution with parameters $(n, 1/\mu)$ (see Section 2.11), and we therefore get

$$\begin{aligned} 1 - F^{(n)}(t) &= \Pr(S_n > t) = \Pr(T_1 + T_2 + \cdots + T_n > t) \\ &\geq \Pr(U_1 + U_2 + \cdots + U_n > t) = \sum_{j=0}^{n-1} \frac{(t/\mu)^j}{j!} e^{-t/\mu} \end{aligned}$$

Hence

$$F^n(t) \leq 1 - \sum_{j=0}^{n-1} \frac{(t/\mu)^j}{j!} e^{-t/\mu} \text{ for } t < \mu \tag{7.38}$$

For a renewal (failure) process where the interoccurrence times have an IFR distribution with mean μ, equation (7.38) provides a conservative bound for the probability that the nth failure will occur before time t, when $t < \mu$. □

7.3.3 The Distribution of $N(t)$

From the strong law of large numbers, that is, with probability 1,

$$\frac{N(t)}{t} \rightarrow \frac{1}{\mu} \text{ as } t \rightarrow \infty \tag{7.39}$$

When t is large, $N(t) \approx t/\mu$. This means that $N(t)$ is approximately a linear function of t when t is large. In Fig. 7.7 the number of renewals $N(t)$ is plotted as a function of t for a simulated renewal process where the underlying distribution is Weibull with parameters $\lambda = 1$ and $\alpha = 3$.

From the definition of $N(t)$ and S_n, it follows that

$$\Pr(N(t) \geq n) = \Pr(S_n \leq t) = F^{(n)}(t) \tag{7.40}$$

and

$$\begin{aligned} \Pr(N(t) = n) &= \Pr(N(t) \geq n) - \Pr(N(t) \geq n+1) \\ &= F^{(n)}(t) - F^{(n+1)}(t) \end{aligned} \tag{7.41}$$

For large values of n we can apply (7.36) and obtain

$$\Pr(N(t) \leq n) \approx \Phi\left(\frac{(n+1)\mu - t}{\sigma}\right) \tag{7.42}$$

and

$$\Pr(N(t) = n) \approx \Phi\left(\frac{t - n\mu}{\sigma\sqrt{n}}\right) - \Phi\left(\frac{t - (n+1)\mu}{\sigma\sqrt{n+1}}\right) \tag{7.43}$$

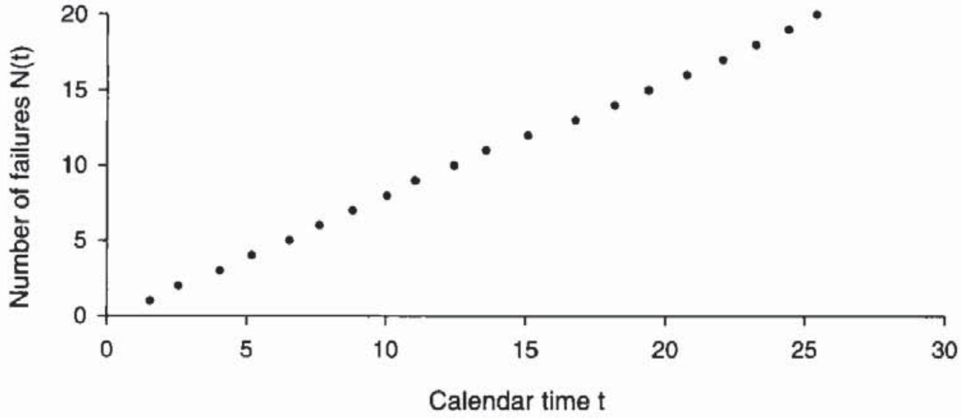

Fig. 7.7 Number of renewals $N(t)$ as a function of t for a simulated renewal process where the underlying distribution is Weibull with parameters $\lambda = 1$ and $\alpha = 3$.

Takács (1956) derived the following alternative approximation formula which is valid when t is large:

$$\Pr(N(t) \le n) \approx \Phi\left(\frac{n - (t/\mu)}{\sigma\sqrt{t/\mu^3}}\right) \tag{7.44}$$

A proof of (7.44) is provided in Ross (1996, p. 109).

7.3.4 The Renewal Function

Since $N(t) \ge n$ if and only if $S_n \le t$, we get that (see Problem 7.4)

$$W(t) = E(N(t)) = \sum_{n=1}^{\infty} \Pr(N(t) \ge n) = \sum_{n=1}^{\infty} \Pr(S_n \le t) = \sum_{n=1}^{\infty} F^{(n)}(t) \tag{7.45}$$

An integral equation for $W(t)$ may be obtained by combining (7.45) and (7.32):

$$\begin{aligned} W(t) &= F_T(t) + \sum_{r=2}^{\infty} F^{(r)}(t) = F_T(t) + \sum_{r=1}^{\infty} F^{(r+1)}(t) \\ &= F_T(t) + \sum_{r=1}^{\infty} \int_0^t F^{(r)}(t-x)\, dF_T(x) \\ &= F_T(t) + \int_0^t \sum_{r=1}^{\infty} F^{(r)}(t-x)\, dF_T(x) \\ &= F_T(t) + \int_0^t W(t-x)\, dF_T(x) \end{aligned} \tag{7.46}$$

This equation is known as the *fundamental renewal equation* and can sometimes be solved for $W(t)$.

Equation (7.46) can also be derived by a more direct argument. By conditioning on the time T_1 of the first renewal, we obtain

$$\begin{aligned} W(t) &= E(N(t)) = E(E(N(t) \mid T_1)) \\ &= \int_0^\infty E(N(t) \mid T_1 = x)\, dF_{T_1}(x) \end{aligned} \tag{7.47}$$

where

$$E(N(t) \mid T_1 = x) = \begin{cases} 0 & \text{when } t < x \\ 1 + W(t - x) & \text{when } t \geq x \end{cases} \tag{7.48}$$

If the first renewal occurs at time x for $x \leq t$, the process starts over again from this point in time. The mean number of renewals in $(0, t]$ is thus 1 plus the mean number of renewals in $(x, t]$, which is $W(t - x)$.

Combining equations (7.47) and (7.48) yields

$$\begin{aligned} W(t) &= \int_0^t (1 + W(t - x))\, dF_T(x) \\ &= F_T(t) + \int_0^t W(t - x)\, dF_T(x) \end{aligned}$$

and thus an alternative derivation of (7.46) is provided.

The exact expression for the renewal function $W(t)$ is often difficult to determine from (7.46). Approximation formulas and bounds may therefore be useful.

Since $W(t)$ is the expected number of renewals in the interval $(0, t]$, the average length μ of each renewal is approximately $t/W(t)$. We should therefore expect that when $t \to \infty$, we get

$$\lim_{t \to \infty} \frac{W(t)}{t} = \frac{1}{\mu} \tag{7.49}$$

This result is known as the *elementary renewal theorem* and is valid for a general renewal process. A proof may, for example, be found in Ross (1996, p. 107).

When the renewals are component failures, the mean number of failures in $(0, t]$ is thus approximately

$$E(N(t)) = W(t) \approx \frac{t}{\mu} = \frac{t}{\text{MTBF}} \quad \text{when } t \text{ is large}$$

where $\mu = \text{MTBF}$ denotes the mean time between failures.

From the elementary renewal theorem (7.49), the mean number of renewals in the interval $(0, t]$ is

$$W(t) \approx \frac{t}{\mu} \quad \text{when } t \text{ is large}$$

The mean number of renewals in the interval $(t, t+u]$ is

$$W(t+u) - W(t) \approx \frac{u}{\mu} \quad \text{when } t \text{ is large, and } u > 0 \tag{7.50}$$

and the underlying distribution $F_T(t)$ is nonlattice. This result is known as *Blackwell's theorem*, and a proof may be found in Feller (1968, Chapter XI).

Blackwell's theorem (7.50) has been generalized by Smith (1958), who showed that when the underlying distribution $F_T(t)$ is nonlattice, then

$$\lim_{t\to\infty} \int_0^t Q(t-x)\, dW(x) = \frac{1}{\mu} \int_0^\infty Q(u)\, du \tag{7.51}$$

where $Q(t)$ is a nonnegative, nonincreasing function which is Riemann integrable over $(0, \infty)$. This result is known as the *key renewal theorem*.

By introducing $Q(t) = \alpha^{-1}$ for $0 < t \le \alpha$ and $Q(t) = 0$ otherwise, in (7.51), we get Blackwell's theorem (7.50).

Let

$$F_e(t) = \frac{1}{\mu} \int_0^t (1 - F_T(u))\, du \tag{7.52}$$

where $F_e(t)$ is a distribution function with a special interpretation that is further discussed on page 267. By using $Q(t) = 1 - F_e(t)$ in (7.51) we get

$$\lim_{t\to\infty} \left(W(t) - \frac{t}{\mu} \right) = \frac{E(T_i^2)}{2\mu^2} - 1 = \frac{\sigma^2 + \mu^2}{2\mu^2} - 1 = \frac{1}{2}\left(\frac{\sigma^2}{\mu^2} - 1 \right)$$

if $E(T_i^2) = \sigma^2 + \mu^2 < \infty$. We may thus use the following approximation when t is large

$$W(t) \approx \frac{t}{\mu} + \frac{1}{2}\left(\frac{\sigma^2}{\mu^2} - 1 \right) \tag{7.53}$$

Upper and lower bounds for the renewal function are supplied on page 262.

7.3.5 The Renewal Density

When $F_T(t)$ has density $f_T(t)$, we may differentiate (7.45) and get

$$w(t) = \frac{d}{dt} W(t) = \frac{d}{dt} \sum_{n=1}^{\infty} F_T^{(n)}(t) = \sum_{n=1}^{\infty} f_T^{(n)}(t) \tag{7.54}$$

This formula can sometimes be used to find the renewal density $w(t)$. Another approach is to differentiate (7.46) with respect to t

$$w(t) = f_T(t) + \int_0^t w(t-x) f_T(x)\, dx \tag{7.55}$$

Yet another approach is to use Laplace transforms. From Appendix B the Laplace transform of (7.55) is

$$w^*(s) = f_T^*(s) + w^*(s) \cdot f_T^*(s)$$

Hence

$$w^*(s) = \frac{f_T^*(s)}{1 - f_T^*(s)} \tag{7.56}$$

Remark: According to (7.5) the probability of a failure (renewal) in a short interval $(t, t+\Delta t]$ is approximately $w(t)\cdot\Delta t$. Since the probability that the *first* failure occurs in $(t, t+\Delta t]$ is approximately $f_T(t)\cdot\Delta t$, we can use (7.55) to conclude that a "later" failure (i.e., not the first) will occur in $(t, t+\Delta t]$ with probability approximately equal to $\left(\int_0^t w(t-x) f_T(x)\,dx\right)\cdot\Delta t$. □

The exact expression for the renewal density $w(t)$ is often difficult to determine from (7.54), (7.55), and (7.56). In the same way as for the renewal function, we therefore have to suffice with approximation formulas and bounds.

From (7.49) we should expect that

$$\lim_{t\to\infty} w(t) = \frac{1}{\mu} \tag{7.57}$$

Smith (1958) has shown that (7.57) is valid for a renewal process with underlying probability density function $f_T(t)$ when there exists a $p > 1$ such that $|f_T(t)|^p$ is Riemann integrable. The renewal density $w(t)$ will therefore approach a constant $1/\mu$ when t is large.

Consider a renewal process where the renewals are component failures. The interoccurrence times $T_1, T_2, \ldots$ then denote the times to failure, and $S_1, S_2, \ldots$ are the times when the failures occur. Let $z(t)$ denote the failure rate [force of mortality (FOM)] function of the time to the first failure T_1. The conditional renewal density (ROCOF) $w_C(t)$ in the interval $(0, T_1)$ must equal $z(t)$ (see page 238). When the first failure has occurred, the component will be renewed or replaced and started up again with the same failure rate (FOM) as for the initial component. The conditional renewal rate (ROCOF) may then be expressed as

$$w_C(t) = z\left(t - S_{N(t-)}\right)$$

where $t - S_{N(t-)}$ is the time since the last failure strictly before time t. The conditional ROCOF is illustrated in Fig. 7.8 when the interoccurrence times are Weibull distributed with scale parameter $\lambda = 1$ and shape parameter $\alpha = 3$. The plot is based on simulated interoccurrence times from this distribution.

Example 7.7

Consider a renewal process where the renewal periods $T_1, T_2, \ldots$ are independent and gamma distributed with parameters $(2, \lambda)$, with probability density function

$$f_T(t) = \lambda^2 t\, e^{-\lambda t} \quad \text{for } t > 0, \lambda > 0$$

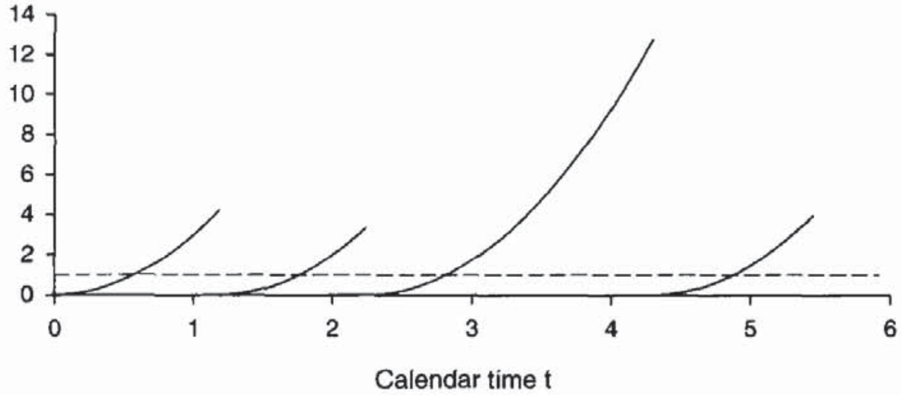

Fig. 7.8 Illustration of the conditional ROCOF (fully drawn line) for simulated data from a Weibull distribution with parameters $\alpha = 3$ and $\lambda = 1$. The corresponding asymptotic renewal density is drawn by a dotted line.

The mean renewal period is $E(T_i) = \mu = 2/\lambda$, and the variance is $\text{var}(T_i) = \sigma^2 = 2/\lambda^2$. The time until the nth renewal, S_n, is gamma distributed (see Section 2.11) with probability density function

$$f^{(n)}(t) = \frac{\lambda}{(2n-1)!}\,(\lambda t)^{2n-1}\,e^{-\lambda t} \quad \text{for } t > 0$$

The renewal density is according to (7.54)

$$\begin{aligned} w(t) &= \sum_{n=1}^{\infty} f^{(n)}(t) = \lambda e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{2n-1}}{(2n-1)!} \\ &= \lambda e^{-\lambda t} \cdot \frac{e^{\lambda t} - e^{-\lambda t}}{2} = \frac{\lambda}{2}\,(1 - e^{-2\lambda t}) \end{aligned}$$

The renewal function is

$$W(t) = \int_0^t w(x)\,dx = \frac{\lambda}{2}\int_0^t (1 - e^{-2\lambda x})\,dx = \frac{\lambda t}{2} - \frac{1}{4}\,(1 - e^{-2\lambda t}) \qquad (7.58)$$

The renewal density $w(t)$ and the renewal function $W(t)$ are illustrated in Fig. 7.9 for $\lambda = 1$. Note that when $t \to \infty$, then

$$\begin{aligned} W(t) &\to \frac{\lambda t}{2} = \frac{t}{\mu} \\ w(t) &\to \frac{\lambda}{2} = \frac{1}{\mu} \end{aligned}$$

in accordance with (7.49) and (7.57), respectively. We may further use (7.53) to find a better approximation for the renewal function $W(t)$. From (7.58) we get the left-hand

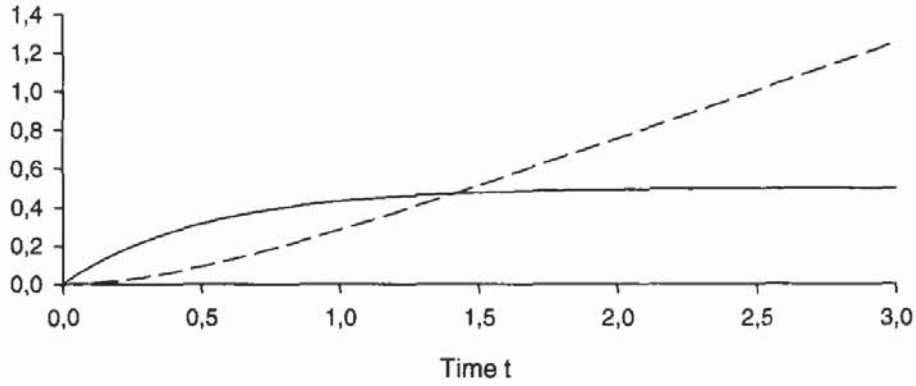

Fig. 7.9 Renewal density $w(t)$ (fully drawn line) and renewal function $W(t)$ (dotted line) for Example 7.7, with ($\lambda = 1$).

