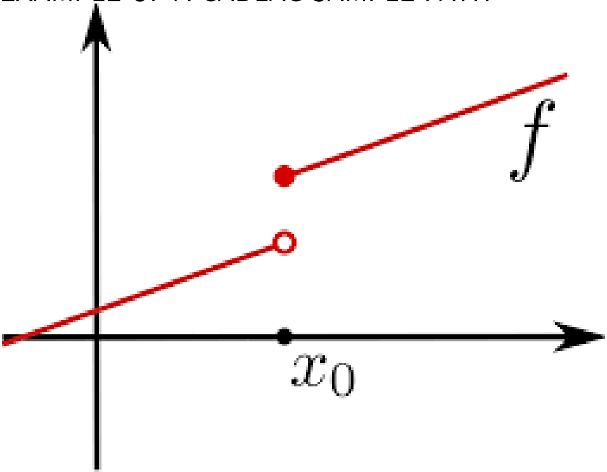
PROCESSES IN CONTINUOUS TIME

Key elements:

- $X = \{X(t) : t \in [0, \tau]\}$
- History (information) at time t is given by $\mathcal{F}_t$, with $\mathcal{F}_s \subset \mathcal{F}_t$ when $s < t$. Typically, $\mathcal{F}_t$ corresponds to observation of $X(u)$ for $0 \leq u \leq t$
- $\{\mathcal{F}_t\}$ is called a *filtration*
- The process X is said to be *adapted* to $\{\mathcal{F}_t\}$ if $X(t) \in \mathcal{F}_t$ for all t.
- The process is called *cadlag* if its *paths* (trajectories on a graph) are right continuous with left hand limits

EXAMPLE OF A CADLAG SAMPLE PATH



MARTINGALES

IN CONTINUOUS TIME	IN DISCRETE TIME
$\mathcal{F}_s \subset \mathcal{F}_t$ whenever $s < t$ $M = \{M(t) : t \in [0, \tau]\}$	$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \cdots$ $M = \{M_n : n = 0, 1, \ldots\}$
Definition: $E(M(t)\vert\mathcal{F}_s) = M(s)$ for all $t > s$	*Definition:* $E(M_n\vert\mathcal{F}_m) = M_m$ for all $n > m$

EQUIVALENT DEFINITIONS OF MARTINGALES

Discrete time:

$$E(\Delta M_n|\mathcal{F}_{n-1}) = E(M_n - M_{n-1}|\mathcal{F}_{n-1}) = 0$$

Continuous time:

$$E(dM(t)|\mathcal{F}_{t-}) = E(M((t+dt)-) - M(t-)|\mathcal{F}_{t-}) = 0$$

Here

- $\mathcal{F}_{t-}$ is the history up to, but not including, time t.
- $dM(t)$ is the increment of $M(t)$ in the time interval $[t, t+dt)$, i.e.,

$$dM(t) = M((t+dt)-) - M(t-)$$

PREDICTABLE PROCESS

Discrete time: The process $H = \{H_n\}$ is *predictable* if the value of H_n is known at time H_{n-1}, i.e.

- $H_n \in \mathcal{F}_{n-1}$.

Continuous time: The process $H = \{H(t)\}$ is *predictable* if, informally, the value of $H(t)$ is known immediately before t. A sufficient condition for predictability of H is that it is

- adapted to $\mathcal{F}_t$
- has *left continuous* sample paths.

STOCHASTIC INTEGRALS

Discrete time: *The transformation of martingale M by predictable H* is

$$Z_n = H_1(M_1 - M_0) + H_2(M_2 - M_1) + \ldots + H_n(M_n - M_{n-1}) = \sum_{i=1}^{n} H_i \Delta M_i$$

The process Z_n is a mean zero martingale.

Continuous time: The *stochastic integral* of a predictable process H with respect to a martingale M is

$$I(t) = \int_0^t H(s)dM(s) =_{\text{def}} \lim_{n\to\infty} \sum_{i=1}^{n} H_i \Delta M_i$$

Here the continuous processes H and M are approximated by discrete processes obtained by partitioning $(0, t]$ into n parts of length t/n and letting

- $H_i = H((i-1)t/n), \qquad M_i = M(it/n)$

The stochastic integral $I(t)$ is a mean zero martingale.

