STK4080/9080 Survival and event history analysis, UiO 2021

Slides 5: Martingales

SOLUTIONS TO EXERCISES

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EXERCISE 1

Throw a die several times. Let Y_i be result in ith throw, and let $X_i = Y_1 + \ldots + Y_i$ be the sum of the i first throws.

(a) Find an expression for $f(x_1, x_2)$.

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$$f(x_1, x_2) = P(X_1 = x_1, X_2 = x_2) = \frac{1}{36} I(1 \leq x_1 \leq 6,\ 1 + x_1 \leq x_2 \leq 6 + x_1)$$

(b) Find an expression for $f(x_2|x_1)$.

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$$f(x_2|x_1) = \frac{f(x_1, x_2)}{f(x_1)} = \frac{1}{6} I(1 + x_1 \leq x_2 \leq 6 + x_1)$$

(c) Find an expression for $E(X_2|x_1)$ as a function of x_1.

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$$E(X_2|x_1) = x_1 + 7/2$$

(d) Find the expectation of the random variable $E(X_2|X_1)$. Find $E(X_2)$ from this. Find also $E(X_2)$ without using the rule of double expectation.

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$$E(X_2|X_1) = X_1 + 7/2$$

$$E(X_2) = E(E(X_2|X_1)) = E(X_1) + 7/2 = 7$$

or

$$E(X_2) = E(Y_1 + Y_2) = E(Y_1) + E(Y_2) = 7/2 + 7/2 = 7$$

(e) Find also $E(X_3|X_2)$, $E(X_3|X_1, X_2)$ and $E(X_3|X_1)$.

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$$E(X_3|X_2) = X_2 + 7/2$$

$$E(X_3|X_1, X_2) = X_2 + 7/2$$

$$E(X_3|X_1) = X_1 + 7$$

EXERCISE 2

Let $X_1, X_2, \ldots$ be independent with $E(X_n) = \mu$ for all n.

Let $S_n = X_1 + \ldots + X_n$.

Find $E(S_4|\mathcal{F}_2)$ and in general $E(S_n|\mathcal{F}_m)$ when $m < n$

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$$
\begin{aligned}
E(S_4|\mathcal{F}_2) &= E(X_1 + X_2 + X_3 + X_4|X_1, X_2) \\
&= X_1 + X_2 + E(X_3) + E(X_4) \\
&= X_1 + X_2 + 2\mu
\end{aligned}
$$

$$E(S_n|\mathcal{F}_m) = S_m + (n - m)\mu$$

EXERCISE 3

Let $X_1, X_2, \ldots$ be independent with

$$P(X_i = 1) = P(X_i = -1) = 1/2$$

Think of X_i as result of a game where one flips a coin and wins 1 unit if "heads" and loses 1 unit if "tails".

Let $M_n = X_1 + \ldots + X_n$ be the gain after n games, $n = 1, 2, \ldots$

Show that M_n is a (mean zero) martingale.

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$$\begin{aligned} E(M_n | \mathcal{F}_{n-1}) &= E(X_1 + \ldots + X_n | X_1, \ldots, X_{n-1}) \\ &= X_1 + \ldots + X_{n-1} + E(X_n) \\ &= M_{n-1} \end{aligned}$$

EXERCISE 4

(a) Let $X_1, X_2, \ldots$ be independent with $E(X_n) = 0$ for all n.

Let $M_n = X_1 + \ldots + X_n$.

Show that M_n is a (mean zero) martingale.

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Same proof as for Exercise 3

(b) Let $X_1, X_2, \ldots$ be independent with $E(X_n) = \mu$ for all n.

Let $S_n = X_1 + \ldots + X_n$.

Compute $E(S_n|\mathcal{F}_{n-1})$ (see Exercise 2). Is $\{S_n\}$ a martingale?

Can you find a simple transformation of S_n which is a martingale?

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$$E(S_n|\mathcal{F}_{n-1}) = S_{n-1} + \mu$$

This is a martingale only if $\mu = 0$.

Easy to see that $M_n = S_n - n\mu$ is a martingale.

(c) Let $X_1, X_2, \ldots$ be independent with $E(X_n) = 1$ for all n.

