STK4080 SURVIVAL AND EVENT HISTORY ANALYSIS

Slides 6: The counting process framework

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Key results on counting processes

- $N(t)$ is a counting process with history $\mathcal{F}_t$
- $P(dN(t) = 1|\mathcal{F}_{t-}) = \lambda(t)dt$
- $\Lambda(t) = \int_0^t \lambda(s)ds$ is predictable *(compensator)*
- $M(t) = N(t) - \Lambda(t)$ is a mean-zero martingale (Doob-Meyer)

Variation processes

Predictable variation process $\langle M \rangle (t)$

- $dM(t) =$ increment in [t,t+dt) $= M((t+dt)-) - M(t-)$
- $d\langle M \rangle (t) = Var(dM(t)|\mathcal{F}_{t-}) =_{count.proc.} \lambda(t)dt$
- $\langle M \rangle (t) = \int_0^t \lambda(s)ds = \Lambda(t)$

Optional variation process $[M](t)$

- $[M](t) = \sum_{s \leq t}(M(s) - M(s-))^2 =_{count.proc.} N(t)$
- $Var(M(t)) = E\langle M \rangle (t) = E[M](t)$
  $=_{count.proc.} E(\Lambda(t)) = E(N(t))$

Counting process formulas for predictable $H(t)$

- $\left\langle \int HdM \right\rangle (t) = \int_0^t H^2(s)\lambda(s)ds$
- $[\int HdM](t) = \int_0^t H^2(s)dN(s)$

Random variables and observed data

$$\begin{aligned} T_i &= \text{survival time for ind. } i \\ C_i &= \text{censoring time for ind. } i \\ \tilde{T}_i &= \min(T_i, C_i) = \text{observed time for ind. } i \\ \tilde{D}_i &= I(T_i = \tilde{T}_i) = \text{indicator for observed time for ind. } i \end{aligned}$$

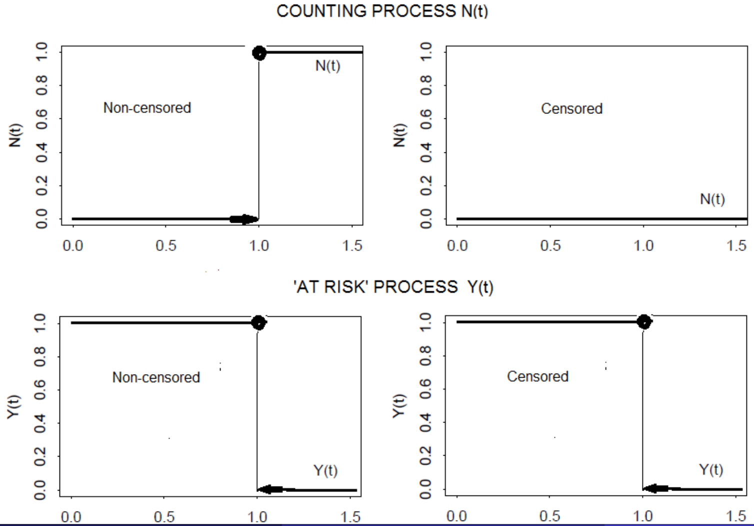
Goal for statistical inference:

Estimate

$$\begin{aligned} \alpha_i(t) &= \text{hazard rate of } T_i \\ A_i(t) &= \int_0^t \alpha_i(s)ds = \text{cumulative hazard rate of } T_i \\ S_i(t) &= e^{-A_i(t)} = \text{survival function of } T_i \end{aligned}$$

Right censored data for individual i: $(\tilde{T}_i, D_i)$

$$N_i(t) = I(\tilde{T}_i \le t, D_i = 1) \text{ (right continuous)}$$
$$Y_i(t) = I(\tilde{T}_i \ge t) = \text{'at risk' indicator (left continuous, predictable)}$$



Independent censoring

Let $\mathcal{F}_t$ be the history of all individuals, their censorings and failures, i.e., $\mathcal{F}_t$ contains $\{(N_i(s), Y_i(s)), s \leq t, i = 1, \ldots, n\}$

Independent censoring means by definition (p. 30-31 in book):

$$\begin{aligned} P(t \leq \tilde{T}_i < t + dt, D_i = 1 | \tilde{T}_i \geq t, \mathcal{F}_{t-}) &= P(t \leq T_i < t + dt | T_i \geq t) \\ &= \alpha_i(t) dt \end{aligned}$$

