Exercise 3.3

a) Let g be a strictly increasing continuously differentiable function. Then we have the Taylor series expansion:

$$g\big(\widehat{A}(t)\big) \approx g\big(A(t)\big) + g'\big(A(t)\big)\big(\widehat{A}(t) - A(t)\big)$$

It follows that the distribution of $g\big(\widehat{A}(t)\big)$ is approximately the same as the distribution of

$$g\big(A(t)\big) + g'\big(A(t)\big)\big(\widehat{A}(t) - A(t)\big) \tag{E.1}$$

If we use that $\widehat{A}(t)$ is approximately normally distributed with mean $A(t)$, we see that (E.1) is approximately normally distributed with mean $g\big(A(t)\big)$ and variance $\big\{g'\big(A(t)\big)\big\}^2 \,\mathrm{Var}\widehat{A}(t)$. The latter is estimated by $\big\{g'\big(\widehat{A}(t)\big)\big\}^2 \,\widehat{\sigma}^2(t)$. This shows the result.

b) By the result in question a we have that

$$\frac{g\big(\widehat{A}(t)\big) - g\big(A(t)\big)}{g'\big(\widehat{A}(t)\big)\widehat{\sigma}(t)}$$

is approximately standard normally distributed. It then follows by a standard argument that

$$g\big(\widehat{A}(t)\big) \pm z_{1-\alpha/2}\; g'\big(\widehat{A}(t)\big)\,\widehat{\sigma}(t)$$

is an approximate $100(1-\alpha)\,\%$ confidence interval for $g(A(t))$. Here $\pm$ means $+$ for the upper limit and $-$ for the lower limit of the confidence interval.

c) With $g(x) = \log x$, we have $g'(x) = 1/x$. Thus an approximate $100(1-\alpha)\,\%$ confidence interval for $\log A(t)$ is

$$\log\,\widehat{A}(t) \pm z_{1-\alpha/2}\,\widehat{\sigma}(t)/\widehat{A}(t)$$

By exponentiating the lower and upper limits of this confidence interval, we get the following confidence interval for $A(t)$:

$$\exp\Big\{\log\,\widehat{A}(t) \pm z_{1-\alpha/2}\,\widehat{\sigma}(t)/\widehat{A}(t)\Big\}$$

i.e.

$$\widehat{A}(t)\exp\Big\{\pm z_{1-\alpha/2}\,\widehat{\sigma}(t)/\widehat{A}(t)\Big\}$$

This is (3.7) in the ABG-book.