side of (7.53):

$$W(t) - \frac{t}{\mu} = W(t) - \frac{\lambda t}{2} \rightarrow -\frac{1}{4} \quad \text{when } t \rightarrow \infty$$

The right hand side of (7.53) is (with $\mu = 2/\mu$ and $\sigma^2 = 2/\lambda^2$)

$$\frac{t}{\mu} + \frac{1}{2}\left(\frac{\sigma^2}{2\mu^2} - 1\right) = \frac{t}{\mu} - \frac{1}{4}$$

We can therefore use the approximation

$$W(t) \approx \frac{\lambda t}{2} - \frac{1}{4} \quad \text{when } t \text{ is large}$$

□

Example 7.8

Consider a renewal process where the renewal periods $T_1, T_2, \ldots$ are independent and Weibull distributed with shape parameter α and scale parameter λ. In this case the renewal function $W(t)$ cannot be deduced directly from (7.45). Smith and Leadbetter (1963) have, however, shown that $W(t)$ can be expressed as an infinite, absolutely convergent series where the terms can be found by a simple recursive procedure. They show that $W(t)$ can be written

$$W(t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot A_k \cdot (\lambda t)^{k\alpha}}{\Gamma(k\alpha + 1)} \tag{7.59}$$

By introducing this expression for $W(t)$ in the fundamental renewal equation, the constants A_k; $k = 1, 2, \ldots$ can be determined. The calculation, which is quite

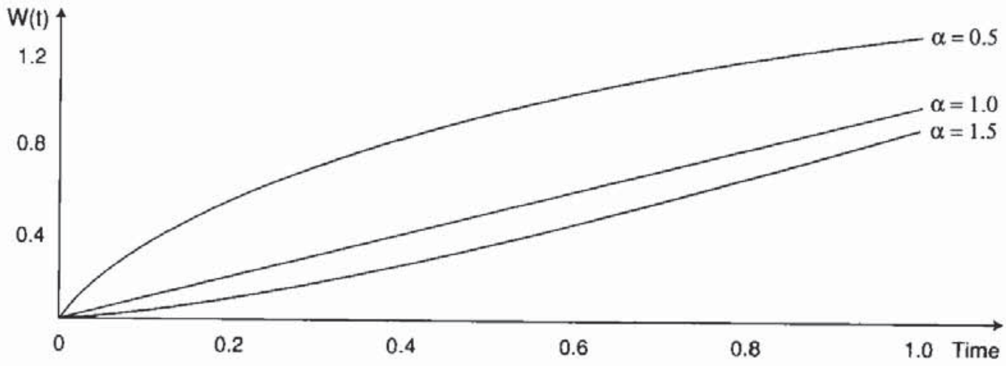

Fig. 7.10 The renewal function for Weibull distributed renewal periods with $\lambda = 1$ and $\alpha = 0.5$, $\alpha = 1$ and $\alpha = 1.5$. (The figure is adapted from Smith and Leadbetter, 1963).

comprehensive, leads to the following *recursion* formula:

$$\begin{aligned} A_1 &= \gamma_1 \\ A_2 &= \gamma_2 - \gamma_1 A_1 \\ A_3 &= \gamma_3 - \gamma_1 A_2 - \gamma_2 A_1 \\ &\vdots \\ A_n &= \gamma_n - \sum_{j=1}^{n-1} \gamma_j A_{n-j} \\ &\vdots \end{aligned} \tag{7.60}$$

where

$$\gamma_n = \frac{\Gamma(n\alpha + 1)}{n!} \quad \text{for } n = 1, 2, \ldots$$

For $\alpha = 1$, the Weibull distribution is an exponential distribution with parameter λ. In this case

$$\gamma_n = \frac{\Gamma(n + 1)}{n!} = 1 \quad \text{for } n = 1, 2, \ldots$$

This leads to

$$\begin{aligned} A_1 &= 1 \\ A_n &= 0 \quad \text{for } n \geq 2 \end{aligned}$$

The renewal function is thus according to (7.59)

$$W(t) = \frac{(-1)^0 A_1 \lambda t}{\Gamma(2)} = \lambda t$$

The renewal function $W(t)$ is illustrated in Fig. 7.10 for $\lambda = 1$ and three values of α. □

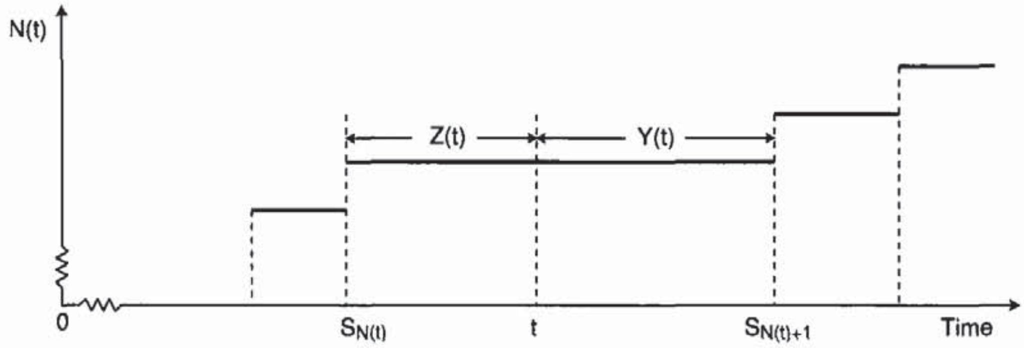

Fig. 7.11 The age $Z(t)$ and the remaining lifetime $Y(t)$.

7.3.6 Age and Remaining Lifetime

The *age* $Z(t)$ of an item operating at time t is defined as

$$Z(t) = \begin{cases} t & \text{for } N(t) = 0 \\ t - S_{N(t)} & \text{for } N(t) > 0 \end{cases} \tag{7.61}$$

The *remaining lifetime* $Y(t)$ of an item that is in operation at time t is given as

$$Y(t) = S_{N(t)+1} - t \tag{7.62}$$

The age $Z(t)$ and the remaining lifetime $Y(t)$ are illustrated in Fig. 7.11. The remaining lifetime is also called the residual life, the excess life, or the forward recurrence time (e.g., see Ross 1996, and Ascher and Feingold 1984). Note that $Y(t) > y$ is equivalent to no renewal in the time interval $(t, t+y]$.

Consider a renewal process where the renewals are component failures, and let T denote the time from start-up to the first failure. The distribution of the remaining life $Y(t)$ of the component at time t is given by

$$\Pr(Y(t) > y) = \Pr(T > y + t \mid T > t) = \frac{\Pr(T > y + t)}{\Pr(T > t)}$$

and the mean remaining lifetime at time t is

$$E(Y(t)) = \frac{1}{\Pr(T > t)} \int_t^{\infty} \Pr(T > u)\, du$$

See also Section 2.7, where $E(Y(t))$ was called the mean residual life (MRL) at time t. When T has an exponential distribution with failure rate λ, the mean remaining lifetime at time t is $1/\lambda$ which is an obvious result because of the memoryless property of the exponential distribution.

Limiting distribution Consider a renewal process with a nonlattice underlying distribution $F_T(t)$. We observe the process at time t. The time till the next failure is the remaining lifetime $Y(t)$. The limiting distribution of $Y(t)$ when $t \to \infty$ is (see Ross 1996, p. 116)

$$\lim_{t\to\infty} \Pr(Y(t) \le t) = F_e(t) = \frac{1}{\mu}\int_0^t (1 - F_T(u))\,du \tag{7.63}$$

which is the same distribution as we used in (7.52). The mean of the limiting distribution $F_e(t)$ of the remaining lifetime is

$$\begin{aligned}
E(Y) &= \int_0^\infty \Pr(Y > y)\,dy = \int_0^\infty (1 - F_e(y))\,dy \\
&= \frac{1}{\mu}\int_0^\infty \int_y^\infty \Pr(T > t)\,dt\,dy = \frac{1}{\mu}\int_0^\infty \int_0^t \Pr(T > t)\,dy\,dt \\
&= \frac{1}{\mu}\int_0^\infty t\Pr(T > t)\,dt = \frac{1}{2\mu}\int_0^\infty \Pr(T > \sqrt{x})\,dx \\
&= \frac{1}{2\mu}\int_0^\infty \Pr(T^2 > x)\,dx = \frac{E(T^2)}{2\mu} = \frac{\sigma^2 + \mu^2}{2\mu}
\end{aligned}$$

where $E(T) = \mu$ and $\text{var}(T) = \sigma^2$, and we assume that $E(T^2) = \sigma^2 + \mu^2 < \infty$.

We have thus shown that the limiting mean remaining life is

$$\lim_{t\to\infty} E(Y(t)) = \frac{\sigma^2 + \mu^2}{2\mu} \tag{7.64}$$

Example 7.7 (Cont.)

Again, consider he renewal process in Example 7.7 where the underlying distribution was a gamma distribution with parameters $(2, \lambda)$, with mean time between renewals $E(T_i) = \mu = 2/\lambda$ and variance $\text{var}(T_i) = 2/\lambda^2$. The mean remaining life of an item that is in operation at time t far from now is from (7.64)

$$E(Y(t)) \approx \frac{\sigma^2 + \mu^2}{2\mu} = \frac{3}{2\lambda} \quad \text{when } t \text{ is large}$$

□

The distribution of the age $Z(t)$ of an item that is in operation at time t can be derived by starting with

$$\begin{aligned}
Z(t) > z &\iff \text{no renewals in } (t - z, t) \\
&\iff Y(t - z) > z
\end{aligned}$$

Therefore

$$\Pr(Z(t) > z) = \Pr(Y(t - z) > z)$$

When the underlying distribution $F_T(t)$ is nonlattice, we can show that the limiting distribution of the age $Z(t)$ when $t \to \infty$ is

$$\lim_{t\to\infty} \Pr(Z(t) \le t) = F_e(t) = \frac{1}{\mu}\int_0^t (1 - F_T(u))\, du \tag{7.65}$$

that is, the same distribution as (7.63). When $t \to \infty$, both the remaining lifetime $Y(t)$ and the age $Z(t)$ at time t will have the same distribution. When t is large, then

$$E(Y(t)) \approx E(Z(t)) \approx \frac{\sigma^2 + \mu^2}{2\mu} \tag{7.66}$$

Let us now assume that a renewal process with a nonlattice underlying distribution has been "running" for a long time, and that we observe the process at a random time, which we denote $t = 0$. The time T_1 to the first renewal after time $t = 0$ is equal to the remaining lifetime of the item that is in operation at time $t = 0$. The distribution of T_1 is equal to (7.63) and the mean time to the first renewal is given by (7.64). Similarly, the age of the item that is in operation at time $t = 0$ has the same distribution and the same mean as the time to the first renewal. For a formal proof, see Ross (1996, p. 131) or Bon (1995, p. 136).

Remark: This result may seem a bit strange. When we observe a renewal process that has been "running" for a long time at a random time t, the length of the corresponding interoccurrence time is $S_{N(t)+1} - S_{N(t)}$, as illustrated in Fig. 7.15, and the mean length of the interoccurrence time is μ. We obviously have that $S_{N(t)+1} - S_{N(t)} = Z(t) + Y(t)$, but $E(Z(t) + E(Y(t)) = (\sigma^2 + \mu^2)/\mu$ is greater than μ. This rather surprising result is known as the *inspection paradox*, and is further discussed by Ross (1996, p. 117), and Bon (1995, p 141). □.

If the underlying distribution function $F_T(t)$ is new better than used (NBU) or new worse than used (NWU) (see Section 2.19), bounds may be derived for the distribution of the remaining lifetime $Y(t)$ of the item that is in operation at time t. Barlow and Proschan (1975, p. 169) have shown that the following apply:

$$\text{If } F_T(t) \text{ is NBU, then } \Pr(Y(t) > y) \le \Pr(T > y) \tag{7.67}$$

$$\text{If } F_T(t) \text{ is NWU, then } \Pr(Y(t) > y) \ge \Pr(T > y) \tag{7.68}$$

Intuitively, these results are obvious. If an item has an NBU life distribution, then a new item should have a higher probability of surviving the interval $(0, y]$ than a used item. The opposite should apply for an item with an NWU life distribution.

When the distributions of $Z(t)$ and $Y(t)$ are to be determined, the following lemma is useful:

Lemma 7.1 If

$$g(t) = h(t) + \int_0^t g(t - x)\, dF(x) \tag{7.69}$$

where the functions h and F are known, while g is unknown, then

$$g(t) = h(t) + \int_0^t h(t-x)\, dW_F(x) \tag{7.70}$$

where

$$W_F(x) = \sum_{r=1}^{\infty} F^{(r)}(x)$$

□

Note that equation (7.70) is a generalization of the fundamental renewal equation (7.46).

Example 7.9
Consider a renewal process with underlying distribution $F_T(t)$. The distribution of the remaining lifetime $Y(t)$ of an item that is in operation at time t can be given by (e.g., see Bon 1995, p. 129)

$$\Pr(Y(t) > y) = \Pr(T > y + t) + \int_0^t \Pr(T > y + t - u)\, dW_F(u) \tag{7.71}$$

By introducing the survivor function $R(t) = 1 - F_T(t)$, and assuming that the renewal density $w_F(t) = dW_F(t)/dt$ exists, (7.71) may be written

$$\Pr(Y(t) > y) = R(y + t) + \int_0^t R(y + t - u) w_F(u)\, du \tag{7.72}$$

If the probability density function $f(t) = dF_T(t)/dt = -dR(t)/dt$ exists, we have from the definition of $f(t)$ that

$$R(t) - R(t + y) \approx f(t) \cdot y \quad \text{when } y \text{ is small}$$

Equation (7.72) may in this case be written

$$\begin{aligned}
\Pr(Y(t) > y) &\approx R(t) - f(t) \cdot y + \int_0^t (R(t-u) - f(t-u) \cdot y) w_F(u)\, du \\
&= R(t) + \int_0^t R(t-u) w_F(u)\, du \\
&\quad - y \left(f(t) + \int_0^t f(t-u) w_F(u)\, du \right) \\
&= \Pr(Y(t) > 0) - w_F(t) \cdot y
\end{aligned} \tag{7.73}$$

The last line in (7.73) follows from Lemma 7.1. Since $\Pr(Y(t) > 0) = 1$, we have the following approximation:

$$\Pr(Y(t) > y) \approx 1 - w_F(t) \cdot y \quad \text{when } y \text{ is small} \tag{7.74}$$

If we observe a renewal process at a random time t, the probability of having a failure (renewal) in a short interval of length y after time t is, from (7.74), approximately $w_F(t) \cdot y$, and it is hence relevant to call $w_F(t)$ the ROCOF. □

7.3.7 Bounds for the Renewal Function

We will now establish some bounds for the renewal function $W(t)$. For this purpose consider a renewal process with interarrival times $T_1, T_2, \ldots$. We stop observing the process at the first renewal after time t, that is, at renewal $N(t)+1$. Since the event $N(t)+1=n$ only depends on $T_1, T_2, \ldots, T_n$, we can use Wald's equation to get

$$E\left(S_{N(t)+1}\right) = E\left(\sum_{i=1}^{N(t)+1}\right) = E(T)\cdot E(N(t)+1) = \mu\,[W(t)+1] \tag{7.75}$$

Since $S_{N(t)+1}$ is the first renewal after t, it can be expressed as

$$S_{N(t)+1} = t + Y(t)$$

The mean value is from (7.75)

$$\mu\,[W(t)+1] = t + E(Y(t))$$

such that

$$W(t) = \frac{t}{\mu} + \frac{E(Y(t))}{\mu} - 1 \tag{7.76}$$

When t is large and the underlying distribution is nonlattice, we can use (7.64) to get

$$W(t) - \frac{t}{\mu} \to \frac{1}{2}\left(\frac{\sigma^2}{\mu^2} - 1\right) \quad \text{when } t \to \infty \tag{7.77}$$

which is the same result as we found in (7.53).

Lorden (1970) has shown that the renewal function $W(t)$ of a *general* renewal process is bounded by

$$\frac{t}{\mu} - 1 \le W(t) \le \frac{t}{\mu} + \frac{\sigma^2}{\mu^2} \tag{7.78}$$

For a proof, see Cocozza-Thivent (1997, p. 170).

In section 2.19 we introduced several families of life distribution. A distribution was said to be "new better than used in expectation" (NBUE) when the mean remaining lifetime of a used item was less, or equal to, the mean life of a new item. In the same way, a distribution was said to be "new worse than used in expectation" (NWUE) when the mean remaining life of a used item was greater, or equal to, the mean life of a new item.

