(when considered as a process in t_0). The variance of the test statistic is estimated by

$$V_{11}(t_0) = \int_0^{t_0} \frac{L^2(t)}{Y_1(t)Y_2(t)} \, \mathrm{d}N_{\bullet}(t) \tag{3.55}$$

If the null hypothesis holds true, we have the decomposition

$$\mathrm{d}N_{\bullet}(t) = \alpha(t)Y_{\bullet}(t)\,\mathrm{d}t + \mathrm{d}M_{\bullet}(t)$$

From (3.54) and (3.55) it follows that under H_0 we may write

$$\begin{aligned} V_{11}(t_0) &= \int_0^{t_0} \frac{L^2(t)}{Y_1(t)Y_2(t)} \left\{\alpha(t)Y_{\bullet}(t)\,\mathrm{d}t + \mathrm{d}M_{\bullet}(t)\right\} \\ &= \langle Z_1 \rangle(t_0) + \int_0^{t_0} \frac{L^2(t)}{Y_1(t)Y_2(t)} \, \mathrm{d}M_{\bullet}(t) \end{aligned}$$

Here the last term on the right-hand side is a mean zero martingale, and hence

$$\mathrm{E}\left\{V_{11}(t_0)\right\} = \mathrm{E}\left\{\langle Z_1 \rangle(t_0)\right\} + 0 = \mathrm{E}\left\{\langle Z_1 \rangle(t_0)\right\}$$

Further by (2.24) in the ABG-book we know that

$$\mathrm{Var}\left\{Z_1(t_0)\right\} = \mathrm{E}\left\{\langle Z_1 \rangle(t_0)\right\}$$

Thus (3.55) is an unbiased variance estimator under H_0.

Exercise 3.11

Under the null hypothesis $Z_1(t_0)$ is a point along the sample path of a mean-zero martingale. For the log-rank test $Z_1(t_0)$ can be rewritten (cf. page 108 in the ABG-book):

$$Z_1(t_0) = N_1(t_0) - \int_0^{t_0} \frac{Y_1(u)}{Y_{\bullet}(u)} \, \mathrm{d}N_{\bullet}(u) = N_1(t_0) - E_1(t_0). \tag{E.21}$$

Since $\mathrm{E}\{Z_1(t_0)\} = 0$, we can take expectations on both sides of (E.21) to get

$$0 = \mathrm{E}\{N_1(t_0)\} - \mathrm{E}\{E_1(t_0)\}.$$

Therefore $\mathrm{E}\{E_1(t_0)] = \mathrm{E}\{N_1(t_0)\}$.

Exercise 3.12

$N(t)$ is a counting process with multiplicative intensity process $\lambda(t) = \alpha(t)Y(t)$.
We want to test the null hypothesis

$$\mathrm{H}_0 : \alpha(t) = \alpha_0(t) \quad \text{for all} \quad t \in [0, t_0]$$

for a known function $\alpha_0(t)$.

a) We introduce $J(t) = I\{Y(t) > 0\}$, the Nelson-Aalen estimator

$$\widehat{A}(t) = \int_0^t \frac{J(s)}{Y(s)}\, \mathrm{d}N(s) \tag{E.22}$$

and

$$A_0^*(t) = \int_0^t J(s)\alpha_0(s)\, \mathrm{d}s \tag{E.23}$$

When the null hypothesis holds true, we have that

$$M(t) = N(t) - \int_0^t \alpha_0(s)Y(s)\, \mathrm{d}s$$

is a mean-zero martingale. Hence under H_0 we have that

$$\begin{aligned}
\widehat{A}(t) - A_0^*(t) &= \int_0^t \frac{J(s)}{Y(s)}\, \mathrm{d}N(s) - \int_0^t J(s)\alpha_0(s)\, \mathrm{d}s \\
&= \int_0^t \frac{J(s)}{Y(s)}\Big(\mathrm{d}N(s) - \alpha_0(s)Y(s)\, \mathrm{d}s\Big) \\
&= \int_0^t \frac{J(s)}{Y(s)}\, \mathrm{d}M(s) \qquad \text{(E.24)}
\end{aligned}$$

From (E.24) we see that under the null hypothes, $\widehat{A}(t) - A_0^*(t)$ is the integral of the predictable process $J(s)/Y(s)$ with respect to the martingale $M(s)$. Therefore $\widehat{A}(t) - A_0^*(t)$ is a stochastic integral, and hence a mean-zero martingale, under H_0.

b) When the null hypothesis holds true, we have that $\lambda(t) = \alpha_0(t)Y(t)$, and it follows by (E.24) and (2.48) in the ABG-book that $\widehat{A}(t) - A_0^*(t)$ has predictable variation process

$$\begin{aligned}
\big\langle \widehat{A} - A_0^* \big\rangle(t) &= \int_0^t \left(\frac{J(s)}{Y(s)}\right)^2 \lambda(s)\, \mathrm{d}s \\
&= \int_0^t \left(\frac{J(s)}{Y(s)}\right)^2 \alpha_0(s)Y(s)\, \mathrm{d}s \\
&= \int_0^t \frac{J(s)}{Y(s)}\, \alpha_0(s)\, \mathrm{d}s \qquad \text{(E.25)}
\end{aligned}$$

In the last equality, we use that $J(s)$ can only take the values 0 and 1, and therefore $J^2(s) = J(s)$.