The book makes the point that

$$\textit{independent censoring} \Longleftrightarrow \lambda_i(t) = \alpha_i(t) Y_i(t) \quad (*)$$

where $\lambda_i(t)$ is the intensity of the observed process $N_i(t)$, recall

$$\lambda_i(t) dt =_{def} P(dN_i(t) = 1 | \mathcal{F}_{t-})$$

(see next page for an argument for $(*)$)

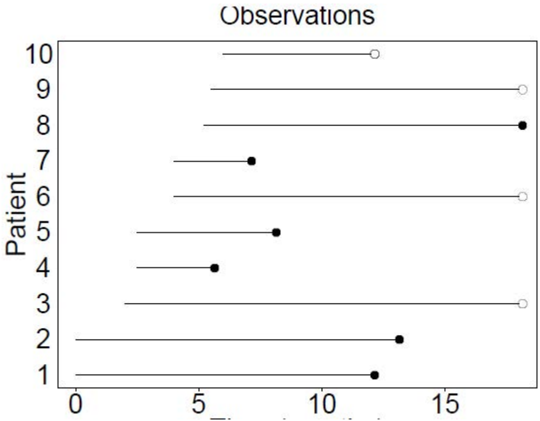
Proof of $(*)$

Claim: independent censoring $\iff \lambda_i(t) = \alpha_i(t) Y_i(t) \quad (*)$

$$
\begin{aligned}
\lambda_i(t)dt &= P(dN_i(t) = 1|\mathcal{F}_{t-}) \\
&= P(t \le \tilde{T}_i < t + dt, D_i = 1|\mathcal{F}_{t-}) \\
&= \text{ \textit{using total probability rule:}} \\
& \quad P(t \le \tilde{T}_i < t + dt, D_i = 1|\tilde{T}_i \ge t, \mathcal{F}_{t-}) P(\tilde{T}_i \ge t|\mathcal{F}_{t-}) \\
&+ P(t \le \tilde{T}_i < t + dt, D_i = 1|\tilde{T}_i < t, \mathcal{F}_{t-}) P(\tilde{T}_i < t|\mathcal{F}_{t-}) \\
&= P(t \le \tilde{T}_i < t + dt, D_i = 1|\tilde{T}_i \ge t, \mathcal{F}_{t-}) P(\tilde{T}_i \ge t|\mathcal{F}_{t-}) + 0 \\
&= \text{\textit{by def. of independent censoring:}} \\
& \quad \alpha_i(t)dt \;\; E(I(\tilde{T}_i \ge t)|\mathcal{F}_{t-}) \\
&= \alpha_i(t)dt \;\; I(\tilde{T}_i \ge t) \; (\textit{the event } \{\tilde{T}_i \ge t\} \textit{ is known at } t-) \\
&= \alpha_i(t)dt \;\; Y_i(t)
\end{aligned}
$$

(the above shows in fact the equivalence in $(*)$)

- *In a clinical study, the patients come under observation some time after the initiating event (i.e. the event defining $t = 0$)*
- *If time t is age, the individuals may be under observation from different ages*



Independent left truncation and right-censoring

The counting process results considered so far generalize almost immediately when **left-truncation** is included.

Under *left-truncation*, the observation for the ith individual is

$$(V_i, \tilde{T}_i, D_i)$$

where V_i is the *left-truncation* time (i.e., the time of entry) for the individual, and $\tilde{T}_i$ and D_i are as before.

We then have **independent left-truncation and right-censoring** provided that the counting process

$$N_i(t) = I\{\tilde{T}_i \leq t, D_i = 1\}$$

has intensity process given by $\lambda_i(t) = \alpha_i(t) Y_i(t)$, where the at risk indicator now takes the form

$$Y_i(t) = I\{V_i < t \leq \tilde{T}_i\}$$

Aggregated processes

If the processes cannot jump simultaneously, the aggregated process

$$N(t) = \sum_{i=1}^{n} N_i(t) = \sum_{i=1}^{n} I(\tilde{T}_i \leq t, D_i = 1)$$

is a counting process with intensity

$$\lambda(t) = \sum_{i=1}^{n} \lambda_i(t) = \sum \alpha_i(t) Y_i(t)$$

(w.r.t. the common history $\mathcal{F}_{t-}$),

If

$$\alpha_i(t) \equiv \alpha(t) \text{ and } Y(t) = \sum_{i=1}^{n} Y_i(t)$$

we get the **multiplicative intensity model** (Chapter 3 in book)

$$\lambda(t) = \alpha(t) Y(t)$$