If we have an NBUE distribution, then $E(Y(t)) \le \mu$, and

$$W(t) = \frac{t + E(Y(t))}{\mu} - 1 \le \frac{t}{\mu} \quad \text{for } t \ge 0$$

and

$$\frac{t}{\mu} - 1 \le W(t) \le \frac{t}{\mu} \tag{7.79}$$

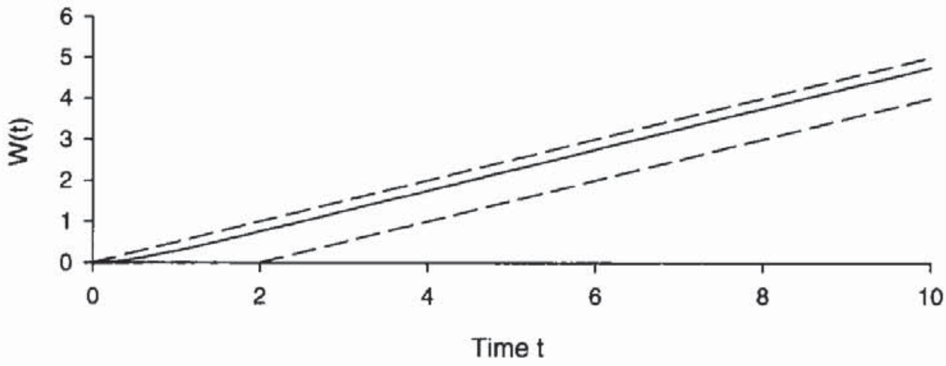

Fig. 7.12 The renewal function $W(t)$ of a renewal process with underlying distribution that is gamma $(2, \lambda)$, together with the bounds for $W(t)$, for $\lambda = 1$.

If we have an NWUE distribution, then $E(Y(t)) \geq \mu$, and

$$W(t) = \frac{t + E(Y(t))}{\mu} - 1 \geq \frac{t}{\mu} \quad \text{for } t \geq 0 \tag{7.80}$$

Further bounds for the renewal function are given by Dohi et al. (2002).

Example 7.7 (Cont.)
Reconsider the renewal process where the underlying distribution has a gamma distribution with parameters $(2, \lambda)$. This distribution has an increasing failure rate and is therefore also NBUE. We can therefore apply the bounds in (7.79). In Fig. 7.12 the renewal function (7.58)

$$W(t) = \frac{\lambda t}{2} - \frac{1}{4}\left(1 - e^{-2\lambda t}\right)$$

is plotted together with the bounds in (7.79)

$$\frac{\lambda t}{2} - 1 \leq W(t) \leq \frac{\lambda t}{2}$$

□

7.3.8 Superimposed Renewal Processes

Consider a series structure of n independent components that are put into operation at time $t = 0$. All the n components are assumed to be new at time $t = 0$. When a component fails, it is replaced with a new component of the same type or restored to an "as good as new" condition. Each component will thus produce a renewal process. The n components will generally be different, and the renewal processes will therefore have different underlying distributions.

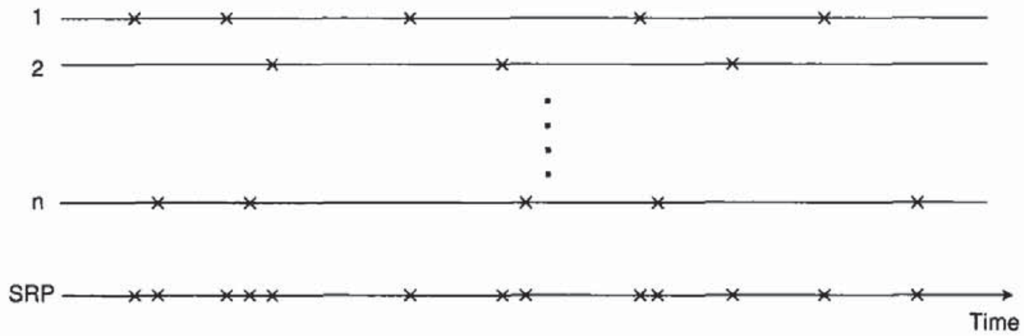

Fig. 7.13 Superimposed renewal process.

The process formed by the union of all the failures is called a *superimposed renewal process* (SRP). The n individual renewal processes and the SRP are illustrated in Fig. 7.13.

In general, the SRP will *not* be a renewal process. However, it has been shown, for example, by Drenick (1960), that superposition of an infinite number of independent *stationary* renewal processes is an HPP. Many systems are composed of a large number of components in series. Drenick's result is often used as a justification for assuming the time between system failures to be exponentially distributed.

Example 7.10
Consider a series structure of two components. When a component fails, it is replaced or repaired to an "as good as new" condition. Each component will therefore produce an ordinary renewal process. The time required to replace or repair a component is considered to be negligible. The components are assumed to fail and be repaired independent of each other. Both components are put into operation and are functioning at time $t = 0$. The series system fails as soon as one of its components fails, and the system failures will produce a superimposed renewal process. Times to failure for selected life distributions with increasing failure rates for the two components and the series system have been simulated on a computer and are illustrated in Fig. 7.14. The conditional ROCOF (when the failure times are given) is also shown in the figure. As illustrated in Fig. 7.14 the system is not restored to an "as good as new" state after each system failure. The system is subject to *imperfect repairs* (see Section 7.5) and the process of system failures is not a renewal process since the times between system failures do not have a common distribution. □

The superimposed renewal process is further discussed, for example, by Cox and Isham (1980), Ascher and Feingold (1984), and Thompson (1988).

7.3.9 Renewal Reward Processes

Consider a renewal process $\{N(t), t \geq 0\}$, and let $(S_{i-1}, S_i]$ be the duration of the ith renewal cycle, with interoccurrence time $T_i = S_i - S_{i-1}$. Let V_i be a reward associated to renewal T_i, for $i = 1, 2, \ldots$. The rewards $V_1, V_2, \ldots$ are assumed to be independent random variables with the common distribution function $F_V(v)$, and

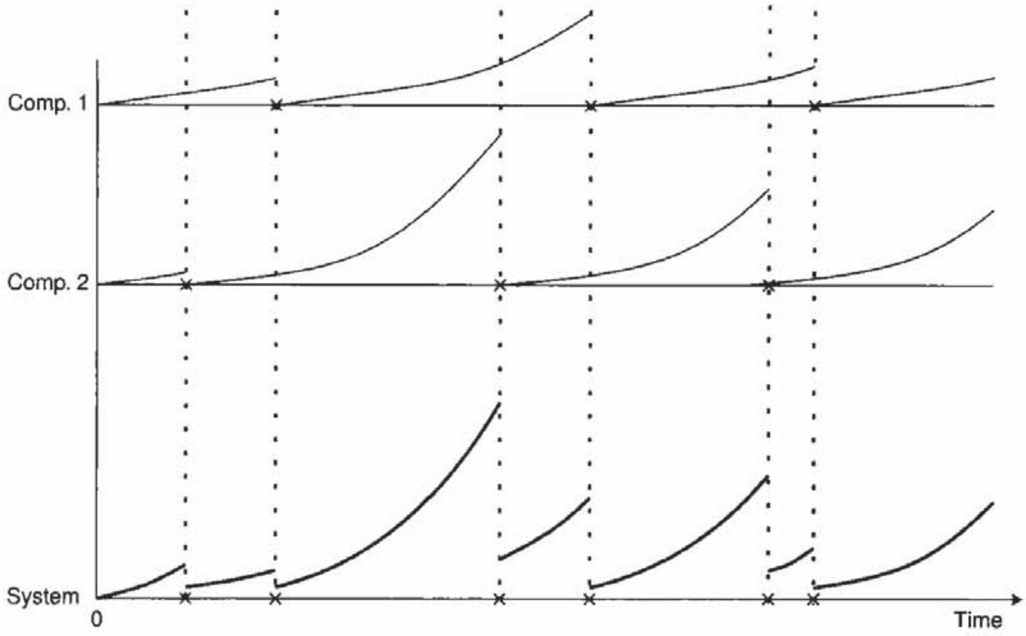

Fig. 7.14 Superimposed renewal process. Conditional ROCOF $w_C(t)$ of a series system with two components that are renewed upon failure.

with $E(T_i) < \infty$. This model is comparable with the compound Poisson process that was described on page 245. The accumulated reward in the time interval $(0, t]$ is

$$V(t) = \sum_{i=1}^{N(t)} V_i \tag{7.81}$$

Let $E(T_i) = \mu_T$ and $E(V_i) = \mu_V$. According to Wald's equation (7.22) the mean accumulated reward is

$$E(V(t)) = \mu_V \cdot E(N(t)) \tag{7.82}$$

According to the elementary renewal theorem (7.49), when $t \to \infty$,

$$\frac{W(t)}{t} = \frac{E(N(t))}{t} \to \frac{1}{\mu_T}$$

Hence

$$\frac{E(V(t))}{t} = \frac{\mu_V \cdot E(N(t))}{t} \to \frac{\mu_V}{\mu_T} \tag{7.83}$$

The same result is true even if the reward V_i is allowed to depend on the associated interoccurrence time T_i for $i = 1, 2, \ldots$. The pairs (T_i, V_i) for $i = 1, 2, \ldots$ are, however, assumed to be independent and identically distributed (for proof, see Ross

1996, p. 133). The reward V_i in renewal cycle i may, for example, be a function of the interoccurrence time T_i, for $i = 1, 2, \ldots$. When t is very large, then

$$V(t) \approx \mu_V \cdot \frac{t}{\mu_T}$$

which is an obvious result.

7.3.10 Delayed Renewal Processes

Sometimes we consider counting processes where the first interoccurrence time T_1 has a distribution function $F_{T_1}(t)$ that is different from the distribution function $F_T(t)$ of the subsequent interoccurrence times. This may, for example, be the case for a failure process where the component at time $t = 0$ is not new. Such a renewal process is called a *delayed* renewal process, or a *modified* renewal process. To specify that the process is not delayed, we sometimes say that we have an *ordinary* renewal process.

Several of the results presented earlier in this section can be easily extended to delayed renewal processes:

The Distribution of $N(t)$ Analogous with (7.41) we get

$$\Pr(N(t) = n) = F_{T_1}^* * F_T^{*(n-1)} - F_{T_1}^* * F_T^{*(n)} \tag{7.84}$$

The Distribution of S_n The Laplace transform of the density of S_n is from (7.34):

$$f^{*(n)}(s) = f_{T_1}^*(s)\left(f_T^*(s)\right)^{n-1} \tag{7.85}$$

The Renewal Function The integral equation (7.46) for the renewal function $W(t)$ becomes

$$W(t) = F_{T_1}(t) + \int_0^t W(t-x)\, dF_T(x) \tag{7.86}$$

and the Laplace transform is

$$W^*(s) = \frac{f_{T_1}^*(s)}{s(1 - f_T^*(s))} \tag{7.87}$$

The Renewal Density Analogous with (7.55) we get

$$w(t) = f_{T_1}(t) + \int_0^t w(t-x) f_T(x)\, dx \tag{7.88}$$

and the Laplace transform is

$$w^*(s) = \frac{f_{T_1}^*(s)}{1 - f_T^*(s)} \tag{7.89}$$

All the limiting properties for ordinary renewal processes, when $t \to \infty$, will obviously also apply for delayed renewal processes.

For more detailed results see, for example, Cocozza-Thivent (1997, section 6.2). We will here briefly discuss a special type of a delayed renewal process, the *stationary* renewal process.

Definition 7.6 A stationary renewal process is a delayed renewal process where the first renewal period has distribution function

$$F_{T_1}(t) = F_e(t) = \frac{1}{\mu} \int_0^t (1 - F_T(x))\, dx \tag{7.90}$$

while the underlying distribution of the other renewal periods is $F_T(t)$. □

Remarks

1. Note that $F_e(t)$ is the same distribution as we found in (7.63).

2. When the probability density function $f_T(t)$ of $F_T(t)$ exists, the density of $F_e(t)$ is

$$f_e(t) = \frac{dF_e(t)}{dt} = \frac{1 - F_T(t)}{\mu} = \frac{R_T(t)}{\mu}$$

3. As pointed out by Cox (1962, p. 28) the stationary renewal process has a simple physical interpretation. Suppose a renewal process is started at time $t = -\infty$, but that the process is not observed before time $t = 0$. Then the first renewal period observed, T_1, is the remaining lifetime of the component in operation at time $t = 0$. According to (7.63) the distribution function of T_1 is $F_e(t)$. A stationary renewal process is called an *equilibrium renewal process* by Cox (1962). This is the reason why we use the subscript e in $F_e(t)$. In Ascher and Feingold (1984) the stationary renewal process is called a renewal process with *asynchronous sampling*, while an ordinary renewal process is called a renewal process with *synchronous sampling*. □

Let $\{N_S(t), t \geq 0\}$ be a stationary renewal process, and let $Y_S(t)$ denote the remaining life of an item at time t. The stationary renewal process has the following properties (e.g., see Ross 1996, p. 131):

$$W_S(t) = t/\mu \tag{7.91}$$

$$\Pr(Y_S(t) \leq y) = F_e(y) \qquad \text{for all } t \geq 0 \tag{7.92}$$

$$\{N_S(t), t \geq 0\} \quad \text{has stationary increments} \tag{7.93}$$

where $F_e(y)$ is defined by equation (7.90).

***Remark*:** A homogeneous Poisson process is a stationary renewal process, because of the memoryless property of exponential distribution. The HPP is seen to fulfill all the three properties (7.91), (7.92), and (7.93). □

Example 7.11

Reconsider the renewal process in Example 7.7 where the interoccurrence times had a gamma distribution with parameters $(2, \lambda)$. The underlying distribution function is then

$$F_T(t) = 1 - e^{-\lambda t} - \lambda t\, e^{-\lambda t}$$

and the mean interoccurrence time is $E(T_i) = 2/\lambda$. Let us now assume that the process has been running for a long time and that when we start observing the process at time $t = 0$, it may be considered as a stationary renewal process.

According to (7.91), the renewal function for this stationary renewal process is $W_S(t) = \lambda t/2$, and the distribution of the remaining life, $Y_S(t)$ is (see 7.92)

$$\begin{aligned}\Pr(Y_S(t) \le y) &= \frac{\lambda}{2}\int_0^y (e^{-\lambda u} + \lambda u\, e^{-\lambda u})\, du \\ &= 1 - \left(1 + \frac{\lambda y}{2}\right) e^{-\lambda y}\end{aligned}$$

The mean remaining lifetime of an item at time t is

$$E(Y_S(t)) = \int_0^\infty \Pr(Y_S(t) > y)\, dy = \int_0^\infty \left(1 + \frac{\lambda y}{2}\right) e^{-\lambda y}\, dy = \frac{3}{2\lambda}$$

□

Delayed renewal processes are used in the next section to analyze alternating renewal processes.

7.3.11 Alternating Renewal Processes

Consider a system that is activated and functioning at time $t = 0$. Whenever the system fails, it is repaired. Let $U_1, U_2, \ldots$ denote the successive times to failure (up-times) of the system. Let us assume that the times to failure are independent and identically distributed with distribution function $F_U(t) = \Pr(U_i \le t)$ and mean $E(U)$ = MTTF (mean time to failure). Likewise we assume the corresponding repair times $D_1, D_2, \ldots$ to be independent and identically distributed with distribution function $F_D(d) = \Pr(D_i \le d)$ and mean $E(D)$ = MTTR (mean time to repair). MTTR denotes the total mean downtime following a failure and will usually involve much more that the active repair time. We therefore prefer to use the term MDT (mean downtime) instead of MTTR[2].

[2]In the rest of this book we are using T to denote time to failure. In this chapter we have already used T to denote interoccurrence time (renewal period), and we will therefore use U to denote the time to failure (up-time). We hope that this will not confuse the reader.

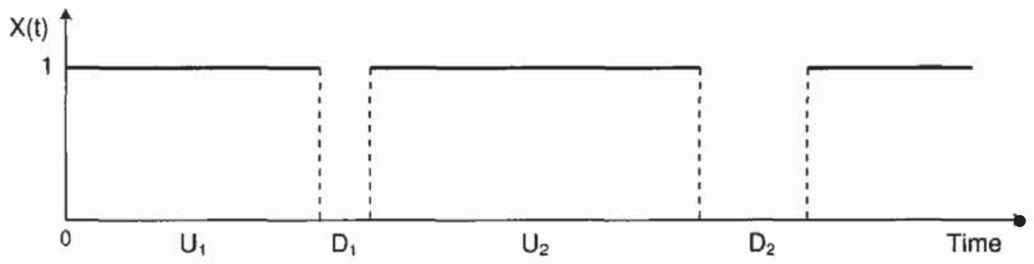

Fig. 7.15 Alternating renewal process.

If we define the completed repairs to be the renewals, we obtain an ordinary renewal process with renewal periods (interoccurrence times) $T_i = U_i + D_i$ for $i = 1, 2, \ldots$. The mean time between renewals is $\mu_T = \text{MTTF} + \text{MDT}$. This resulting process is called an *alternating renewal process* and is illustrated in Fig. 7.15. The underlying distribution function, $F_T(t)$, is the convolution of the distribution functions $F_U(t)$ and $F_D(t)$,

$$F_T(t) = \Pr(T_i \leq t) = \Pr(U_i + D_i \leq t) = \int_0^t F_U(t - x)\, dF_D(x) \tag{7.94}$$

If instead we let the renewals be the events when a failure occurs, we get a *delayed* renewal process where the first renewal period T_1 is equal to U_1 while $T_i = D_{i-1} + U_i$ for $i = 2, 3, \ldots$.