c) We now consider the test statistic

$$Z(t_0) = \int_0^{t_0} L(t)\Big(\mathrm{d}\widehat{A}(t) - \mathrm{d}A_0^*(t)\Big) \tag{E.26}$$

where $L(t)$ is a nonnegative predictable weight process that is 0 whenever $Y(t) = 0$. Under H_0 we have by (E.24) that

$$Z(t_0) = \int_0^{t_0} L(t) \frac{J(t)}{Y(t)} \, \mathrm{d}M(t) \tag{E.27}$$

which is a stochastic integral and hence a mean-zero martingale (when considered as a process in t_0). Hence, under H_0 we have that $\mathrm{E}\{Z(t_0)\} = 0$. Further we see from (E.26) that when $\alpha(t) > \alpha_0(t)$ the test statistic $Z(t_0)$ tends to be positive, while when $\alpha(t) < \alpha_0(t)$ it tends to be negative. Therefore $Z(t_0)$ is a reasonable test statistic.

d) By (E.27) and (2.48) in the ABG-book we have under the null hypothesis that

$$\begin{aligned}
\langle Z \rangle(t_0) &= \int_0^{t_0} \left\{ L(t) \frac{J(s)}{Y(s)} \right\}^2 \lambda(s) \, \mathrm{d}s \\
&= \int_0^{t_0} \left\{ L(t) \frac{J(s)}{Y(s)} \right\}^2 \alpha_0(s) Y(s) \, \mathrm{d}s \qquad \text{(E.28)} \\
&= \int_0^{t_0} \frac{L^2(s)}{Y(s)} \alpha_0(s) \, \mathrm{d}s \qquad \text{(E.29)}
\end{aligned}$$

The final equality uses the assumption that $L(t) = 0$ whenever $Y(t) = 0$.

We note that since $\alpha_0(t)$ is known, $\langle Z \rangle(t_0)$ can be computed from data. Further by (2.24) in the ABG-book

$$\mathrm{E}\{\langle Z \rangle(t_0)\} = \mathrm{Var}\{Z(t_0)\}$$

so $\langle Z \rangle(t_0)$ is an unbiased estimator for the variance og $Z(t_0)$ under H_0.

e) Using the martingale central limit theorem, one may prove that under the null hypothesis $Z(t_0)$ is approximately normally distributed with mean zero and a variance that may be estimated by $\langle Z \rangle(t_0)$.

Therefore the standardized test statistic

$$\frac{Z(t_0)}{\sqrt{\langle Z \rangle(t_0)}}$$

is approximately standard normally distributed when H_0 holds true.

f) When $L(t) = Y(t)$, we have from (E.22), (E.23), and (E.26) that

$$\begin{aligned} Z(t_0) &= \int_0^{t_0} Y(t)\Big(\,\mathrm{d}\widehat{A}(t) - \,\mathrm{d}A_0^*(t)\Big) \\ &= \int_0^{t_0} Y(s)\left(\frac{J(s)}{Y(s)}\,\mathrm{d}N(s) - J(s)\alpha_0(s)\,\mathrm{d}s\right) \\ &= \int_0^{t_0} J(s)\Big(\,\mathrm{d}N(s) - \alpha_0(s)Y(s)\,\mathrm{d}s\Big) \\ &= N(t_0) - \int_0^{t_0} Y(s)\alpha_0(s)\,\mathrm{d}s \\ &= N(t_0) - E(t_0). \end{aligned}$$

Under the null hypothesis $\mathrm{E}\{Z(t_0)\} = 0$. Therefore $\mathrm{E}\{N(t_0)\} = \mathrm{E}\{E(t_0)\}$, so $E(t_0)$ may be interpreted as the expected number of events under the null hypothesis.

g) By (E.29) we have when $L(t) = Y(t)$ that

$$\langle Z\rangle(t_0) = \int_0^{t_0} \frac{Y^2(t)}{Y(t)}\alpha_0(t)\,\mathrm{d}t = \int_0^{t_0} Y(t)\alpha_0(t)\,\mathrm{d}t = E(t_0)$$

Combining this with the results of parts e) and f), we find that the standardized test statistic

$$\frac{N(t_0) - E(t_0)}{\sqrt{E(t_0)}}$$

is approximately standard normally distributed under H_0.

Chapter 4

Exercise 4.1

Each counting process $N_i(t)$ has intensity process $\lambda_i(t) = Y_i(t)\exp\big(\boldsymbol{\beta}^\top \mathbf{x}_i\big)\alpha_0(t)$. Let $T_1 < T_2 < \cdots$ be the failure times, let $\mathcal{R}_j$ denote the risk set at time T_j, and let i_j denote the index of the individual who has an event at time T_j (assuming no tied failure times). Then the partial likelihood is given by

$$L(\boldsymbol{\beta}) = \prod_{T_j} \frac{\exp\big(\boldsymbol{\beta}^\top \mathbf{x}_{i_j}\big)}{\sum_{\ell\in\mathcal{R}_j}\exp\big(\boldsymbol{\beta}^\top \mathbf{x}_\ell\big)}$$

It follows that the log partial likelihood takes the form

$$\ell(\boldsymbol{\beta}) = \sum_{T_j}\left\{\boldsymbol{\beta}^\top \mathbf{x}_{i_j} - \log\Big(\sum_{\ell\in\mathcal{R}_j}\exp\big(\boldsymbol{\beta}^\top \mathbf{x}_\ell\big)\Big)\right\}$$