Let $M_n = X_1 \cdot X_2 \cdot \ldots \cdot X_n$.

Show that M_n is a martingale. What is $E(M_n)$?

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\begin{aligned}
E(M_n|\mathcal{F}_{n-1}) &= E(X_1 \cdots X_n | X_1, \ldots, X_{n-1}) \\
&= X_1 \cdots X_{n-1} E(X_n) \\
&= M_{n-1}
\end{aligned}
$$

$E(M_n) = 1$

EXERCISE 5

Show that for a martingale $\{M_n\}$ is

(a) $E(M_n|\mathcal{F}_m) = M_m$ for all $m < n$.

This is essentially Exercise 2.1 in book.

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We use induction on k to prove that

$$(*) \quad E(M_n|\mathcal{F}_{n-k}) = M_{n-k} \text{ for } k = 1, 2, \ldots, n-1$$

$k = 1$ is definition of martingale.

Suppose (*) holds for k. Then

$$E(M_n|\mathcal{F}_{n-k-1}) = E\{E(M_n|\mathcal{F}_{n-k})|\mathcal{F}_{n-k-1}\} = E(M_{n-k}|\mathcal{F}_{n-k-1}) = M_{n-k-1}$$

Hence (*) holds for all the required k.

(b) $E(M_m|\mathcal{F}_n) = M_m$ for all $m < n$

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This holds trivially since $M_m \in \mathcal{F}_n$

(c) Give verbal (intuitive) interpretations of each of (a) and (b).

EXERCISE 6

Define the *martingale differences* by

$$\Delta M_n = M_n - M_{n-1}$$

(a) Show that the definition of martingale, $E(M_n|\mathcal{F}_{n-1}) = M_{n-1}$, is equivalent to

$$E(M_n - M_{n-1}|\mathcal{F}_{n-1}) = 0, \text{ i.e. } E(\Delta M_n|\mathcal{F}_{n-1}) = 0 \quad (1)$$

(*Hint: Why is* $E(M_{n-1}|\mathcal{F}_{n-1}) = M_{n-1}$*?*)

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The following are clearly equivalent by 'Hint' which is obvious:

$$E(M_n|\mathcal{F}_{n-1}) = M_{n-1}$$
$$E(M_n|\mathcal{F}_{n-1}) - M_{n-1} = 0$$
$$E(M_n|\mathcal{F}_{n-1}) - E(M_{n-1}|\mathcal{F}_{n-1}) = 0$$
$$E(M_n - M_{n-1}|\mathcal{F}_{n-1}) = 0$$

(b) Show that for a martingale we have

$$\text{Cov}(M_{n-1} - M_{n-2}, M_n - M_{n-1}) = 0 \text{ for all } n \tag{2}$$

i.e.

$$\text{Cov}(\Delta M_{n-1}, \Delta M_n) = 0$$

Explain in word what this means.

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$$\begin{aligned}
\text{Cov}(M_{n-1} - M_{n-2}, M_n - M_{n-1}) &= E\{(M_{n-1} - M_{n-2})(M_n - M_{n-1})\} \\
&= E[E\{(M_{n-1} - M_{n-2})(M_n - M_{n-1})|\mathcal{F}_{n-1}\}] \\
&= E[(M_{n-1} - M_{n-2})E\{(M_n - M_{n-1})|\mathcal{F}_{n-1}\}] \\
&= 0
\end{aligned}$$

(c) Show that (1) and (2) automatically hold when $M_n = X_1 + \ldots + X_n$ for independent $X_1, X_2, \ldots.$

Note that in this case the differences $\Delta M_n = M_n - M_{n-1} = X_n$ are independent.

Thus (2) shows that martingale differences correspond to a weakening of the independent increments property

EXERCISE 7

Suppose that you use the martingale betting strategy, but that you in any case must stop after n games.

Calculate the expected gain. Can you "beat the game"?

(Hint: You either win 1 or lose $2^n - 1$ units.)

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You lose all n times with probability $1/2^n$. The loss is then $1+2+4+\cdots+2^{n-1} = 2^n - 1$.