In this case the distribution function $F_{T_1}(t)$ of the first renewal period is given by

$$F_{T_1}(t) = \Pr(T_1 \leq t) = \Pr(U_1 \leq t) = F_U(t) \tag{7.95}$$

while the distribution function $F_T(t)$ of the other renewal periods is given by (7.94).

Example 7.12

Consider the alternating renewal process described above, and let the renewals be the completed repairs such that we have an ordinary renewal process. Let a reward V_i be associated to the ith interoccurrence time, and assume that this reward is defined such that we earn one unit per unit of time the system is functioning in the time period since the last failure. When the reward is measured in time units, then $E(V_i) = \mu_V =$ MTTF. The average availability $A_{\text{av}}(0, t)$ of the component in the time interval $(0, t)$ has been defined as the mean fraction of time in the interval $(0, t)$ where the system is functioning. From (7.83) we therefore get

$$A_{\text{av}}(0, t) \to \frac{\mu_V}{\mu_T} = \frac{\text{MTTF}}{\text{MTTF} + \text{MDT}} \quad \text{when } t \to \infty \tag{7.96}$$

which is the same result we obtain in Section 9.4 based on heuristic arguments. □

Availability The availability $A(t)$ of an item (component or system) was defined as the probability that the item is functioning at time t, that is, $A(t) = \Pr(X(t) = 1)$, where $X(t)$ denotes the state variable of the item.

As above, consider an alternating renewal process where the renewals are completed repairs, and let $T = U_1 + D_1$. The availability of the item is then

$$A(t) = \Pr(X(t) = 1) = \int_0^\infty \Pr(X(t) = 1 \mid T = x)\, dF_T(x)$$

Since the component is assumed to be "as good as new" at time $T = U_1 + D_1$, the process repeats itself from this point in time and

$$\Pr(X(t) = 1 \mid T = x) = \begin{cases} A(t-x) & \text{for} \quad t > x \\ \Pr(U_1 > t \mid T = x) & \text{for} \quad t \le x \end{cases}$$

Therefore

$$A(t) = \int_0^t A(t-x)\, dF_T(x) + \int_t^\infty \Pr(U_1 > t \mid T = x)\, dF_T(x)$$

But since $D_1 > 0$, then

$$\begin{aligned} \int_t^\infty \Pr(U_1 > t \mid U_1 + D_1 = x)\, dF_T(x) &= \int_0^\infty \Pr(U_1 > t \mid T = x)\, dF_T(x) \\ &= \Pr(U_1 > t) = 1 - F_U(t) \end{aligned}$$

Hence

$$A(t) = 1 - F_U(t) + \int_0^t A(t-x)\, dF_T(x) \tag{7.97}$$

We may now apply Lemma 7.1 and get

$$A(t) = 1 - F_T(t) + \int_0^t (1 - F_T(t-x))\, dW_{F_T}(x) \tag{7.98}$$

where

$$W_{F_T}(t) = \sum_{n=1}^\infty F_T^{(n)}(t)$$

is the renewal function for a renewal process with underlying distribution $F_T(t)$.

When $F_U(t)$ is a nonlattice distribution, the key renewal theorem (7.51) can be used with $Q(t) = 1 - F_U(t)$ and we get

$$\int_0^t (1 - F_U(t-x))\, dW_{F_T}(x) \underset{t\to\infty}{\to} \frac{1}{E(T)} \int_0^\infty (1 - F_U(t))\, dt = \frac{E(U)}{E(U) + E(D)}$$

Since $F_T(t) \to 1$ when $t \to \infty$, we have thus shown that

$$A = \lim_{t\to\infty} A(t) = \frac{E(U)}{E(U) + E(D)} = \frac{\text{MTTF}}{\text{MTTF+MDT}} \tag{7.99}$$

Notice that this is the same result as we got in (7.96) by using results from renewal reward processes.

Example 7.13

Consider a parallel structure of n components that fail and are repaired independent of each other. Component i has a time to failure (up-time) U_i which is exponentially distributed with failure rate λ_i and a time to repair (downtime) D_i which is also exponentially distributed with repair rate μ_i, for $i = 1, 2, \ldots$. The parallel structure will fail when all the n components are in a failed state at the same time. Since the components are assumed to be independent, a system failure must occur in the following way: Just prior to the system failure, $(n-1)$ components must be in a failed state, and then the functioning component must fail.

Let us now assume that the system has been in operation for a long time, such that we can use limiting (average) availabilities. The probability that component i is in a failed state is then approximately:

$$\bar{A}_i \approx \frac{\text{MDT}}{\text{MTTF} + \text{MDT}} = \frac{1/\mu_i}{1/\lambda_i + 1/\mu_i} = \frac{\lambda_i}{\lambda_i + \mu_i}$$

Similarly, the probability that component i is functioning is approximately

$$A_i \approx \frac{\mu_i}{\lambda_i + \mu_i}$$

The probability that a functioning component i will fail within a very short time interval of length Δt is approximately

$$\Pr(\Delta t) \approx \lambda_i \, \Delta t$$

The probability of system failure in the interval $(t, t + \Delta t)$, when t is large, is

$$\begin{aligned}
\Pr(\text{System failure in}(t, t+\Delta t)) &= \sum_{i=1}^{n} \left[\frac{\mu_i}{\lambda_i + \mu_i} \prod_{j \neq i} \frac{\lambda_j}{\lambda_j + \mu_j} \right] \cdot \lambda_i \, \Delta t + o(\Delta t) \\
&= \sum_{i=1}^{n} \left[\frac{\lambda_i}{\lambda_i + \mu_i} \prod_{j \neq i} \frac{\lambda_j}{\lambda_j + \mu_j} \right] \cdot \mu_i \, \Delta t + o(\Delta t) \\
&= \prod_{j=1}^{n} \frac{\lambda_j}{\lambda_j + \mu_j} \sum_{i=1}^{n} \mu_i \Delta t + o(\Delta t)
\end{aligned}$$

Since Δt is assumed to be very small, no more than one system failure will occur in the interval. We can therefore use Blackwell's theorem (7.50) to conclude that the above expression is just Δt times the reciprocal of the mean time between system failures, MTBF_S, that is,

$$\text{MTBF}_S = \left[\prod_{j=1}^{n} \frac{\lambda_j}{\lambda_j + \mu_j} \sum_{i=1}^{n} \mu_i \right]^{-1} \tag{7.100}$$

When the system is in a failed state, all the n components are in a failed state. Since the repair times (downtimes) are assumed to be independent with repair rates μ_i for $i = 1, 2, \ldots, n$, the system downtime will be exponential with repair rate $\sum_{i=1}^{n} \mu_i$, and the mean downtime to repair the system is

$$\text{MDT}_S = \frac{1}{\sum_{i=1}^{n} \mu_i}$$

The mean up-time, or the mean time to failure, MTTF_S, of the system is equal to $\text{MTBF}_S - \text{MDT}_S$:

$$\text{MTTF}_S = \left[\prod_{j=1}^{n} \frac{\lambda_j}{\lambda_j + \mu_j} \sum_{i=1}^{n} \mu_i \right]^{-1} - \frac{1}{\sum_{i=1}^{n} \mu_i} \tag{7.101}$$

$$= \frac{1 - \prod_{j=1}^{n} \lambda_j / (\lambda_j + \mu_j)}{\prod_{j=1}^{n} \lambda_j / (\lambda_j + \mu_j) \sum_{i=1}^{n} \mu_i} \tag{7.102}$$

To check that the above calculations are correct, we may calculate the average unavailability:

$$\bar{A}_S = \frac{\text{MDT}_S}{\text{MTTF}_S + \text{MDT}_S} = \prod_{j=1}^{n} \frac{\lambda_j}{\lambda_j + \mu_j}$$

which is in accordance with the results obtained in Chapter 9. [Example 7.13 is adapted from Example 3.5(B) in Ross (1996)]. □

Mean Number of Failures/Repairs First, let the renewals be the events where a repair is completed. Then we have an ordinary renewal process with renewal periods $T_1, T_2, \ldots$ which are independent and identically distributed with distribution function (7.94).

Assume that U_i and D_i both are continuously distributed with densities $f_U(t)$ and $f_D(t)$, respectively. The probability density function of the T_i's is then

$$f_T(t) = \int_0^t f_U(t - x) f_D(x) \, dx \tag{7.103}$$

According to Appendix B the Laplace transform of (7.103) is

$$f_T^*(s) = f_U^*(s) \cdot f_D^*(s)$$

Let $W_1(t)$ denote the renewal function, that is, the mean number of completed repairs in the time interval $(0, t]$. According to (7.84)

$$W_1^*(s) = \frac{f_U^*(s) \cdot f_D^*(s)}{s(1 - f_U^*(s) \cdot f_D^*(s))} \tag{7.104}$$

In this case both the U_i's and the D_i's were assumed to be continuously distributed. This, however, turns out *not* to be essential. Equation (7.104) is also valid for discrete distributions, or for a mixture of discrete and continuous distributions. In this case we may use that

$$\begin{aligned} f_U^*(s) &= E(e^{-sU_i}) \\ f_D^*(s) &= E(e^{-sD_i}) \end{aligned}$$

The mean number of completed repairs in $(0, t]$ can now, at least in principle, be determined for any choice of life- and repair time distributions.

Next, let the renewals be the events where a failure occurs. In this case we get a delayed renewal process. The renewal periods $T_1, T_2, \ldots$ are independent and $F_{T_1}(t)$ is given by (7.95) while the distribution of $T_2, T_3, \ldots$ is given by (7.94).

Let $W_2(t)$ denote the renewal function, that is, the mean number of failures in $(0, t]$ under these conditions. According to (7.87) the Laplace transform is

$$W_2^*(s) = \frac{f_U^*(s)}{s(1 - f_U^*(s) \cdot f_D^*(s))} \tag{7.105}$$

which, at least in principle, can be inverted to obtain $W_2(t)$.

Availability at a Given Point of Time By taking Laplace transforms of (7.98) we get

$$A^*(s) = \frac{1}{s} - F_U^*(s) + \left(\frac{1}{s} - F_U^*(s)\right) \cdot w_{F_T}^*(s)$$

Since

$$F^*(s) = \frac{1}{s} f^*(s)$$

then

$$A^*(s) = \frac{1}{s}(1 - f_U^*(s)) \cdot (1 + w_{F_T}^*(s))$$

If we have an ordinary renewal process (i.e., the renewals are the events where a repair is completed), then

$$w_{F_T}^*(s) = s W_1^*(s)$$

Hence

$$A^*(s) = \frac{1}{s}(1 - f_U^*(s)) \cdot \left(1 + \frac{f_U^*(s) \cdot f_D^*(s)}{1 - f_U^*(s) \cdot f_D^*(s)}\right)$$

that is,

$$A^*(s) = \frac{1 - f_U^*(s)}{s\left(1 - f_U^*(s) \cdot f_D^*(s)\right)} \tag{7.106}$$

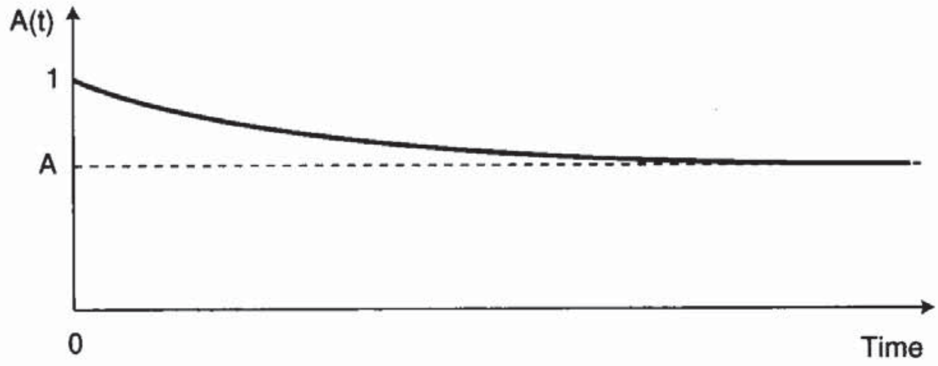

Fig. 7.16 Availability of a component with exponential life and repair times.

The availability $A(t)$ can in principle be determined from (7.106) for any choice of life and repair time distributions.

Example 7.14 Exponential Lifetime–Exponential Repair Time
Consider an alternating renewal process where the component up-times $U_1, U_2, \ldots$ are independent and exponentially distributed with failure rate λ. The corresponding downtimes are also assumed to be independent and exponentially distributed with the repair rate $\mu = 1/\text{MDT}$.

Then

$$\begin{aligned} f_U(t) &= \lambda e^{-\lambda t} \quad \text{for } t > 0 \\ f_U^*(s) &= \frac{\lambda}{\lambda + s} \end{aligned}$$

and

$$\begin{aligned} f_D(t) &= \mu e^{-\mu t} \quad \text{for } t > 0 \\ f_D^*(s) &= \frac{\mu}{\mu + s} \end{aligned}$$

The availability $A(t)$ is then obtained from (7.106):

$$\begin{aligned} A^*(s) &= \frac{1 - \lambda/(\lambda + s)}{s[1 - (\lambda/(\lambda + s)) \cdot (\mu/(\mu + s)]} \\ &= \frac{\mu}{\lambda + \mu} \cdot \frac{1}{s} + \frac{\lambda}{\lambda + \mu} \cdot \frac{1}{s + (\lambda + \mu)} \end{aligned} \tag{7.107}$$

Equation (7.107) can be inverted (see Appendix B) and we get

$$A(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu)t} \tag{7.108}$$

which is the same result as we get in Chapter 8. The availability $A(t)$ is illustrated in Fig. 7.16.

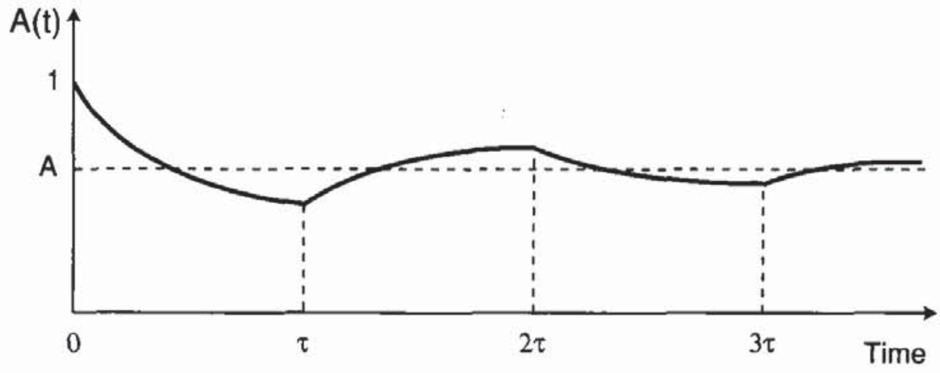

Fig. 7.17 The availability of a component with exponential lifetimes and constant repair time (τ).

