Exercise 1.1

Throughout we assume $t > 0$.

a) For the exponential distribution we have $S(t) = e^{-\gamma t}$. Therefore:

$$f(t) = -S'(t) = \gamma e^{-\gamma t}$$

$$\alpha(t) = -\frac{S'(t)}{S(t)} = \frac{\gamma e^{-\gamma t}}{e^{-\gamma t}} = \gamma$$

b) For the Weibull distribution we have $\alpha(t) = bt^{k-1}$. Therefore:

$$A(t) = \int_0^t \alpha(u)\,\mathrm{d}u = \int_0^t bu^{k-1}\,\mathrm{d}u = (b/k)t^k$$

$$S(t) = e^{-A(t)} = e^{-(b/k)t^k}$$

$$f(t) = -S'(t) = bt^{k-1}e^{-(b/k)t^k}$$

c) For the Gamma distribution we have $f(t) = \frac{\gamma^k}{\Gamma(k)}t^{k-1}e^{-\gamma t}$.
Therefore,making the substitution $u = \gamma v$, we obtain:

$$S(t) = \int_t^\infty f(v)\,\mathrm{d}v = \int_t^\infty \frac{\gamma^k}{\Gamma(k)}v^{k-1}e^{-\gamma v}\,\mathrm{d}v = \frac{1}{\Gamma(k)}\int_{\gamma t}^\infty u^{k-1}e^{-u}\,\mathrm{d}u = \frac{\Gamma(k,\gamma t)}{\Gamma(k)}$$

$$\alpha(t) = -\frac{S'(t)}{S(t)} = \frac{f(t)}{S(t)} = \frac{\gamma^k}{\Gamma(k,\gamma t)}t^{k-1}e^{-\gamma t}$$

Exercise 1.2

ξ_p is defined by the relation $F(\xi_p) = P(T \le \xi_p) = p$, or equivalently $S(\xi_p) = 1 - p$.

a) From the relation $S(t) = e^{-A(t)}$ we have that

$$S(\xi_p) = e^{-A(\xi_p)} = 1 - p,$$

which gives:

$$A(\xi_p) = -\log(1-p).$$

b) For the exponential distribution we have $A(t) = \gamma t$, which gives

$$A(\xi_p) = \gamma\xi_p = -\log(1-p)$$

$$\xi_p = -\frac{1}{\gamma}\log(1-p)$$

For the Weibull distribution we have $A(t) = (b/k)t^k$, which gives

$$A(\xi_p) = \frac{b}{k}\xi_p^k = -\log(1-p)$$

$$\xi_p = \left(-\frac{k}{b}\log(1-p)\right)^{\frac{1}{k}}$$

Exercise 1.3

We consider a survival time T with survival function $S(t) = P(T > t)$ that satisfy $S(\infty) = 0$.

a) We may write the survival time as

$$T = \int_0^\infty I(T > u)\,\mathrm{d}u$$

Hence we have that

$$\begin{aligned}
\mathrm{E}(T) &= \mathrm{E}\left\{\int_0^\infty I(T > u)\,\mathrm{d}u\right\} \\
&= \int_0^\infty \mathrm{E}\{I(T > u)\}\,\mathrm{d}u \\
&= \int_0^\infty P(T > u)\,\mathrm{d}u \\
&= \int_0^\infty S(u)\,\mathrm{d}u
\end{aligned}$$

b) For the exponential distribution we have $S(t) = e^{-\gamma t}$, which gives:

$$\mathrm{E}(T) = \int_0^\infty S(u)\,\mathrm{d}u = \int_0^\infty e^{-\gamma u}\,\mathrm{d}u = \left[-\frac{1}{\gamma}e^{-\gamma u}\right]_0^\infty = \frac{1}{\gamma}$$

For the Weibull distribution we have $S(t) = e^{-(b/k)t^k}$.
Therefore, making the substitution $v = (b/k)u^k$, we obtain:

$$\begin{aligned}
\mathrm{E}(T) &= \int_0^\infty S(u)\,\mathrm{d}u \\
&= \int_0^\infty e^{-(b/k)u^k}\,\mathrm{d}u \\
&= \frac{1}{k}\left(\frac{k}{b}\right)^{\frac{1}{k}} \int_0^\infty v^{\frac{1}{k}-1}e^{-v}\,\mathrm{d}v \\
&= \left(\frac{k}{b}\right)^{\frac{1}{k}} \frac{1}{k}\Gamma\left(\frac{1}{k}\right) \\
&= \left(\frac{k}{b}\right)^{\frac{1}{k}} \Gamma\left(\frac{1}{k} + 1\right)
\end{aligned}$$