You otherwise win 1, which must have probability $1 - 1/2^n$. Hence expected gain is:

$$-(2^n - 1) \cdot (1/2^n) + 1 \cdot (1 - 1/2^n) = -1 + 1/2^n + 1 - 1/2^n = 0$$

so you do not "beat the game"

EXERCISE 8

Do Exercise 2.6 in book:

Show that the stopped process M^T is a martingale. *(Hint: Find a predictable process H such that $M^T = H \bullet M$).*

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We have

$$M_n^T = M_{n \wedge T} \equiv \begin{cases} M_n \text{ if } n \leq T \\ M_T \text{ if } n > T \end{cases}$$

Want to write

$$(*) \quad M_n^T = H_1(M_1 - M_0) + H_2(M_2 - M_1) + \ldots + H_n(M_n - M_{n-1})$$

Let

$$H_i = \begin{cases} 1 \text{ if } T \geq i \\ 0 \text{ otherwise} \end{cases}$$

Then H_i is known at time $i - 1$ and we may check that (*) does the job.

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This suggests that $E(M_T) \equiv E(M_T^T) = E(M_0^T) = E(M_0) = 0$.

But this does obviously not hold for the martingale strategy case (where indeed the process is a martingale, stopped at a stopping time).

What is the clue here? *We can conclude from the above that $E(M_n^T) = 0$ for all n, but not that this n can be replaced by a random T. But see next slide in lecture!*

EXERCISE 9

Suppose $X_n = U_1 + \ldots + U_n$ where $U_1, \ldots, U_n$ are independent with $E(U_i) = \mu$.

Find the Doob decomposition of the process X and identify the predictable part and the innovation part.

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We have

$$E(X_n|\mathcal{F}_{n-1}) = E(X_{n-1} + U_n|\mathcal{F}_{n-1}) = X_{n-1} + \mu$$

Using this, the Doob decomposition is $X = X^* + M$, where

$$\begin{aligned} X_n^* &= \sum_{k=1}^{n}[E(X_k|\mathcal{F}_{k-1}) - X_{k-1}] = n\mu \\ M_n &= X_0 + \sum_{k=1}^{n}[X_k - E(X_k|\mathcal{F}_{k-1})] = X_n - n\mu \end{aligned}$$

and the innovation is hence

$$X_n - (X_{n-1} + \mu) = X_n - X_{n-1} - \mu = U_n - \mu$$

Thus the decomposition is $X_n = n\mu + (X_n - n\mu)$, where $n\mu$ is the predictable part and $X_n - n\mu$ is the innovation (sometimes called "surprise")

EXERCISE 10

Prove the results on the covariances (see Exercise 2.2 in book)

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Let $0 \leq s < t < u < v \leq \tau$

$$\begin{aligned}
\text{Cov}(M(t) - M(s), M(v) - M(u)) &= E\{(M(t) - M(s))(M(v) - M(u))\} \\
&= E[E\{(M(t) - M(s))(M(v) - M(u))|\mathcal{F}_u\}] \\
&= E[(M(t) - M(s))E\{(M(v) - M(u))|\mathcal{F}_u\}] \\
&= 0
\end{aligned}$$

POISSON PROCESSES

$N(t) = \#$ events in $[0, t]$

Characterizing properties:

- $N(t) - N(s)$ is Poisson-distributed with parameter $\lambda(t-s)$
- $N(t)$ has independent increments, i.e. number of events in disjoint intervals are independent.

EXERCISE 11

Show that

$$M(t) = N(t) - \lambda t$$

is a martingale. Identify the compensator of $N(t)$.

Hint:

For $t > s$ is $E(N(t)|\mathcal{F}_s) = N(s) + \lambda(t-s)$

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The compensator must be λt by Doob-Meyer (and λt is increasing, predictable.

