Exercise 3.6

We have that $\widehat{S}(t)$ is approximately normally distributed with mean $S(t)$ and a variance that may be estimated by $\widehat{\tau}^2(t)$. By an argument similar to the one in exercise 3.3, we have that

$$g\big(\widehat{S}(t)\big) \pm z_{1-\alpha/2}\, g'\big(\widehat{S}(t)\big)\,\widehat{\tau}(t)$$

is an approximate $100(1-\alpha)\,\%$ confidence interval for $g(S(t))$. Here g is a strictly increasing continuously differentiable function.

For $g(x) = -\log\big(-\log x\big)$, we have $g'(x) = -1/(x\log x)$. We note that $g'(x) > 0$ for $x \in (0,1)$. Thus

$$-\log\big(-\log \widehat{S}(t)\big) \pm z_{1-\alpha/2}\,\frac{-1}{\widehat{S}(t)\log\widehat{S}(t)}\,\widehat{\tau}(t) \tag{E.2}$$

is an approximate $100(1-\alpha)\,\%$ confidence interval for $-\log\big(\log S(t)\big)$. If we exponentiate the lower and upper limits of (E.2), we find that

$$\frac{-1}{\log\widehat{S}(t)}\exp\left\{\pm z_{1-\alpha/2}\,\frac{-\widehat{\tau}(t)}{\widehat{S}(t)\log\widehat{S}(t)}\right\} \tag{E.3}$$

is an approximate confidence interval for $\exp\big\{-\log\big(-\log S(t)\big)\big\} = -1/\log S(t)$. Now (E.2) may equivalently be written as

$$\frac{-1}{\log\widehat{S}(t)\,\exp\left\{\pm z_{1-\alpha/2}\,\frac{\widehat{\tau}(t)}{\widehat{S}(t)\log\widehat{S}(t)}\right\}} \tag{E.4}$$

We may now use the transformation $h(x) = -1/x$ for the lower and upper limit of (E.4) to find that

$$\log\widehat{S}(t)\,\exp\left\{\pm\, z_{1-\alpha/2}\,\frac{\widehat{\tau}(t)}{\widehat{S}(t)\log\widehat{S}(t)}\right\} \tag{E.5}$$

is an approximate confidence interval for $\log S(t)$. Finally we exponentiate the lower and upper limit of (E.5) to see that

$$\widehat{S}(t)^{\exp\left\{\pm z_{1-\alpha/2}\,\widehat{\tau}(t)/\left(\widehat{S}(t)\log\widehat{S}(t)\right)\right\}}$$

is an approximate $100(1-\alpha)\,\%$ confidence interval for $S(t)$. This shows (3.30) in the ABG-book.