DISCRETE TIME PROCESSES

Consider a discrete time process $X_1, X_2, \ldots, X_n$.

$$\mathcal{F}_n = \text{"information contained in } X_1, X_2, \ldots, X_n\text{"}$$

or "the history at time n" (which may contain also other random variables)

We assume $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \cdots \subset \mathcal{F}$ where
$\mathcal{F}$ = information in all random variables of the application

CONDITIONAL EXPECTATION GIVEN THE HISTORY.

Let $Y \in \mathcal{F}$. Then the following holds:

- $E(Y|\mathcal{F}_n) \in \mathcal{F}_n$
- $E[E(Y|\mathcal{F}_n)] = E[Y]$
- If $Z \in \mathcal{F}_n$, then $E(ZY|\mathcal{F}_n) = ZE(Y|\mathcal{F}_n)$
- If Y is *independent* of $\mathcal{F}_n$, then $E(Y|\mathcal{F}_n) = E(Y)$

MARTINGALES IN DISCRETE TIME

A stochastic process $M = \{M_0, M_1, M_2, \ldots\}$ is called a *martingale* if

$$E(M_n|\mathcal{F}_{n-1}) = M_{n-1} \text{ for } n = 1, 2, \ldots \quad (1)$$

$$E(M_0) = E(M_1) = \cdots = \text{(usually) } 0$$

Define the *martingale differences* by

$$\Delta M_n = M_n - M_{n-1}$$

Then the definition of martingale, $E(M_n|\mathcal{F}_{n-1}) = M_{n-1}$,
is equivalent to

$$E(M_n - M_{n-1}|\mathcal{F}_{n-1}) = 0, \text{ i.e. } E(\Delta M_n|\mathcal{F}_{n-1}) = 0 \quad (2)$$

TRANSFORMATION OF A MARTINGALE

Let $M = \{M_0, M_1, \ldots\}$ be a zero-mean martingale. Then a transformation is given as

$$\begin{aligned} Z_n &= H_1(M_1 - M_0) + H_2(M_2 - M_1) + \ldots + H_n(M_n - M_{n-1}) \\ &\equiv H_1 \Delta M_1 + H_2 \Delta M_2 + \ldots + H_n \Delta M_n \end{aligned}$$

written $Z = H \bullet M$.

If H is **predictable**, i.e., $H_n \in \mathcal{F}_{n-1}$ for each n,
then $Z = H \bullet M$ is a (zero-mean) martingale.

Proof:

$$\begin{aligned} E(Z_n - Z_{n-1} | \mathcal{F}_{n-1}) &= E(H_n(M_n - M_{n-1}) | \mathcal{F}_{n-1}) \\ &= H_n \, E(M_n - M_{n-1} | \mathcal{F}_{n-1}) \\ &= 0 \end{aligned}$$

THE DOOB DECOMPOSITION

Theorem: Let $X = \{X_0, X_1, X_2, \ldots\}$ be adapted to the history $\{\mathcal{F}_n\}$, where $X_0 = 0$. Then there exist (uniquely given) a zero-mean martingale M and a predictable process X^* starting with $X_0^* = 0$ such that

$$X_n = X_n^* + M_n \text{ for } n = 0, 1, 2, \ldots$$

Proof: Let

$$\begin{aligned} X_n^* &= \sum_{k=1}^{n} [E(X_k|\mathcal{F}_{k-1}) - X_{k-1}] \\ &= \text{sum of } \textit{predictions} \text{ for next state} \\ M_n &= \sum_{k=1}^{n} [X_k - E(X_k|\mathcal{F}_{k-1})] = \text{sum of } \textit{innovations} \end{aligned}$$

The process X_n^* is predictable since each term is in $\mathcal{F}_{n-1}$ (why?). Finally, M_n is a martingale since

$$\begin{aligned} E(M_n - M_{n-1}|\mathcal{F}_{n-1}) &= E[X_n - E(X_n|\mathcal{F}_{n-1})|\mathcal{F}_{n-1}] \\ &= E(X_n|\mathcal{F}_{n-1}) - E(X_n|\mathcal{F}_{n-1}) = 0 \end{aligned}$$