THE DOOB-MEYER DECOMPOSITION

The *adapted* process $X = \{X(t) : t \in [0, \tau]\}$ is a *submartingale* if

$$E(X(t)|\mathcal{F}_s) \geq X(s) \text{ for all } t > s$$

Every increasing process – e.g. counting process – is a submartingale.
Doob-Meyer decomposition for submartingales:

$$X(t) = X^*(t) + M(t) \quad \text{(uniquely)}$$

- X^* is an increasing *predictable* process, called the **compensator** of X
- M is a mean zero martingale

VARIATION PROCESSES IN DISCRETE TIME

The *predictable variation* process is defined by:

$$\langle M \rangle_n = \sum_{i=1}^{n} E\{(M_i - M_{i-1})^2 | \mathcal{F}_{i-1}\} = \sum_{i=1}^{n} \mathrm{Var}(\Delta M_i | \mathcal{F}_{i-1})$$

The *optional variation* process is defined by:

$$[M]_n = \sum_{i=1}^{n} (M_i - M_{i-1})^2 = \sum_{i=1}^{n} (\Delta M_i)^2$$

Properties:

$$E(\langle M \rangle_n) = E([M]_n) = Var(M_n)$$

If H is predictable:

$$\begin{aligned} \langle H \bullet M \rangle &= H^2 \bullet \langle M \rangle \\ [H \bullet M] &= H^2 \bullet [M] \end{aligned}$$

VARIATION PROCESSES IN CONTINUOUS TIME

The *predictable variation* process is defined by:

$$\langle M \rangle (t) = \lim_{n\to\infty} \sum_{i=1}^{n} \text{Var}(\Delta M_i | \mathcal{F}_{(i-1)t/n})$$

where $[0, t]$ is partitioned into n parts of length t/n and $M_i = M(it/n)$.
Thus:

$$d\langle M \rangle (t) = \text{Var}(dM(t)|\mathcal{F}_{t-})$$

The *optional variation* process is defined by:

$$[M](t) = \lim_{n\to\infty} \sum_{i=1}^{n} (\Delta M_i)^2 = \sum_{s\le t} (M(s) - M(s-))^2$$

The last equality is valid for processes of finite variation (holds for our applications) and means the sum of squares of all jumps of $M(t)$.

VARIATION PROCESSES IN CONTINUOUS TIME (cont.)

Properties:

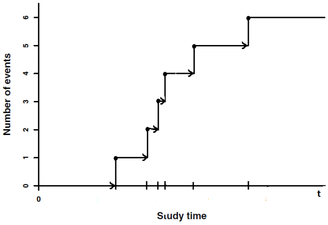
$$E(\langle M \rangle (t)) = E([M] (t)) = Var(M(t))$$

If H is predictable:

$$\left\langle \int H dM \right\rangle = \int H^2 d \langle M \rangle$$

$$\left[\int H dM \right] = \int H^2 d [M]$$

COUNTING PROCESS BASICS



- $N(t) = \#$ events in $(0, t]$
- $dN(t) = \#$ events in $[t, t + dt) = \begin{cases} 1 & \text{if event at time } t \\ 0 & \text{if no event at time } t \end{cases}$
- **Intensity process** $\lambda(t)$ (*predictable*) is defined by:
  $P(dN(t) = 1|\mathcal{F}_{t-}) \equiv P(\text{event in } [t, t + dt) \mid \mathcal{F}_{t-}) = \lambda(t)dt$

Poisson process: $P(dN(t) = 1|\mathcal{F}_{t-}) = P(dN(t) = 1) =$ **deterministic** $\lambda(t)dt$

Example of a random intensity: $P(dN(t) = 1|\mathcal{F}_{t-}) = (N(t-) + \beta)\alpha(t)dt$
$=$ **random** $\lambda(t)dt$