The limiting availability is

$$A = \lim_{t\to\infty} A(t) = \frac{\mu}{\lambda+\mu} = \frac{1/\lambda}{1/\lambda + 1/\mu} = \frac{\text{MTTF}}{\text{MTTF+MDT}}$$

By inserting $f_U^*(s)$ and $f_D^*(s)$ into (7.104) we get the Laplace transform of the mean number of renewals $W(t)$,

$$\begin{aligned} W^*(s) &= \frac{(\lambda/(\lambda+s))\cdot(\mu/(\mu+s))}{s[1-(\lambda/(\lambda+s))\cdot(\mu/(\mu+s))]} \\ &= \frac{\lambda\mu}{\lambda+\mu}\cdot\frac{1}{s^2} - \frac{\lambda\mu}{(\lambda+\mu)^2}\cdot\frac{1}{s} + \frac{\lambda\mu}{(\lambda+\mu)^2}\cdot\frac{1}{s+(\lambda+\mu)} \end{aligned}$$

By inverting this expression we get the mean number of completed repairs in the time interval $(0, t]$

$$W(t) = \frac{\lambda\mu}{\lambda+\mu}t - \frac{\lambda\mu}{(\lambda+\mu)^2} + \frac{\lambda\mu}{(\lambda+\mu)^2}\, e^{-(\lambda+\mu)t} \tag{7.109}$$

□

Example 7.15 Exponential Lifetime–Constant Repair Time
Consider an alternating renewal process where the system up-times $U_1, U_2, \ldots$ are independent and exponentially distributed with failure rate λ. The downtimes are assumed to be constant and equal to τ with probability 1: $\Pr(D_i = \tau) = 1$ for $i = 1, 2, \ldots$. The corresponding Laplace transforms are

$$\begin{aligned} f_U^*(s) &= \frac{\lambda}{\lambda+s} \\ f_D^*(s) &= E(e^{-sD}) = e^{-s\tau}\cdot\Pr(D=\tau) = e^{-s\tau} \end{aligned}$$

Hence the Laplace transform of the availability (7.106) becomes

$$\begin{aligned}
A^*(s) &= \frac{1-\lambda/(\lambda+s)}{s[1-(\lambda/(\lambda+s))\,e^{-s\tau}]} = \frac{1}{s+\lambda-\lambda e^{-s\tau}} \\
&= \frac{1}{\lambda+s}\cdot\left[\frac{1}{1-(\lambda/(\lambda+s))\,e^{-s\tau}}\right] = \frac{1}{\lambda+s}\sum_{\nu=0}^{\infty}\left(\frac{\lambda}{\lambda+s}\right)^{\nu} e^{-s\nu\tau} \\
&= \frac{1}{\lambda}\sum_{\nu=0}^{\infty}\left(\frac{\lambda}{\lambda+s}\right)^{\nu+1} e^{-s\nu\tau} \qquad (7.110)
\end{aligned}$$

The availability then becomes

$$A(t) = \mathcal{L}^{-1}(A^*(s)) = \sum_{\nu=0}^{\infty}\frac{1}{\lambda}\mathcal{L}^{-1}\left[\left(\frac{\lambda}{\lambda+s}\right)^{\nu+1} e^{-s\nu\tau}\right]$$

According to Appendix B

$$\mathcal{L}^{-1}\left[\left(\frac{\lambda}{\lambda+s}\right)^{\nu+1}\right] = \frac{\lambda^{\nu+1}}{\nu!}t^{\nu}e^{-\lambda t} = f(t)$$

$$\mathcal{L}^{-1}(e^{-s\nu\tau}) = \delta(t-\nu\tau)$$

where $\delta(t)$ denotes the Dirac delta-function. Thus

$$\begin{aligned}
\mathcal{L}^{-1}\left[(\frac{\lambda}{\lambda+s})^{\nu+1}\cdot e^{-s\nu\tau}\right] &= \mathcal{L}^{-1}\left[(\frac{\lambda}{\lambda+s})^{\nu+1}\right] * \mathcal{L}^{-1}(e^{-s\nu\tau}) \\
&= \int_0^{\infty}\delta(t-\nu\tau-x)f(x)\,dx \\
&= f(t-\nu\tau)\cdot u(t-\nu\tau)
\end{aligned}$$

where

$$u(t-\nu\tau) = \begin{cases} 1 & \text{if} \quad t \geq \nu\tau \\ 0 & \text{if} \quad t < \nu\tau \end{cases}$$

Hence the availability is

$$A(t) = \sum_{\nu=0}^{\infty}\frac{\lambda^{\nu}}{\nu!}(t-\nu\tau)^{\nu}e^{-\lambda(t-\nu\tau)}u(t-\nu\tau) \qquad (7.111)$$

The availability $A(t)$ is illustrated in Fig. 7.17.

The limiting availability is then according to (7.99)

$$A = \lim_{t\to\infty}A(t) = \frac{\text{MTTF}}{\text{MTTF+MDT}} = \frac{1/\lambda}{(1/\lambda)+\tau} = \frac{1}{1+\lambda\tau} \qquad (7.112)$$

The Laplace transform for the renewal density is

$$\begin{aligned} w^*(s) &= \frac{f_T^*(s) \cdot f_D^*(s)}{1 - f_T^*(s) \cdot f_D^*(s)} = \frac{\lambda e^{-s\tau}/(\lambda + s)}{1 - \lambda e^{-s\tau}/(\lambda + s)} \\ &= \frac{1}{\lambda + s - \lambda e^{-s\tau}} \lambda e^{-s\tau} = \lambda \cdot A^*(s)\, e^{-s\tau} \end{aligned}$$

where $A^*(s)$ is given by (7.110).

Then the renewal density becomes

$$w(t) = \lambda \mathcal{L}^{-1}(A^*(s) \cdot e^{-s\tau}) = \lambda \int_0^{\infty} \delta(t - \tau - x) A(x)\, dx$$

that is,

$$w(t) = \begin{cases} \lambda \cdot A(t - \tau) & \text{if} \quad t \geq \tau \\ 0 & \text{if} \quad t < \tau \end{cases} \tag{7.113}$$

Hence the mean number of completed repairs in the time interval $(0, t]$ for $t > \tau$ is

$$W(t) = \int_0^t w(u)\, du = \lambda \int_{\tau}^t A(u - \tau)\, du = \lambda \int_0^{t-\tau} A(u)\, du \tag{7.114}$$

□

7.4 NONHOMOGENEOUS POISSON PROCESSES

In this section the homogeneous Poisson process is generalized by allowing the rate of the process to be a function of time.

7.4.1 Introduction and Definitions

Definition 7.7 A counting process $\{N(t), t \geq 0\}$ is a nonhomogeneous (or nonstationary) Poisson process with rate function $w(t)$ for $t \geq 0$, if

1. $N(0) = 0$.
2. $\{N(t), t \geq 0\}$ has independent increments.
3. $\Pr(N(t + \Delta t) - N(t) \geq 2) = o(\Delta t)$, which means that the system will not experience more than one failure at the same time.
4. $\Pr(N(t + \Delta t) - N(t) = 1) = w(t)\Delta t + o(\Delta t)$. □

The basic "parameter" of the NHPP is the ROCOF function $w(t)$. This function is also called the *peril rate* of the NHPP. The *cumulative rate* of the process is

$$W(t) = \int_0^t w(u)\, du \tag{7.115}$$

This definition of course covers the situation in which the rate is a function of some observed explanatory variable that is a function of the time t.

It is important to note that the NHPP model does not require stationary increments. This means that failures may be more likely to occur at certain times than others, and hence the interoccurrence times are generally neither independent nor identically distributed. Consequently, statistical techniques based on the assumption that the data are independent and identically distributed cannot be applied to an NHPP.

The NHPP is often used to model trends in the interoccurrence times, that is, improving (*happy*) or deteriorating (*sad*) systems. It seems intuitive that a happy system will have a decreasing ROCOF function, while a sad system will have an increasing ROCOF function. Several studies of failure data from practical systems have concluded that the NHPP was an adequate model, and that the systems that were studied approximately satisfied the properties of the NHPP listed in Definition 7.7.

Due to the assumption of independent increments, the number of failures in a specified interval $(t_1, t_2]$ will be independent of the failures and interoccurrence times prior to time t_1. When a failure has occurred at time t_1, the conditional ROCOF $w_C(t \mid \mathcal{H}_t)$ (see page 254) in the next interval will be $w(t)$ and independent of the history $\mathcal{H}_{t_1}$ up to time t_1, and especially the case when no failure has occurred before t_1, in which case $w(t) = z(t)$, that is, the failure rate function (FOM) for $t < t_1$. A practical implication of this assumption is that the conditional (ROCOF), $w_C(t)$, is the same just before a failure and immediately after the corresponding repair. This assumption has been termed *minimal repair* (see Ascher and Feingold, 1984, p. 51). When replacing failed parts that may have been in operation for a long time, by new ones, an NHPP clearly is not a realistic model. For the NHPP to be realistic, the parts put into service should be identical to the old ones, and hence should be aged outside the system under identical conditions for the same period of time.

Now consider a system consisting of a large number of components. Suppose that a critical component fails and causes a system failure and that this component is immediately replaced by a component of the same type, thus causing a negligible system downtime. Since only a small fraction of the system is replaced, it seems natural to assume that the systems's reliability after the repair essentially is the same as immediately before the failure. In other words, the assumption of *minimal repair* is a realistic approximation. When an NHPP is used to model a repairable system, the system is treated as a *black box* in that there is no concern about how the system "looks inside."

A car is a typical example of a repairable system. Usually the operating time of a car is expressed in terms of the mileage indicated on the speedometer. Repair actions will usually not imply any extra mileage. The repair "time" is thus negligible. Many repairs are accomplished by adjustments or replacement of single components. The minimal repair assumption is therefore often applicable and the NHPP may be accepted as a realistic model, at least as a first order approximation.

Consider an NHPP with ROCOF $w(t)$, and suppose that failures occur at times $S_1, S_2, \ldots$. An illustration of $w(t)$ is shown in Fig. 7.18.

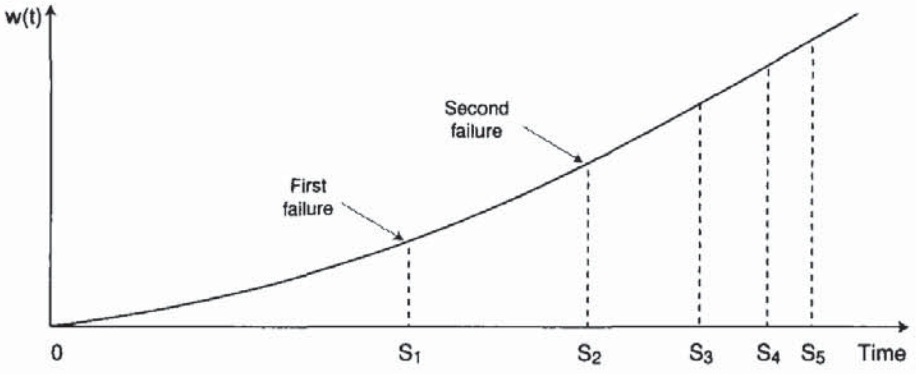

Fig. 7.18 The ROCOF $w(t)$ of an NHPP and random failure times.

7.4.2 Some Results

From the definition of the NHPP it is straightforward to verify (e.g., see Ross 1996, p. 79) that the number of failures in the interval $(0, t]$ is Poisson distributed:

$$\Pr(N(t) = n) = \frac{[W(t)]^n}{n!} e^{-W(t)} \quad \text{for } n = 0, 1, 2, \ldots \tag{7.116}$$

The mean number of failures in $(0, t]$ is therefore

$$E(N(t)) = W(t)$$

and the variance is $\text{var}(N(t)) = W(t)$. The cumulative rate $W(t)$ of the process (7.115) is therefore the mean number of failures in the interval $(0, t]$ and is sometimes called the *mean value function* of the process. When n is large, $\Pr(N(t) \le n)$ may be determined by normal approximation:

$$\begin{aligned} \Pr(N(t) \le n) &= \Pr\left(\frac{N(t) - W(t)}{\sqrt{W(t)}} \le \frac{n - W(t)}{\sqrt{W(t)}}\right) \\ &= \Phi\left(\frac{n - W(t)}{\sqrt{W(t)}}\right) \end{aligned} \tag{7.117}$$

From (7.116) it follows that the number of failures in the interval $(v, t + v]$ is Poisson distributed:

$$\begin{aligned} \Pr(N(t+v) - N(v) = n) &= \frac{[W(t+v) - W(v)]^n}{n!} e^{-[W(t+v)-W(v)]} \\ &\quad \text{for } n = 0, 1, 2, \ldots \end{aligned}$$

and that the mean number of failures in the interval $(v, t + v]$ is

$$E(N(t+v) - N(v)) = W(t+v) - W(v) = \int_v^{t+v} w(u)\, du \tag{7.118}$$

The probability of no failure in the interval (t_1, t_2) is

$$\Pr(N(t_2) - N(t_1) = 0) = e^{-\int_{t_1}^{t_2} w(t)\, dt}$$

Let S_n denote the time until failure n for $n = 0, 1, 2, \ldots$, where $S_0 = 0$. The distribution of S_n is given by:

$$\Pr(S_n > t) = \Pr(N(t) \le n-1) = \sum_{k=0}^{n-1} \frac{W(t)^k}{k!} e^{-W(t)} \tag{7.119}$$

When $W(t)$ is small, this probability may be determined from standard tables of the Poisson distribution. When $W(t)$ is large, the probability may be determined by normal approximation; see (7.117):

$$\begin{aligned} \Pr(S_n > t) &= \Pr(N(t) \le n-1) \\ &\approx \Phi\left(\frac{n-1-W(t)}{\sqrt{W(t)}}\right) \end{aligned} \tag{7.120}$$

Time to First Failure Let T_1 denote the time from $t = 0$ until the first failure. The survivor function of T_1 is

$$R_1(t) = \Pr(T_1 > t) = \Pr(N(t) = 0) = e^{-W(t)} = e^{-\int_0^t w(t)\,dt} \tag{7.121}$$

Hence the failure rate (FOM) function $z_{T_1}(t)$ of the first interoccurrence time T_1 is equal to the ROCOF $w(t)$ of the process. Note, however, the different meaning of the two expressions. $z_{T_1}(t)\Delta t$ approximates the (conditional) probability that the *first* failure occurs in $(t, t+\Delta t]$, while $w(t)\Delta t$ approximates the (unconditional) probability that a failure, not necessarily the first, occurs in $(t, t+\Delta t]$.

A consequence of (7.121) is that the distribution of the first interoccurrence time, that is, the time from $t = 0$ until the system's first failure, will determine the ROCOF of the entire process. Thompson (1981) claims that this is a nonintuitive fact which is casting doubt on the NHPP as a realistic model for repairable systems. Use of an NHPP model implies that if we are able to estimate the failure rate (FOM) function of the time to the *first* failure, such as for a specific type of automobiles, we at the same time have an estimate of the ROCOF of the entire life of the automobile.

Time Between Failures Assume that the process is observed at time t_0, and let $Y(t_0)$ denote the time until the next failure. In the previous sections $Y(t_0)$ was called the remaining lifetime, or the forward recurrence time. By using (7.116), we can express the distribution of $Y(t_0)$ as

$$\begin{aligned} \Pr(Y(t_0) > t) &= \Pr(N(t+t_0) - N(t_0) = 0) = e^{-[W(t+t_0)-W(t_0)]} \\ &= e^{-\int_{t_0}^{t+t_0} w(u)\,du} = e^{-\int_0^t w(u+t_0)\,du} \end{aligned} \tag{7.122}$$

Note that this result is independent of whether t_0 denotes a failure time or an arbitrary point in time.

Assume that t_0 is the time, S_{n-1}, at failure $n-1$. In this case $Y(t_0)$ denotes the time between failure $n-1$ and failure n (i.e., the nth interoccurrence time $T_n = S_n - S_{n-1}$). The failure rate (FOM) function of the nth interoccurrence time T_n is from (7.122):

$$z_{t_0}(t) = w(t+t_0) \quad \text{for } t \ge 0 \tag{7.123}$$

Notice that this is a conditional failure rate, given that $S_{n-1} = t_0$. The mean time between failure $n - 1$ (at time t_0) and failure n, MTBF_n, is

$$\begin{aligned} \text{MTBF}_n &= E(T_n) = \int_0^\infty \Pr(Y_{t_0} > t)\,dt \\ &= \int_0^\infty e^{-\int_0^t w(u+t_0)\,du}\,dt \end{aligned} \tag{7.124}$$

Example 7.16

Consider an NHPP with ROCOF $w(t) = 2\lambda^2 t$, for $\lambda > 0$ and $t \geq 0$. The mean number of failures in the interval $(0, t)$ is $W(t) = E(N(t)) = \int_0^t w(u)\,du = (\lambda t)^2$. The distribution of the time to the first failure, T_1, is given by the survivor function

$$R_1(t) = e^{-W(t)} = e^{-(\lambda t)^2} \quad \text{for } t \geq 0$$

that is, a Weibull distribution with scale parameter λ and shape parameter $\alpha = 2$. If we observe the process at time t_0, the distribution of the time $Y(t_0)$ till the next failure is from (7.122):

$$\Pr(Y(t_0) > t) = e^{-\int_0^t w(u+t_0)\,du} = e^{-\lambda^2(t^2+2t_0 t)}$$

If t_0 is the time of failure $n - 1$, the time to the next failure, $Y(t_0)$, is the nth interoccurrence time T_n and the failure rate (FOM) function of T_n is

$$z_{t_0}(t) = 2\lambda^2(t + t_0)$$

which is linearly increasing with the time t_0 of failure $n - 1$. Notice again that this is a conditional rate, given that failure $n - 1$ occurred at time $S_{n-1} = t_0$. The mean time between failure $n - 1$ and failure n is from (7.124):

$$\text{MTBF}_n = \int_0^\infty e^{-\lambda^2(t^2+2t_0 t)}\,dt$$

□

Relation to the Homogeneous Poisson Process Let $\{N(t), t \geq 0\}$ be an NHPP with ROCOF $w(t) > 0$ such that the inverse $W^{-1}(t)$ of the cumulative rate $W(t)$ exists, and let $S_1, S_2, \ldots$ be the times when the failures occur.

Consider the time-transformed occurrence times $W(S_1), W(S_2), \ldots$, and let $\{N^*(t), t \geq 0\}$ denote the associated counting process. The distribution of the (transformed) time $W(S_1)$ till the first failure is from (7.121)

$$\Pr(W(S_1) > t) = \Pr(S_1 > W^{-1}(t)) = e^{-W(W^{-1}(t))} = e^{-t}$$

that is, an exponential distribution with parameter 1.

The new counting process is defined by

$$N(t) = N^*(W(t)) \quad \text{for } t \geq t$$

hence

$$N^*(t) = N(W^{-1}(t)) \quad \text{for } t \geq 0$$

and we get from (7.116)

$$\begin{aligned} \Pr(N^*(t) = n) &= \Pr(N(W^{-1}(t)) = n) \\ &= \frac{[W(W^{-1}(t))]^n}{n!} e^{-W(W^{-1}(t))} = \frac{1^n}{n!} e^{-t} \end{aligned}$$

that is, the Poisson distribution with rate 1. We have thereby shown that an NHPP with cumulative rate $W(t)$ (where the inverse of $W(t)$ exists) can be transformed into an HPP with rate 1, by time-transforming the failure occurrence times $S_1, S_2, \dots$ to $W(S_1), W(S_2), \dots$.

