$$
\begin{aligned}
&= N^2(s) + 2N(s)\int_s^t \lambda(u)\,\mathrm{d}u + \mathrm{Var}\left\{N(t)-N(s)\right\} + \left\{\mathrm{E}\left(N(t)-N(s)\right)\right\}^2 \\
&\quad -2N(s)\int_0^t \lambda(u)\,\mathrm{d}u - 2\left(\int_s^t \lambda(u)\,\mathrm{d}u\right)\left(\int_0^t \lambda(u)\,\mathrm{d}u\right) \\
&\quad +\left(\int_0^t \lambda(u)\,\mathrm{d}u\right)^2 - \int_0^t \lambda(u)\,\mathrm{d}u \\
&= N^2(s) + 2N(s)\int_s^t \lambda(u)\,\mathrm{d}u + \int_s^t \lambda(u)\,\mathrm{d}u + \left(\int_s^t \lambda(u)\,\mathrm{d}u\right)^2 \\
&\quad -2N(s)\int_0^t \lambda(u)\,\mathrm{d}u - 2\left(\int_s^t \lambda(u)\,\mathrm{d}u\right)\left(\int_0^t \lambda(u)\,\mathrm{d}u\right) \\
&\quad +\left(\int_0^t \lambda(u)\,\mathrm{d}u\right)^2 - \int_0^t \lambda(u)\,\mathrm{d}u \\
&= N^2(s) - 2N(s)\int_0^s \lambda(u)\,\mathrm{d}u + \left(\int_0^t \lambda(u)\,\mathrm{d}u - \int_s^t \lambda(u)\,\mathrm{d}u\right)^2 - \int_0^s \lambda(u)\,\mathrm{d}u \\
&= N^2(s) - 2N(s)\int_0^s \lambda(u)\,\mathrm{d}u + \left(\int_0^s \lambda(u)\,\mathrm{d}u\right)^2 - \int_0^s \lambda(u)\,\mathrm{d}u \\
&= \left(N(s) - \int_0^s \lambda(u)\,\mathrm{d}u\right)^2 - \int_0^s \lambda(u)\,\mathrm{d}u \\
&= M^2(s) - \int_0^s \lambda(u)\,\mathrm{d}u
\end{aligned}
$$

Thus $M^2(t) - \int_0^t \lambda(u)\,\mathrm{d}u$ is a martingale.

Exercise 2.12

$W(t)$ is a Wiener process. The increment $W(t)-W(s)$ over $(s,t]$ is normally distributed with mean zero and variance ts, and increments over disjoint intervals are independent.

$V(t)$ is a strictly increasing continuous function with $V(0)=0$.

Consider the process $U(t) = W(V(t))$ and let $\mathcal{F}_t$ be generated by $U(s)$ for $s \leq t$.

a) Note that $U(t)-U(s)$ is the increment of $W(\cdot)$ over the interval $(V(s), V(t)]$. Hence $U(t)-U(s)$ is normally distributed with mean zero and variance $V(t)-V(s)$. Further the increments over disjoint intervals are independent.

Hence we have for $s < t$

$$\begin{aligned} \mathrm{E}\{U(t) \,|\, \mathcal{F}_s\} &= \mathrm{E}\{U(s) + U(t) - U(s) \,|\, U(s)\} \\ &= U(s) + \mathrm{E}\{U(t) - U(s) \,|\, U(s)\} \\ &= U(s) + \mathrm{E}\{U(t) - U(s)\} \\ &= U(s) + 0 \\ &= U(s) \end{aligned}$$

Thus $U(t)$ is a martingale.

b) We have

$$\begin{aligned} &\mathrm{E}\{U^2(t) - V(t) \,|\, \mathcal{F}_s\} \\ &= \mathrm{E}\{(U(s) + U(t) - U(s))^2 \,|\, U(s)\} - V(t) \\ &= U^2(s) + 2U(s)\mathrm{E}\{U(t) - U(s) \,|\, U(s)\} + \mathrm{E}\{(U(t) - U(s))^2 \,|\, U(s)\} - V(t) \\ &= U^2(s) + 2U(s)\mathrm{E}\{U(t) - U(s)\} + \mathrm{E}\{(U(t) - U(s))^2\} - V(t) \\ &= U^2(s) + 2U(s) \cdot 0 + \mathrm{Var}\{U(t) - U(s)\} - V(t) \\ &= U^2(s) + V(t) - V(s) - V(t) \\ &= U^2(s) - V(s) \end{aligned}$$

This shows that $U^2(t) - V(t)$ is a mean-zero martingale.