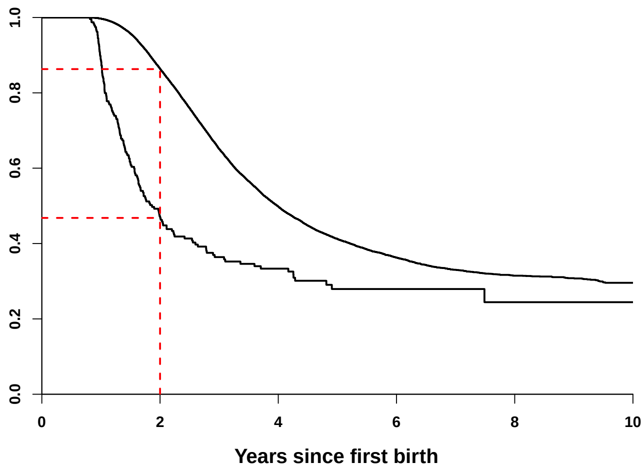
Exercise 1.4

a) From the figure below we see that if the first child survived one year, we have:

$$\hat{P}(\text{second child within 2 years}) = 1 - \hat{S}(2) \approx 1 - 0.85 = 0.15$$

Further, if the first child died within one year:

$$\hat{P}(\text{second child within 2 years}) = 1 - \hat{S}(2) \approx 1 - 0.45 = 0.55$$

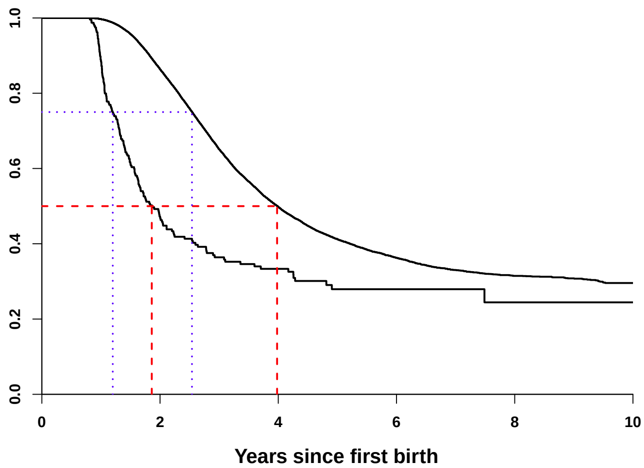


b) From the figure below we see that if the first child survived one year, the lower quartile and the median becomes:

$$\hat{\xi}_{0.25} \approx 2.5 \text{ years} \qquad \hat{\xi}_{0.50} \approx 4.0 \text{ years}$$

Further, if the first child died within one year we have:

$$\hat{\xi}_{0.25} \approx 1.2 \text{ years} \qquad \hat{\xi}_{0.50} \approx 1.9 \text{ years}$$



Exercise 1.5

We have covariates $x_{i1}, \ldots, x_{ip}$ for individual i; $i = 1, 2$. The hazard rate for the ith individual is given by Cox's regression model

$$\alpha_i(t) = \alpha_0(t) \exp\left\{\beta_1 x_{i1} + \cdots + \beta_p x_{ip}\right\}$$

a) The hazard ratio becomes

$$\frac{\alpha_2(t)}{\alpha_1(t)} = \frac{\exp\left\{\beta_1 x_{21} + \cdots + \beta_p x_{2p}\right\}}{\exp\left\{\beta_1 x_{11} + \cdots + \beta_p x_{1p}\right\}} = \exp\left\{\beta_1(x_{21} - x_{11}) + \cdots + \beta_p(x_{2p} - x_{1p})\right\}.$$

Thus the hazard ratio does not depend on t.

b) If $x_{2j} = x_{1j} + 1$ and $x_{2\ell} = x_{1\ell}$ for $\ell \neq j$, the hazard ratio may be written

$$\frac{\alpha_2(t)}{\alpha_1(t)} = e^{\beta_j}.$$

Thus e^{β_j} is the hazard ratio for one unit's increase in the jth covariate when all the other covariates remain the same.