7.4.3 The Nelson-Aalen Estimator

Let $\{N(t), t \geq 0\}$ be an NHPP with ROCOF $w(t)$. We want to find an estimate of the mean number of failures $W(t) = E(N(t)) = \int_0^t w(u)\,du$ in the interval $(0, t)$. An obvious estimate is $\widehat{W}(t) = N(t)$.

Assume that we have n different and independent NHPPs $\{N_i(t), t \geq 0\}$ for $i = 1, 2, \dots, n$ with a common ROCOF $w(t)$. This is, for example, the situation when we observe failures of the same type of repairable equipment installed in different places. Each installation will then produce a separate NHPP with rate $w(t)$. An obvious estimator for $W(t)$ is now

$$\widehat{W}(t) = \frac{1}{n}\sum_{i=1}^{n} N_i(t) = \frac{\text{Total number of failures in}(0, t]}{\text{Total number of processes in}(0, t]}$$

This estimator may be written in an alternative way:

$$\widehat{W}(t) = \frac{1}{n}\sum_{i=1}^{n} N_i(t) = \sum_{\{i;\, T_i \leq t\}} \frac{1}{n} = \sum_{i \geq 1} \frac{1}{n}\, 1_{\{T_i \leq t\}}$$

where $T_1, T_2, \dots$ denotes the failure times (for all the processes), and $1_{\{T_i \leq t\}}$ is an indicator that is equal to 1 when $T_i \leq t$, and 0 otherwise.

In practice, the various processes will not be observed in the same interval. As an illustration, let us assume that n_1 processes are observed in the interval $(0, t_1]$, and that n_2 processes are observed for $t \geq t_1$. Let $N(t)$ denote the total number of failures in $(0, t]$, irrespective of how many processes are active. It seems now natural to estimate $W(t)$ by

$$\widehat{W}(t) = \frac{N(t)}{n_1} = \sum_{\{i;\, T_i \leq t_1\}} \frac{1}{n_1} \quad \text{when } 0 \leq t \leq t_1$$

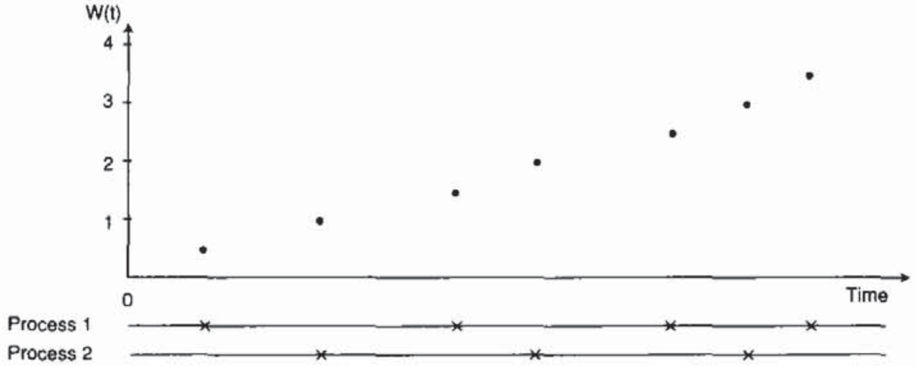

Fig. 7.19 The Nelson-Aalen estimator for two simultaneous processes.

When $t > t_1$, an estimator for $\int_{t_1}^{t} w(u)\,du = W(t) - W(t_1)$ is

$$\frac{N(t) - N(t_1)}{n_2} = \sum_{\{i;\; t_1 < T_i \le t\}} \frac{1}{n_2} \quad \text{when } t > t_1$$

The total estimator will then be

$$\widehat{W}(t) = \sum_{\{i;\; T_i \le t_1\}} \frac{1}{n_1} + \sum_{\{i;\; t_1 < T_i \le t\}} \frac{1}{n_2}$$

Assume now that we have a set of independent NHPPs with a common ROCOF $w(t)$. Let $Y(s)$ denote the number of active processes immediately before time s. From the arguments above, it seems natural to use the following estimator for $W(t)$,

$$\widehat{W}(t) = \sum_{\{i;\; T_i \le t\}} \frac{1}{Y(T_i)} = \sum_{i \ge 1} \frac{1}{Y(T_i)} 1_{\{T_i \le t\}} \tag{7.125}$$

This nonparametric estimator is called the Nelson-Aalen estimator for $W(t)$. Note that when there is only one sample, then the Nelson-Aalen estimator coincides with $N(t)$, which is plotted in Fig. 7.3 A simple example of the Nelson-Aalen estimator for two simultaneous processes is illustrated in Fig. 7.19.

The estimator (7.125) was introduced by Aalen (1975, 1978) for counting processes in general and generalizes the Nelson (1969) estimator which is further discussed in Chapter 8. It may be shown (see the discussion in Andersen and Borgan, 1985) that $\widehat{W}(t)$ is an approximately unbiased estimator of $W(t)$ and that the variance can be estimated (almost unbiasedly) by

$$\text{var}(\widehat{W}(t)) \approx \hat{\sigma}^2(t) = \sum_{\{i;\; T_i \le t\}} \frac{1}{Y(T_i)^2} \tag{7.126}$$

$\widehat{W}(t)$ may further be shown (see Andersen and Borgan, 1985) to be asymptotically normally distributed with mean $W(t)$ and a variance estimated by $\hat{\sigma}^2(t)$. Hence an approximate $100(1-\alpha)\%$ pointwise confidence interval for $W(t)$, is given by

$$\widehat{W}(t) - u_{\alpha/2}\hat{\sigma}(t) \le W(t) \le \widehat{W}(t) + u_{\alpha/2}\hat{\sigma}(t)$$

where u_α denotes the upper $100\alpha\%$ percentile of the standard normal distribution $\mathcal{N}(0, 1)$.

7.4.4 Parametric NHPP Models

Several parametric models have been established to describe the ROCOF of an NHPP. We will discuss some of these models:

1. The power law model
2. The linear model
3. The log-linear model

All three models may be written in the common form (see Atwood, 1992)

$$w(t) = \lambda_0\, g(t; \vartheta) \tag{7.127}$$

where λ_0 is a common multiplier, and $g(t; \vartheta)$ determines the shape of the ROCOF $w(t)$. The three models may be parameterized in various ways. In this section we shall use the parameterization of Crowder et al. (1991), although the parametrization of Atwood (1992) may be more logical.

The Power Law Model In the power law model the ROCOF of the NHPP is defined as

$$w(t) = \lambda\beta t^{\beta-1} \quad \text{for } \lambda > 0,\ \beta > 0 \text{ and } t \geq 0 \tag{7.128}$$

This NHPP is sometimes referred to as a *Weibull process*, since the ROCOF has the same functional form as the failure rate (FOM) function of the Weibull distribution. Also note that the first arrival time T_1 of this process is Weibull distributed with shape parameter β and scale parameter λ. However, according to Ascher and Feingold (1984), one should avoid the name Weibull process in this situation, since it gives the wrong impression that the Weibull distribution can be used to model trend in interoccurrence times of a repairable system. Hence such notation may lead to confusion.

A repairable system modeled by the power law model is seen to be improving (happy) if $0 < \beta < 1$, and deteriorating (sad) if $\beta > 1$. If $\beta = 1$ the model reduces to an HPP. The case $\beta = 2$ is seen to give a linearly increasing ROCOF. This model was studied in Example 7.16.

The power law model was first proposed by Crow (1974) based on ideas of Duane (1964). A goodness-of-fit test for the power law model based on total time on test (TTT) plots (see Chapter 11) is proposed and discussed by Klefsjö and Kumar (1992).

Assume that we have observed an NHPP in a time interval $(0, t_0]$ and that failures have occurred at times $s_1, s_2, \ldots, s_n$. Maximum likelihood estimates $\hat{\beta}$ and $\hat{\lambda}$ of β and λ, respectively, are given by

$$\hat{\beta} = \frac{n}{n \ln t_0 - \sum_{i=1}^{n} \ln s_i} \tag{7.129}$$

and

$$\hat{\lambda} = \frac{n}{t_0^{\hat{\beta}}} \tag{7.130}$$

The estimates are further discussed by Crowder et al. (1991, p. 171) and Cocozza-Thivent (1997, p. 64).

A $(1 - \varepsilon)$ confidence interval for β is given by (see Cocozza-Thivent 1997, p. 65)

$$\left(\frac{\hat{\beta}}{2n} z_{(1-\varepsilon/2),2n}, \frac{\hat{\beta}}{2n} z_{(1+\varepsilon/2),2n} \right) \tag{7.131}$$

where $z_{\varepsilon,\nu}$ denotes the upper $100\varepsilon\%$ percentile of the chi-square (χ^2) distribution with ν degrees of freedom (tables are given in Appendix F).

The Linear Model In the linear model the ROCOF of the NHPP is defined by

$$w(t) = \lambda(1 + \alpha t) \quad \text{for } \lambda > 0 \text{ and } t \geq 0 \tag{7.132}$$

The linear model has been discussed by Vesely (1991) and Atwood (1992). A repairable system modeled by the linear model is deteriorating if $\alpha > 0$, and improving when $\alpha < 0$. When $\alpha < 0$, then $w(t)$ will sooner or later become less than zero. The model should only be used in time intervals where $w(t) > 0$.

The Log-Linear Model In the log-linear model or *Cox-Lewis* model, the ROCOF of the NHPP is defined by

$$w(t) = e^{\alpha + \beta t} \quad \text{for } -\infty < \alpha, \beta < \infty \text{ and } t \geq 0 \tag{7.133}$$

A repairable system modeled by the log-linear model is improving (happy) if $\beta < 0$, and deteriorating (sad) if $\beta > 0$. When $\beta = 0$ the log-linear model reduces to an HPP.

The log-linear model was proposed by Cox and Lewis (1966) who used the model to investigate trends in the interoccurrence times between failures in air-conditioning equipment in aircrafts. The first arrival time T_1 has failure rate (FOM) function $z(t) = e^{\alpha + \beta t}$ and hence has a truncated Gumbel distribution of the smallest extreme (i.e., a Gompertz distribution; see Section 2.17).

Assume that we have observed an NHPP in a time interval $(0, t_0]$ and that failures have occurred at times $s_1, s_2, \ldots, s_n$. Maximum likelihood estimates $\hat{\alpha}$ and $\hat{\beta}$ of α and β, respectively, are found by solving

$$\sum_{i=1}^{n} s_i + \frac{n}{\beta} - \frac{n t_0}{1 - e^{-\beta t_0}} = 0 \tag{7.134}$$

to give $\hat{\beta}$, and then taking

$$\hat{\alpha} = \ln \left(\frac{n \hat{\beta}}{e^{\hat{\beta} t_0} - 1} \right) \tag{7.135}$$

The estimates are further discussed by Crowder et al. (1991, p. 167).

7.4.5 Statistical Tests of Trend

The simple graph in Fig. 7.3 clearly indicates an increasing rate of failures, that is, a deteriorating or *sad* system. The next step in an analysis of the data may be to perform a *statistical test* to find out whether the observed trend is *statistically significant* or just accidental. A number of tests have been developed for this purpose, that is, for testing the null hypothesis

H_0: "No trend" (or more precisely that the interoccurrence times are independent and identically exponentially distributed, that is, an HPP)

against the alternative hypothesis

H_1: "Monotonic trend" (i.e., the process is an NHPP that is either *sad* or *happy*)

Among these are two nonparametric tests that we will discuss:

1. The Laplace test
2. The military handbook test

These two tests are discussed in detail by Ascher and Feingold (1984) and Crowder et al. (1991). It can be shown that the Laplace test is optimal when the true failure mechanism is that of a log-linear NHPP model (see Cox and Lewis, 1966), while the military handbook test is optimal when the true failure mechanism is that of a power law NHPP model (see Bain et al. 1985).

The Laplace Test The test statistic for the case where the system is observed until n failures have occurred is

$$U = \frac{\frac{1}{n-1}\sum_{j=1}^{n-1} S_j - (S_n/2)}{S_n/\sqrt{12(n-1)}} \tag{7.136}$$

where $S_1, S_2, \ldots$ denote the failure times. For the case where the system is observed until time t_0, the test statistic is

$$U = \frac{\frac{1}{n}\sum_{j=1}^{n} S_j - (t_0/2)}{t_0/\sqrt{12n}} \tag{7.137}$$

In both cases, the test statistic U is approximately standard normally $\mathcal{N}(0, 1)$ distributed when the null hypothesis H_0 is true. The value of U is seen to indicate the direction of the trend, with $U < 0$ for a *happy* system and $U > 0$ for a *sad* system. Optimal properties of the Laplace test have, for example, been investigated by Gaudoin (1992).

Military Handbook Test The test statistic of the so-called military handbook test (see MIL-HDBK-189) for the case where the system is observed until n failures have occurred is

$$Z = 2\sum_{i=1}^{n-1} \ln \frac{S_n}{S_i} \tag{7.138}$$

For the case where the system is observed until time t_0, the test statistic is

$$Z = 2 \sum_{i=1}^{n} \ln \frac{t_0}{S_i} \tag{7.139}$$

The asymptotic distribution of Z is in the two cases a χ^2 distribution with $2(n-1)$ and $2n$ degrees of freedom, respectively.

The hypothesis of no trend (H_0) is rejected for *small* or *large* values of Z. Low values of Z correspond to deteriorating systems, while large values of Z correspond to improving systems.

7.5 IMPERFECT REPAIR PROCESSES

In the previous sections we studied two main categories of models that can be used to describe the occurrence of failures of repairable systems: renewal processes and nonhomogeneous Poisson processes. The homogeneous Poisson process may be considered a special case of both models. When using a renewal process, the repair action is considered to be *perfect*, meaning the the system is "as good as new" after the repair action is completed. When we use the NHPP, we assume the the repair action is *minimal*, meaning that the reliability of the system is the same immediately after the repair action as it was immediately before the failure occurred. In this case we say the the system is "as bad as old" after the repair action. The renewal process and the NHPP may thus be considered as two extreme cases. Systems subject to normal repair will be somewhere between these two extremes. Several models have been suggested for the *normal*, or *imperfect*, repair situation, a repair that is somewhere between a minimal repair and a renewal.

In this section we will consider a system that is put into operation at time t. The initial failure rate (FOM) function of the system is denoted $z(t)$, and the conditional ROCOF of the system is denoted $w_C(t)$. The conditional ROCOF was defined by (7.7).

When the system fails, a repair action is initiated. The repair action will bring the system back to a functioning state and may involve a repair or a replacement of the system component that produced the system failure. The repair action may also involve maintenance and upgrading of the rest of the system and even replacement of the whole system. The time required to perform the repair action is considered to be negligible. Preventive maintenance, except for preventive maintenance carried out during a repair action, is disregarded.

A high number of models have been suggested for modeling imperfect repair processes. Most of the models may be classified in two main groups: (i) models where the repair actions reduce the rate of failures (ROCOF), and (ii) models where the repair actions reduce the (virtual) age of the system. A survey of available models are provided, for example, by Pham and Wang (1996), Hokstad (1997), and Akersten (1998).

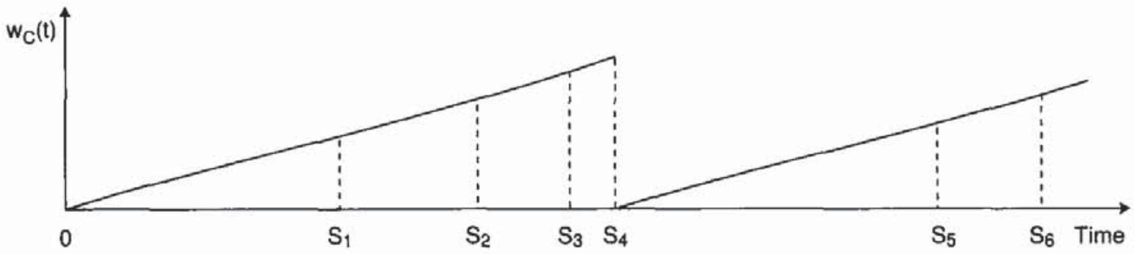

Fig. 7.20 An illustration of a possible shape of the conditional ROCOF of Brown and Proschan's imperfect repair model.

7.5.1 Brown and Proschan's Model

One of the best known imperfect repair models is described by Brown and Proschan (1983). Brown and Proschan's model is based on the following repair policy: A system is put into operation at time $t = 0$. Each time the system fails, a repair action is initiated, that with probability p is a *perfect* repair that will bring the system back to an "as good as new" condition. With probability $1 - p$ the repair action will be a *minimal* repair, leaving the system in an "as bad as old" condition. The renewal process and the NHPP are seen to be special cases of Brown and Proschan's model, when $p = 1$ and $p = 0$, respectively. Brown and Proschan's model may therefore be regarded as a mixture of the renewal process and the NHPP. Note that the probability p of a perfect repair is independent of the time elapsed since the previous failure and also of the age of the system. Let us, as an example, assume that $p = 0.02$. This means that we for most failures will make do with a minimal repair, and on the average renew (or replace) the system once for every 50 failures. This may be a realistic model, but the problem is that the renewals will come at random, meaning that we have the same probability of renewing a rather new system as an old system. Fig. 7.20 illustrates a possible shape of the conditional ROCOF.