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\begin{aligned}
E(M(t)|\mathcal{F}_s) &= E\{N(t) - \lambda t|\mathcal{F}_s\} \\
&= E\{N(t)|\mathcal{F}_s\} - \lambda t \\
&= E\{N(t) - N(s) + N(s)|\mathcal{F}_s\} - \lambda t \\
&= E\{N(t) - N(s)|\mathcal{F}_s\} + N(s) - \lambda t \\
&= E\{N(t) - N(s)\} + N(s) - \lambda t \\
&= \lambda t - \lambda s + N(s) - \lambda t \\
&= N(s) - \lambda s = M(s)
\end{aligned}
$$

EXERCISE 12

This is Exercise 2.3 in book, plus a new question (d). (a) was done as our Exercise 3(a)

Thus let $X_1, X_2, \ldots$ be independent with $E(X_n) = 0$ and $\mathrm{Var}(X_n) = \sigma^2$ for all n. Let $M_n = X_1 + \ldots + X_n$.

(a) Show that M_n is a (mean zero) martingale.

(b) Compute $\langle M \rangle_n$

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$$\langle M \rangle_n = \sum_{i=1}^{n} Var(\Delta M_i | \mathcal{F}_{i-1})$$

Here $\Delta M_i = M_i - M_{i-1} = X_i$, so we get

$$\langle M \rangle_n = \sum_{i=1}^{n} Var(\Delta M_i | \mathcal{F}_{i-1}) = \sum_{i=1}^{n} Var(X_i | \mathcal{F}_{i-1}) = \sum_{i=1}^{n} Var(X_i) = n\sigma^2$$

since X_i is independent of $\mathcal{F}_{i-1}$. (Why?)

(c) Compute $[M]_n$

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$$[M]_n = \sum_{i=1}^{n} (\Delta M_i)^2 = \sum_{i=1}^{n} X_i^2$$

(d) Compute $E\langle M\rangle_n$, $E[M]_n$ and $\mathrm{Var}(M_n)$.

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$$E\langle M\rangle_n = E[M]_n = \mathrm{Var}(M_n) = n\sigma^2$$

(and these are equal in general, see next exercise)

EXERCISE 13

Consider a general discrete time martingale M

(a) $M_n^2 - \langle M \rangle_n$ is a mean zero martingale - Exercise 2.4 in book
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Write

$$\begin{aligned} M_n^2 &= (M_{n-1} + M_n - M_{n-1})^2 \\ &= M_{n-1}^2 + 2M_{n-1}(M_n - M_{n-1}) + (M_n - M_{n-1})^2 \end{aligned}$$

Thus

$$E\{M_n^2 - \langle M \rangle_n | \mathcal{F}_{n-1}\}$$

$$= E\{M_{n-1}^2 + 2M_{n-1}(M_n - M_{n-1}) + (M_n - M_{n-1})^2 - \langle M \rangle_n | \mathcal{F}_{n-1}\}$$

$$= M_{n-1}^2 + 2M_{n-1}E\{(M_n - M_{n-1})|\mathcal{F}_{n-1}\} + E\{(M_n - M_{n-1})^2|\mathcal{F}_{n-1}\}$$

$$-\sum_{i=1}^{n} E\{(M_i - M_{i-1})^2|\mathcal{F}_{i-1}\}$$

$$= M_{n-1}^2 + 2 \cdot 0 - \sum_{i=1}^{n-1} E\{(M_i - M_{i-1})^2|\mathcal{F}_{i-1}\}$$

$$= M_{n-1}^2 - \langle M \rangle_{n-1}$$

(b) $M_n^2 - [M]_n$ is a mean zero martingale - See book p. 45

(c) Use (a) and (b) to prove that

$$Var(M_n) = E\langle M\rangle_n = E[M]_n$$

What is the intuitive essence of this result?

EXERCISE 14

Consider again the Poisson process, where we have shown that

$$M(t) = N(t) - \lambda t$$

is a martingale.

Prove now that:

$$M^2(t) - \lambda t$$

is a martingale (Exercise 2.10 in book)

Then prove that $\langle M \rangle (t) = \lambda t$

What is $[M] (t)$?

Hint:

For $t > s$ is $E(N(t)|\mathcal{F}_s) = N(s) + \lambda(t - s)$

For $t > s$ is $E(N^2(t)|\mathcal{F}_s) = N(s)^2 + 2N(s)\lambda(t-s) + \lambda(t-s) + (\lambda(t-s))^2$