Extra exercises in STK4080 2021

Solution to Exercise E2.1

Assume that $k = 2$ and T_1, T_2 are independent and exponentially distributed with

$$f_{T_j}(t) = \lambda_j e^{-\lambda_j t}, \quad t \geq 0$$

Let $T = \min(T_1, T_2)$, $H = \text{argmin}_j(T_j)$.

a) Calculate the cumulative incidence functions

$$P(T \leq t, H = j) \quad \text{for } j = 1, 2$$

Solution:

$$\begin{aligned}
P(T \leq t, H = 1) &= P(T_1 \leq t, T_1 < T_2) \\
&= \int_0^\infty P(T_1 \leq t, T_1 < T_2 | T_2 = u) f_{T_2}(u) du \\
&= \int_0^\infty P(T_1 \leq t, T_1 < u) \lambda_2 e^{-\lambda_2 u} du \\
&= \int_0^t P(T_1 < u) \lambda_2 e^{-\lambda_2 u} du + \int_t^\infty P(T_1 \leq t) \lambda_2 e^{-\lambda_2 u} du \\
&= \int_0^t (1 - e^{-\lambda_1 u}) \lambda_2 e^{-\lambda_2 u} du + \int_t^\infty P(T_1 \leq t) \lambda_2 e^{-\lambda_2 u} du \\
&= \int_0^t \lambda_2 e^{-\lambda_2 u} du - \int_0^t \lambda_2 e^{-(\lambda_1 + \lambda_2) u} du + P(T_1 \leq t) \cdot e^{-\lambda_2 t} \\
&= 1 - e^{-\lambda_2 t} - \frac{\lambda_2}{\lambda_1 + \lambda_2}(1 - e^{-(\lambda_1 + \lambda_2)t}) + (1 - e^{-\lambda_1 t}) e^{-\lambda_2 t} \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}(1 - e^{-(\lambda_1 + \lambda_2)t})
\end{aligned}$$

where we used the well known result that $P(T_j \leq t) = 1 - e^{-\lambda_j t}$.

By symmetry,

$$P(T \leq t, H = 2) = \frac{\lambda_2}{\lambda_1 + \lambda_2}(1 - e^{-(\lambda_1 + \lambda_2)t})$$

b) Show that the cause-specific hazard rates are

$$\alpha_j(t) = \lambda_j \quad \text{for } j = 1, 2$$

Solution: From the theory of competing risks, it is known that if the $T_1, T_2, \ldots$ are independent, then the cause-specific hazards are equal to the ordinary hazards for each of the T_j. This can be shown directly as follows for the case with two risks:

$$\begin{aligned}
\alpha_1(t) dt &=_{def} P(t \leq T < t + dt, H = 1 | T \geq t) \\
&= \frac{P(t \leq T < t + dt, H = 1)}{P(T \geq t)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{P(t \le T_1 < t + dt, T_1 < T_2)}{P(T_1 \ge t, T_2 \ge t)} \\
&= \frac{P(t \le T_1 < t + dt, t < T_2)}{P(T_1 \ge t, T_2 \ge t)} \\
&= \frac{P(t \le T_1 < t + dt)P(T_2 > t)}{P(T_1 \ge t)P(T_2 \ge t)} \\
&= \frac{P(t \le T_1 < t + dt)}{P(T_1 \ge t)}
\end{aligned}
$$

which is the ordinary hazard rate of the lifetime T_1, which is λ_1 in our case.

c) Discuss the property that cause specific hazards are not influenced by the corresponding one for the other risk, while this is the case for the cumulative incidence function. (This is a general property of competing risks which one should be aware of).

Solution: It is seen that the cause-specific hazards are not influenced by the size of the hazard for the other risk. On the other hand, for a given λ_1, the size of λ_2 may have a big influence on the cumulative incidence function for risk 1.

d) Use the recipe for simulation and analysis in R given on p. 10-14/24 in Slides 9 to illustrate the above results. Choose reasonable values for λ_1, λ_2 and include also a censoring time C.

(Do this yourself!)