The data obtained from a repairable system is usually limited to the times between failures, $T_1, T_2, \ldots$. The detailed repair modes associated to each failure are in general not recorded. Bases on this "masked" data set, Lim (1998) has developed a procedure for estimating p and the other parameters of Brown and Proschan's model.

Brown and Proschan's model was extended by Block et al. (1985) to age-dependent repair, that is, when the item fails at time t, a perfect repair is performed with probability $p(t)$ and a minimal repair is performed with probability $1 - p(t)$. Let Y_1 denote the time from $t = 0$ until the first perfect repair. When a perfect repair is carried out, the process will start over again, and we get a sequence of times between perfect repairs $Y_1, Y_2, \ldots$ that will form a renewal process. Assume that $F(t)$ is the distribution of the time to the first failure T_1, and let $f(t)$ and $R(t) = 1 - F(t)$ be the corresponding probability density function and the survivor function, respectively. The failure rate (FOM) function of T_1 is then $z(t) = f(t)/R(t)$, and we know from Chapter 2 that the distribution function may be written as

$$F(t) = 1 - e^{-\int_0^t z(x)\,dx} = 1 - e^{-\int_0^t [f(x)/R(x)]\,dx}$$

The distribution of Y_i is given by (see Block et al. 1985)

$$F_p(t) = \Pr(Y_i \le t) = 1 - e^{-\int_0^t [p(x)f(x)/R(x)]\,dx} = 1 - e^{-\int_0^t z_p(x)\,dx} \quad (7.140)$$

Hence, the time between renewals, Y has failure rate (FOM) function

$$z_p(t) = \frac{p(t)f(t)}{R(t)} = p(t)z(t) \quad (7.141)$$

Block et al. (1985) also supply an explicit formula for the renewal function and discuss the properties of of $F_p(t)$.

Failure Rate Reduction Models Several models have been suggested where each repair action results in a reduction of the conditional ROCOF. The reduction may be a fixed reduction, a certain percentage of the actual value of the rate of failures, or a function of the history of the process. Models representing the first two types were proposed by Chan and Shaw (1993). Let $z(t)$ denote the failure rate (FOM) function of the time to the first failure. If all repairs were minimal repairs, the ROCOF of the process would be $w_1(t) = z(t)$. Consider the failure at time S_i, and let S_{i-} denote the time immediately before time S_i. In the same way, let S_{i+} denote the time immediately after time S_i. The models suggested by Chan and Shaw (1993) may then be expressed by the conditional ROCOF as

$$w_C(S_{i+}) = \begin{cases} w_C(S_{i-}) - \Delta & \text{for a fixed reduction } \Delta \\ w_C(S_{i-})(1-\rho) & \text{for a proportional reduction } 0 \le \rho \le 1 \end{cases} \quad (7.142)$$

Between two failures, the conditional ROCOF is assumed to be vertically parallel to the initial ROCOF $w_1(t)$. The parameter ρ in (7.142) is an index representing the efficiency of the repair action. When $\rho = 0$, we have minimal repair, and the NHPP is therefore a special case of Chan and Shaw's proportional reduction model. When $\rho = 1$, the repair action will bring the conditional ROCOF down to zero, but this will not represent a renewal process since the interoccurrence times will not be identically distributed, except for the special case when $w_1(t)$ is a linear function. The conditional ROCOF of Chan and Shaw's proportional reduction model is illustrated in Fig. 7.21 for some possible failure times and with $\rho = 0.30$.

Chan and Shaw's model (7.142) has been generalized by Doyen and Gaudoin (2002a,b). They propose a set of models where the proportionality factor ρ depends on the history of the process. In their models the conditional ROCOF is expressed as

$$w_C(S_{i+}) = w_C(S_{i-}) - \varphi(i, S_1, S_2, \ldots, S_i) \quad (7.143)$$

where $\varphi(i, S_1, S_2, \ldots, S_i)$ is the reduction of the conditional ROCOF resulting from the repair action. Between two failures they assume that the conditional ROCOF is vertically parallel to the initial ROCOF $w_1(t)$. These assumptions lead to the conditional ROCOF

$$w_C(t) = w_1(t) - \sum_{i=1}^{N(t)} \varphi(i, S_1, S_2, \ldots, S_i) \quad (7.144)$$

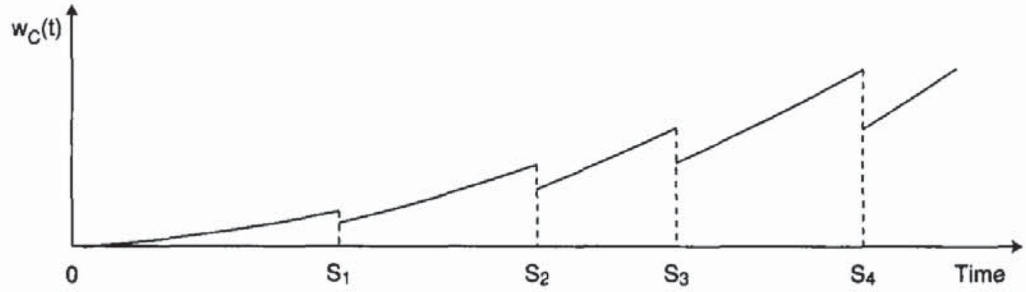

Fig. 7.21 The conditional ROCOF of Chan and Shaw's proportional reduction model for some possible failure times ($\rho = 0.30$).

When we, as in Chan and Shaw's model (7.142), assume a proportional reduction after each repair action, the conditional ROCOF in the interval $(0, S_1)$ becomes $w_C(t) = w_1(t)$. In the interval $[S_1, S_2)$ the conditional ROCOF is $w_C(t) = w_1(t) - \rho\, w_1(S_1)$. In the third interval $[S_2, S_3)$ the conditional ROCOF is

$$\begin{aligned} w_C(t) &= w_1(t) - \rho\, w_1(S_1) - \rho\, [w_1(S_2) - \rho\, w_1(S_1)] \\ &= w_1(t) - \rho \left[(1-\rho)^0 w_1(S_2) + (1-\rho)^1 w_1(S_1)\right] \end{aligned}$$

It is now straightforward to continue this derivation and show that the conditional ROCOF of Chan and Shaw's proportional reduction model (7.142) may be written as

$$w_C(t) = w_1(t) - \rho \sum_{i=0}^{N(t)} (1-\rho)^i\, w_1(S_{N(t)-i}) \tag{7.145}$$

This model is called arithmetic reduction of intensity with infinite memory (ARI_∞) by Doyen and Gaudoin (2002a).

In the model (7.142) the reduction is proportional to the conditional ROCOF just before time t. Another approach is to assume that a repair action can only reduce a proportion of the wear that has accumulated since the previous repair action. This can be formulated as:

$$w_C(S_{i+}) = w_C(S_{i-}) - \rho \left[w_C(S_{i-}) - w_C(S_{i-1+})\right] \tag{7.146}$$

The conditional ROCOF of this model is

$$w_C(t) = w_1(t) - \rho\, w_1(S_{N(t)}) \tag{7.147}$$

This model is called arithmetic reduction of intensity with memory one (ARI_1) by Doyen and Gaudoin (2002a). If $\rho = 0$, the system is "as bad as old" after the repair action and the NHPP is thus a special case of the ARI_1 model. If $\rho = 1$, the conditional ROCOF is brought down to zero by the repair action, but the process is not a renewal process, since the interoccurrence times are not identically distributed. For the ARI_1 model, there exists a deterministic function $w_{\min}(t)$ that is always smaller than the

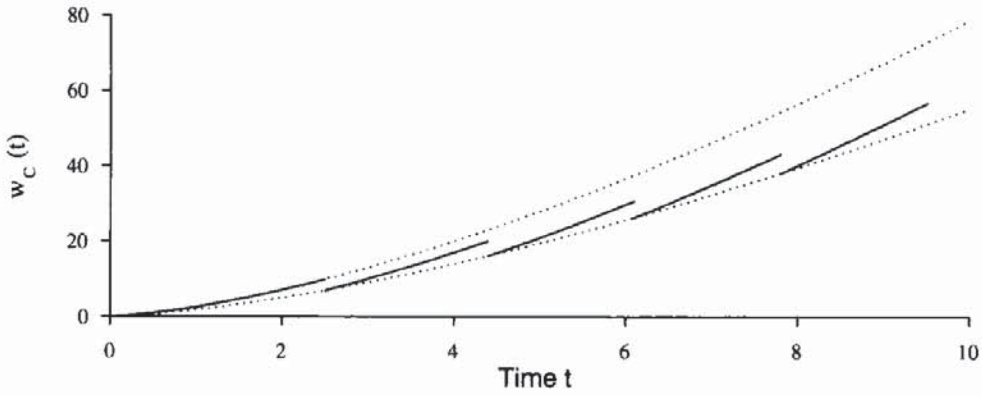

Fig. 7.22 The ARI_1 model for some possible failure times. The "underlying" ROCOF $w_1(t)$ is a power law model with shape parameter $\beta = 2.5$, and the parameter $\rho = 0.30$. The upper dotted curve is $w_1(t)$, and the lower dotted curve is the minimal wear intensity $(1 - \rho)w_1(t)$.

conditional ROCOF such that there is a nonzero probability that the ROCOF will be excessively close to $w_{\min}(t)$.

$$w_{\min}(t) = (1 - \rho)\, w_1(t)$$

This intensity is a minimal wear intensity, that is to say, a maximal lower boundary for the conditional ROCOF. The ARI_1 model is illustrated in Fig. 7.22 for some possible failure times.

The two models ARI_∞ and ARI_1 may be considered as two extreme cases. To illustrate the difference, we may consider the conditional ROCOF as an index representing the wear of the system. By the ARI_∞ model, every repair action will reduce, by a specified percentage ρ, the total accumulated wear of the system since the system was installed. By the ARI_1 model the repair action will only reduce, by a percentage ρ, the wear that has been accumulated since the previous repair action. This is why Doyen and Gaudoin (2002a) say that the ARI_∞ has infinite memory, while the ARI_1 has memory one (one period).

Doyen and Gaudoin (2002a) have also introduced a larger class of models in which only the first m terms of the sum in (7.145) are considered. They call this model the arithmetic reduction of intensity model of memory m (ARI_m), and the corresponding conditional ROCOF is

$$w_C(t) = w_1(t) - \rho \sum_{i=0}^{\min\{m-1, N(t)\}} (1 - \rho)^i\, w_C(S_{N(t)-i}) \qquad (7.148)$$

The ARI_m model has a minimal wear intensity:

$$w_{\min}(t) = (1 - \beta)^m w_1(t)$$

In all these models we note that the parameter ρ may be regarded as an index of the efficiency of the repair action.

- $0 < \rho < 1$: The repair action is efficient.
- $\rho = 1$: Optimal repair. The conditional ROCOF is put back to zero (but the repair effect is different from the "as good as new" situation.
- $\rho = 0$: The repair action has no effect on the wear of the system. The system state after the repair action is "as bad as old."
- $\rho < 0$: The repair action is harmful to the system, and will introduce extra problems.

Age Reduction Models Malik (1979) proposed a model where each repair action reduces the *age* of the system by a time that is proportional to the operating time elapsed from the previous repair action. The age of the system is hence considered as a virtual concept.

To establish a model, we assume that a system is put into operation at time $t = 0$. The initial ROCOF $w_1(t)$ is equal to the failure rate (FOM) function $z(t)$ of the interval until the first system failure. $w_1(t)$ is then the ROCOF of a system where all repairs are minimal repairs. The first failure occurs at time S_1, and the conditional ROCOF just after the repair action is completed is

$$w_C(S_{1+}) = w_1(S_1 - \vartheta)$$

where $S_1 - \vartheta$ is the new virtual age of the system. After the next failure, the conditional ROCOF will be $w_C(S_{2+}) = w_1(S_2 - 2\vartheta)$ and so on. The conditional ROCOF at time t is

$$w_C(t) = w_1(t - N(t)\vartheta)$$

We may now let ϑ be a function of the history and get

$$w_C(t) = w_1\left(t - \sum_{i=1}^{N(t)} \vartheta(i, S_1, S_2, \ldots S_i)\right) \tag{7.149}$$

Between two consecutive failures we assume that the conditional ROCOF is horizontally parallel with the initial ROCOF $w_1(t)$.

Doyen and Gaudoin (2002a) propose an age reduction model where the repair action reduces the virtual age of the system with an amount proportional to its age just before the repair action. Let ρ denote the percentage of reduction of the virtual age. In the interval $(0, S_1)$ the conditional ROCOF is $w_C(t) = w_1(t)$. Just after the first failure (when the repair is completed) the virtual age is $S_1 - \rho S_1$, and in the interval (S_1, S_2) the conditional ROCOF is $w_C(t) = w_1(t - \rho S_1)$. Just before the second failure at time S_2, the virtual age is $S_2 - \rho S_1$, and just after the second failure the virtual age is $S_2 - \rho S_1 - \rho(S_2 - \rho S_1)$. In the interval (S_2, S_3) the conditional ROCOF is $w_C(t) = w_1(t - \rho S_1 - \rho(S_2 - \rho S_1))$ which may be written as $w_C(t) =$

$w_1(t-\rho(1-\rho)^0S_2-\rho(1-\rho)^1S_1)$. By continuing this argument, it is easy to realize that the conditional ROCOF of this age reduction model is

$$w_C(t) = w_1\left(t-\rho\sum_{i=0}^{N(t)}(1-\rho)^i S_{N(t)-i}\right) \tag{7.150}$$

This model is by Doyen and Gaudoin (2002a) called arithmetic reduction of age with infinite memory (ARA_∞). The same model has also been introduced by Yun and Choung (1999). We note that when $\rho = 0$, we get $w_C(t) = w_1(t)$ and we have an NHPP. When $\rho = 1$, we get $w_C(t) = w_1(t - S_{N(t)})$ which represents that the repair action leaves the system in an "as good as new" condition. The NHPP and the renewal process are therefore special cases of the $\text{ARA}_{-\infty}$ model.

Malik (1979) introduced a model in which the repair action at time S_i reduces the last operating time from $S_i - S_{i-1}$ to $\rho(S_i - S_{i-1})$ where as before, $0 \le \rho \le 1$. Using this model, Shin et al. (1996) developed an optimal maintenance policy and derived estimates for the various parameters. The corresponding conditional ROCOF is

$$w_C(t) = w_1(t - \rho S_{N(t)})$$

The minimal wear intensity is equal to $w_1((1-\rho)t)$. This model is by Doyen and Gaudoin (2002a) called arithmetic reduction of age with memory one (ARA_1).

In analogy with the failure rate reduction models, we may define a model called arithmetic reduction of age with memory m by

$$w(t) = w_1\left(t-\rho\sum_{i=0}^{\min\{m-1,N(t)\}}(1-\rho)^i S_{N(t)-i}\right)$$

The minimal wear intensity is

$$w_{\min}(t) = w_1((1-\beta)^m t)$$

Kijima and Sumita (1986) introduced a model which they called the generalized renewal process for modeling the imperfect repair process. This model has later been extended by Kaminskiy and Krivstov (1998). The model is an age reduction model that is similar to the models described by Doyen and Gaudoin (2002a). Estimation of the parameters of the generalized renewal model is discussed by Yañes et al. (2002).

Trend Renewal Process Let $S_1, S_2, \ldots$ denote the times when failure occur in an NHPP with ROCOF $w(t)$, and let $W(t)$ denote the mean number of failures in the interval $(0, t]$. On page 281 we showed that the time-transformed process with occurrence times $W(S_1), W(S_2), \ldots$ is an HPP with rate 1. In the transformed process, the mean time between failures (and renewals) will then be 1. Lindqvist (1993, 1998) generalized this model, by replacing the HPP with rate 1 with a renewal process with

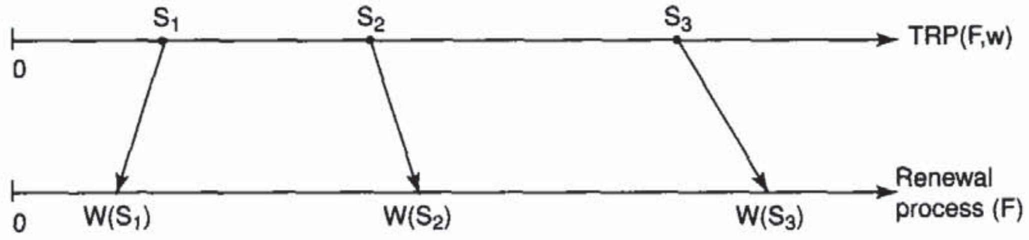

Fig. 7.23 Illustration of the transformation of a TRP(F, w) to a renewal process. (Adapted from Lindqvist 1999).

underlying distribution $F(\cdot)$ with mean 1. He called the resulting process a trend-renewal process, TRP(F, w). To specify the process we need to specify the rate $w(t)$ of the initial NHPP and the distribution $F(t)$.

