Problem 3 Poisson processes with unobs. heterog.

a) Intensity: $\lambda(t)dt = P(dN(t) = 1 \mid \mathcal{F}_{t-})$
i.e., the probability of having an event in a small interval $[t, t+dt)$, given the history up to and not including t.

For $N_i(t)$, events can only occur when $t \leq \tau_i$, but otherwise at rate θ_i at each time.

i.e.
$$\lambda_i(t)\,dt = I(t \leq \tau_i)\,\theta_i\,dt$$

$$\text{so } \lambda_i(t) = I(t \leq \tau_i)\,\theta_i$$

For general formula:
$\lambda_i(t;\theta)$ equals $I(t \leq \tau_i)\theta_i$ at each of the $N_i(\tau_i)$ jumps of the process (i.e., when $\Delta N_i(t) = 1$).
This gives the first factor, $\theta_i^{N_i(\tau_i)}$
The second factor of ABG (5.4) has a factor from each process. From $N_i(t)$ it is
$$e^{-\int_0^{\tau} \lambda_i(t)\,dt} = e^{-\int_0^{\tau} I(t \leq \tau_i)\theta_i\,dt} = \underline{e^{-\theta_i \tau_i}}$$
This completes the result

b) Given Z_i, the likelihood contribution is
- by (a) -

$$(Z_i\theta)^{N_i(\tau_i)} e^{-Z_i\theta\tau_i}$$

$$= \theta^{N_i(\tau_i)} Z_i^{N_i(\tau_i)} e^{-Z_i\theta\tau_i}$$

Taking expectation we get the unconditional contribution:

$$\theta^{N_i(\tau_i)} E\left[Z_i^{N_i(\tau_i)} e^{-Z_i\theta\tau_i}\right]$$

$$= \theta^{N_i(\tau_i)} \cdot \frac{\Gamma(N_i(\tau_i)+\frac{1}{\delta})}{\Gamma(\frac{1}{\delta})} \cdot \frac{\left(\frac{1}{\delta}\right)^{\frac{1}{\delta}}}{\left(\theta\tau_i+\frac{1}{\delta}\right)^{N_i(\tau_i)+\frac{1}{\delta}}}$$

Use appendix (last formula) with

$a = N_i(\tau_i)$
$b = \theta\tau_i$
$k = \gamma = \frac{1}{\delta}$

c) The likelihood for all the data is a product of the above for $i=1,\dots,n$,
i.e.

$$\theta^{N} \prod_{i=1}^{n} \frac{\Gamma(N_i(\tau_i)+\frac{1}{\delta})}{\Gamma(\frac{1}{\delta})} \cdot \frac{1}{\delta^{\frac{n}{\delta}} \prod_{i=1}^{n}\left(\theta\tau_i+\frac{1}{\delta}\right)^{N_i(\tau_i)+\frac{1}{\delta}}}$$

Taking the log, we get the announced $l(\theta,\delta)$

$$U_1(\theta,\delta) = \frac{\partial}{\partial\theta} l(\theta,\delta) = \frac{N}{\theta} - \sum_{i=1}^{n}\left(N_i(\tau_i)+\frac{1}{\delta}\right)\frac{\tau_i}{\theta\tau_i+\frac{1}{\delta}}$$

If all $\tau_i = \tau$:

$$U_1(\theta,\delta) = \frac{N}{\theta} - \left[\sum_{i=1}^{n}\left(N_i(\tau) + \frac{1}{\delta}\right)\cdot \frac{\tau}{\theta\tau + \frac{1}{\delta}}\right]$$

$$= \frac{N}{\theta} - \left(N + \frac{n}{\delta}\right)\cdot \frac{\tau}{\theta\tau + \frac{1}{\delta}}$$

Set equal 0:

$$\frac{N}{\theta} = \frac{\tau\left(N + \frac{n}{\delta}\right)}{\theta\tau + \frac{1}{\delta}}$$

$$\cancel{N\theta\tau} + \frac{N}{\delta} = \cancel{\theta\tau N} + \theta\tau\frac{n}{\delta}$$

$$N = \theta\tau n$$

$$\hat{\theta} = \frac{N}{\tau n} = \frac{\#\text{events}}{\text{total time}} = \frac{\text{occurences}}{\text{exposure}}$$

By inserting $\hat{\theta} = \frac{N}{\tau n}$ in the log likelihood, we get a profile likelihood for δ, which can be maximized (where the maximum may be at the value 0 for δ).

d) Usual procedure is

$$\text{"} f(z_i \mid \text{data}) \propto f(\text{data} \mid z_i)\, f(z_i) \text{"}$$

Here

$$f(\text{data} \mid z_i) = \Theta(z_i \theta)^{N_i(\tau_i)} e^{-z_i \theta \tau_i}$$

from a)
with $\theta_i = z_i \theta$

$$f(z_i) = \frac{\left(\frac{1}{\delta}\right)^{\frac{1}{\delta}}}{\Gamma\left(\frac{1}{\delta}\right)} z_i^{\frac{1}{\delta}-1} e^{-\frac{1}{\delta} z_i}$$

Appendix

Multiply and collect terms with z_i

$$\propto z_i^{N_i(\tau_i) + \frac{1}{\delta} - 1} e^{-z_i\left(\theta z_i + \frac{1}{\delta}\right)}$$

$$\sim \text{gamma}\left(N_i(\tau_i) + \frac{1}{\delta},\ \theta z_i + \frac{1}{\delta}\right)$$

by appendix

which proves (*)

The recovered z_i are then computed as the expected value of this, where ML estimates for θ, δ are substituted:

$$\hat{z}_i = \frac{N_i(\tau_i) + \frac{1}{\hat{\delta}}}{\hat{\theta} z_i + \frac{1}{\hat{\delta}}}$$