If we have a TRP(F, w) with failure times $S_1, S_2, \ldots$, the time-transformed process with occurrence times $W(S_1), W(S_2), \ldots$ will be a renewal process with underlying distribution $F(t)$. The transformation is illustrated in Fig. 7.23. The requirement that $F(t)$ has mean value 1 is made for convenience. The scale is then taken care of by the rate $w(t)$. Lindqvist (1998) shows that the conditional ROCOF of the TRP(F, w) is

$$w_C^{\text{TRP}}(t) = z\left(W(t) - W(S_{N(t-)})\right) w(t) \tag{7.151}$$

where $z(t)$ is the failure rate (FOM) function of the distribution $F(t)$. The conditional ROCOF of the TRP(F, w) is hence a product of a factor, $w(t)$, that depends on the age t of the system, and a factor that depends on the (transformed) time from the previous failure. When both the failure rate (FOM) function $z(t)$ and the initial ROCOF $w(t)$ are increasing functions, then the conditional ROCOF (7.151) at time t after a failure at time s_0 is

$$z\left(W(t + s_0) - W(s_0)\right) w(t + s_0)$$

To check the properties of the TRP we may look at some special cases:

- If $z(t) = \lambda$ and $w(t) = \beta$ are both constant, the conditional ROCOF is also constant, $w_C(t) = \lambda \cdot \beta$. Hence the HPP is a special case of the TRP.

- If $z(t) = \lambda$ is constant, the conditional ROCOF is $w_C(t) = \lambda \cdot w(t)$, and the NHPP is hence a special case of the TRP.

- If $z(0) = 0$, the conditional ROCOF is equal to 0 just after each failure, that is, $w_C(S_{N(t_+)}) = 0$.

- If $w(t) = \beta$ is constant, we have an ordinary renewal process, $w_C(t) = z(t - S_{N(t_-)})$.

- If $z(0) > 0$, the conditional ROCOF just after a failure is $z(0) \cdot w(S_{N(t_+)})$ and is increasing with t when $w(t)$ is an increasing function

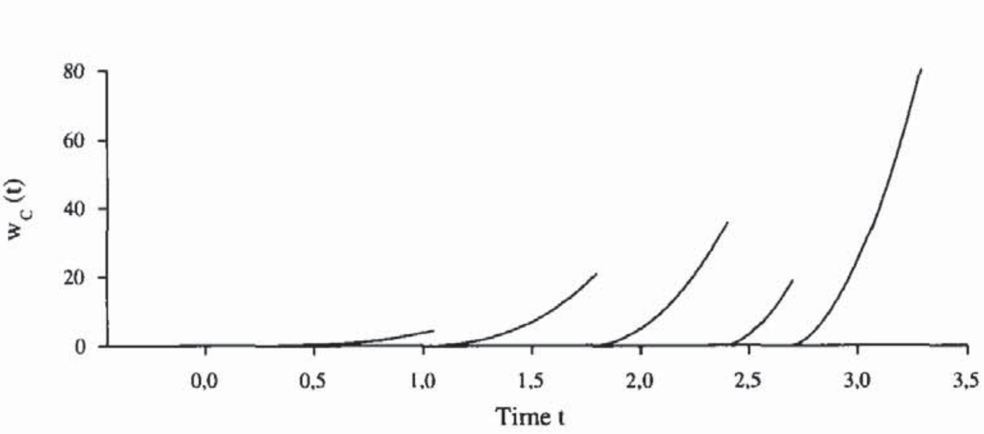

Fig. 7.24 Illustration of the conditional ROCOF $w_C(t)$ in Example 7.17 for some possible failure times.

- If $z(t)$ is the failure rate (FOM) function of a Weibull distribution with shape parameter α and $w(t)$ is a power law (Weibull) process with shape parameter β, the conditional ROCOF will have a Weibull form with shape parameter $\alpha\beta - 1$.

Example 7.17

Consider a trend renewal process with initial ROCOF $w(t) = 2\theta^2 t$, that is, a linearly increasing ROCOF, and a distribution $F(t)$ with failure rate (FOM) function $z(t) = 2.5\,\lambda^{2.5} \cdot t^{1.5}$, that is, a Weibull distribution with shape parameter $\alpha = 2.5$ and scale parameter λ. For the mean value of $F(t)$ to be equal to 1, the scale parameter must be $\lambda \approx 0.88725$. The conditional ROCOF in the interval until the first failure is from (7.151)

$$w_C(t) = 5\,\lambda^{2.5}\,\theta^5 \cdot t^4 \quad \text{for } 0 \le t < S_1$$

Just after the first failure, $w_C(S_{1+}) = 0$. Generally, we can find $w_C(t)$ from (7.151). Between failure n and failure $n+1$, the conditional ROCOF is

$$w_C(t) = 5\,\lambda^{2.5}\,\theta^5 \cdot (t^2 - S_n^2)^{1.5} \cdot t \quad \text{for } S_n \le t < S_{n+1}$$

The conditional ROCOF $w_C(t)$ is illustrated for some possible failure times $S_1, S_2, \ldots$ in Fig. 7.24. □

The trend renewal process is further studied by Lindqvist (1993, 1998) and Elvebakk (1999) who also provides estimates for the parameters of the model.

7.6 MODEL SELECTION

A simple framework for model selection for a repairable system is shown in Fig. 7.25. The figure is inspired by a figure in Ascher and Feingold (1984), but new aspects have been added.

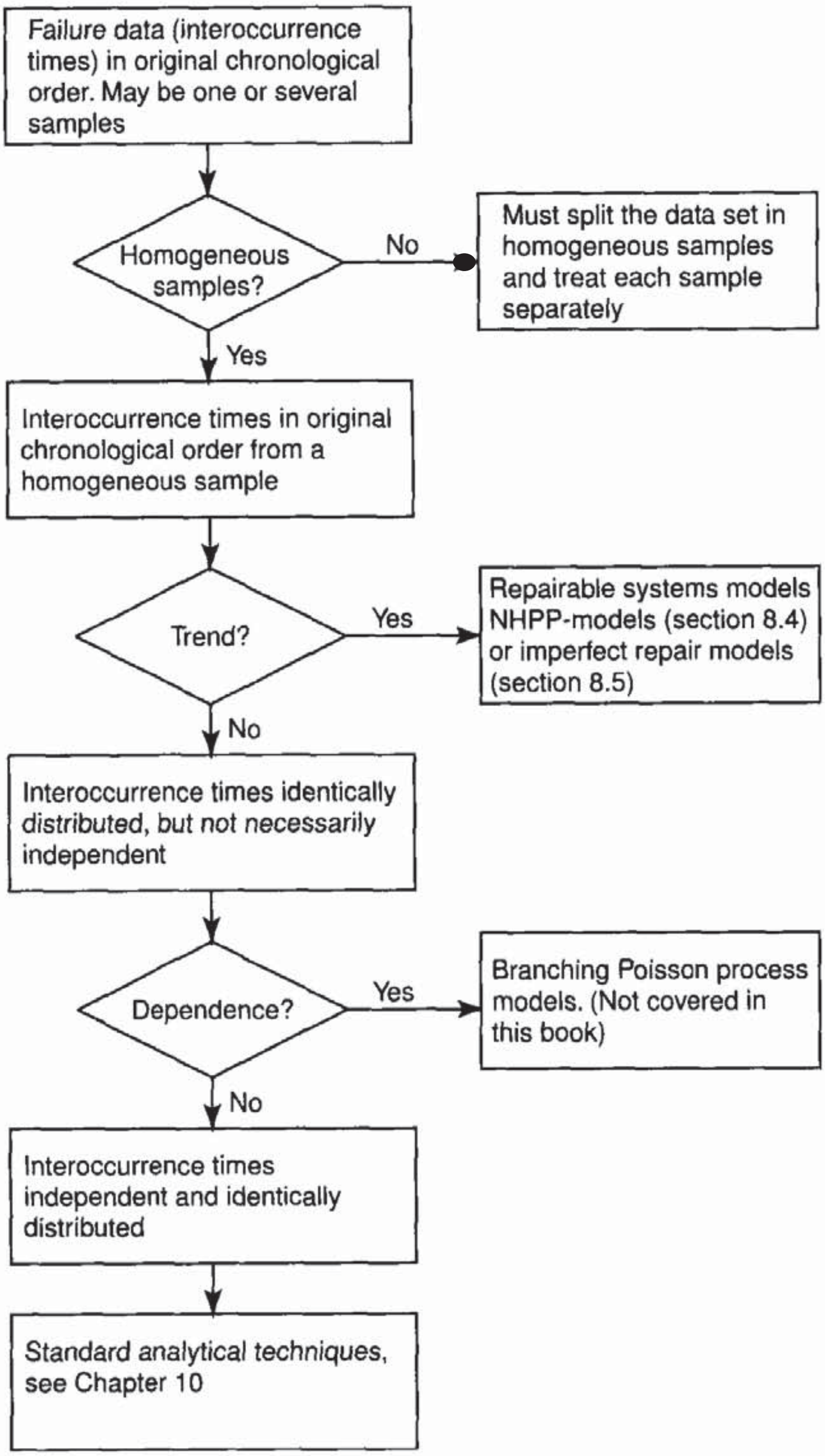

Fig. 7.25 Model selection framework.

We will illustrate the model selection framework by a simple example. In OREDA (2002), failure data from 449 pumps were collected from 61 different installations. A total of 524 critical failures were recorded, that is, on the average 1.17 failures per pump. To get adequate results we have to merge failure data from several valves. It is important that the data that are merged are homogeneous, meaning that the valves are of the same type and that the operational and environmental stresses are comparable. Since there are very few data from each valve, this analysis will have to be qualitative. The total data set should be split into homogeneous subsets and each subset has to be analyzed separately. A very simple problem related to inhomogeneous samples is illustrated in Section 2.9.

We now continue with a subset of the data that is deemed to be homogeneous. The next step is to check whether or not there is a trend in the ROCOF. This may be done by establishing a Nelson-Aalen plot as described in Section 7.4.3 on page 282. If the plot is approximately linear we conclude that the ROCOF is close to constant. If the plot is convex (concave) we conclude that the ROCOF is increasing (decreasing). The ROCOF may also be increasing in one part of the lifelength and decreasing in another part.

If we conclude that the ROCOF is increasing or decreasing, we may use either a NHPP or one of the imperfect repair models described in Section 7.5. Which model to use must (usually) be decided by a qualitative analysis of the repair actions, whether it is a minimal repair or and age, or failure rate, reduction repair. In some cases we may have close to minimal repairs during a period followed by a major overhaul. In the Norwegian offshore sector, such overhauls are often carried our during annual revision stops. When we have decided a model, we may use the methods described in this chapter to analyze the data. More detailed analyses are described, for example, in Crowder et al. (1991).

If no trend in the ROCOF is detected, we conclude that the intervals between failures are identically distributed, but not necessarily independent. The next step is then to check whether or not the data may be considered as independent. Several plotting techniques and formal tests are available. These methods are, however, not covered in this book. An introduction to such methods may, for example, be found in Crowder et al. (1991).

If we can conclude that the intervals between failures are independent and identically distributed, we have a renewal process, and we can use the methods described in Chapter 11 to analyze the data.

If the intervals are dependent, we have to use methods that are not described in this book. Please consult, for example, Crowder et al. (1991) for relevant approaches.

PROBLEMS

7.1 Consider a homogeneous Poisson process (HPP) $\{N(t), t \geq 0\}$ and let $t, s \geq 0$. Determine

$$E(N(t) \cdot N(t+s))$$

7.2 Consider an HPP $\{N(t), t \geq 0\}$ with rate $\lambda > 0$. Verify that

$$\Pr(N(t) = k \mid N(s) = n) = \binom{n}{k} \left(\frac{t}{s}\right)^k \left(1 - \frac{t}{s}\right)^{n-k} \quad \text{for } 0 < t < s \text{ and } 0 \leq k \leq n$$

7.3 Let T_1 denote the time to the first occurrence of an HPP $\{N(t), t \geq 0\}$ with rate λ. Show that

$$\Pr(T_1 \leq s \mid N(t) = 1) = \frac{s}{t} \quad \text{for } s \leq t$$

7.4 Let $\{N(t), t \geq 0\}$ be a counting process, with possible values $0, 1, 2, 3, \ldots$. Show that the mean value of $N(t)$ can be written

$$E(N(t)) = \sum_{n=1}^{\infty} \Pr(N(t) \geq n) = \sum_{n=0}^{\infty} \Pr(N(t) > n) \tag{7.152}$$

7.5 Let $S_1, S_2, \ldots$ be the occurrence times of an HPP $\{N(t), t \geq 0\}$ with rate λ. Assume that $N(t) = n$. Show that the random variables $S_1, S_2, \ldots, S_n$ have the joint probability density function

$$f_{S_1,\ldots,S_n|N(t)=n}(s_1, \ldots, s_n) = \frac{n!}{t^n} \quad \text{for } 0 < s_1 < \cdots < s_n \leq t$$

7.6 Consider a renewal process $\{N(t), t \geq 0\}$. Is it true that

(a) $N(t) < r$ if and only if $S_r > t$?

(b) $N(t) \leq r$ if and only if $S_r \geq t$?

(c) $N(t) > r$ if and only if $S_r < t$?

7.7 Consider a nonhomogeneous Poisson process (NHPP) with rate

$$w(t) = \lambda \cdot \frac{t+1}{t} \quad \text{for } t \geq 0$$

(a) Make a sketch of $w(t)$ as a function of t.

(b) Make a sketch of the cumulative ROCOF, $W(t)$, as a function of t.

7.8 Consider an NHPP $\{N(t), t \geq 0\}$ with rate:

$$w(t) = \begin{cases} 6 - 2t & \text{for } 0 \leq t \leq 2 \\ 2 & \text{for } 2 < t \leq 20 \\ -18 + t & \text{for } t > 20 \end{cases}$$

(a) Make a sketch of $w(t)$ as a function of t.

(b) Make a sketch of the corresponding cumulative ROCOF, $W(t)$, as a function of t.

(c) Estimate the number of failures/events in the interval (0, 12)

7.9 In Section 7.3.8 it is claimed that the superposition of independent renewal processes is generally *not* a renewal process. Explain why the superposition of independent homogeneous Poisson processes (HPP) is a renewal process. What is the renewal density of this superimposed process?

7.10 Table 7.2 shows the intervals in operating hours between successive failures of air-conditioning equipment in a Boeing 720 aircraft. The data are from Proschan (1963).

Table 7.2 Time Between Failures in Operating Hours of Air-conditioning Equipment.

413	14	58	37	100	65	9	169
447	184	36	201	118	34	31	18
18	67	57	62	7	22	34	

First interval is 413, the second is 14, and so on. Source: Proschan (1963).

(a) Establish the Nelson-Aalen plot ($N(t)$ plot) of the data set. Describe (with words) the shape of the ROCOF.

7.11 Atwood (1992) uses the following parametrization for the power law model, the linear model and the log-linear model:

$$w(t) = \lambda_0\,(t/t_0)^\beta \qquad \text{(power law model)}$$
$$w(t) = \lambda_0[1 + \beta(t - t_0)] \qquad \text{(linear model)}$$
$$w(t) = \lambda_0\,e^{\beta(t-t_0)} \qquad \text{(log-linear model)}$$

(a) Discuss the meaning of t_0 item[(b)] Show that Atwood's parameterization is compatible with the parameterization used in Section 7.4.4.

(c) Show that $w(t) = \lambda_0$ when $t = t_0$ for all the three models.

(d) Show that $w(t)$ is increasing if $\beta > 0$, is constant if $\beta = 0$, and decreasing if $\beta < 0$, for all the three models.

7.12 Use the MIL-HDBK test described in Section 7.4.5 to check if the "increasing trend" of the data in Example 7.1 is significant (5% - level).

7.13 Table 7.3 shows the intervals in days between successive failures of a piece of software developed as part of a large data system. The data are from Jelinski and Moranda (1972).

Table 7.3 Intervals in Days Between Successive Failures of a Piece of Software.

9	12	11	4	7	2	5	8	5	7
1	6	1	9	4	1	3	3	6	1
11	33	7	91	2	1	87	47	12	9
135	258	16	35						

First interval is 9, the second is 12, and so on. Source: Jelinski and Moranda (1972).

(a) Establish the Nelson-Aalen plot ($N(t)$ plot) of the data set. Is the ROCOF increasing or decreasing?

(b) Assume that the ROCOF follows a log-linear model, and find the maximum likelihood estimates (MLE) for the parameters of this model.

(c) Draw the estimated cumulative ROCOF in the same diagram as the Nelson-Aalen plot. Is the fit acceptable?

(d) Use the Laplace test to determine whether the ROCOF is decreasing or not (use a 5% level of significance).