Extra exercises in STK4080 2021

Exercise E5.1

Let the situation be as on pages 15-17 in Slides 15.

Assume now that $\alpha(t) = \lambda$ is a constant, and let $Z \sim \text{Gamma}(1/\delta, 1/\delta)$.

a) Write down the resulting expressions for $A(t), \mu(t), S(t)$ and $f(t)$.

Let $(\tilde{T}_i, D_i); i = 1, 2, \ldots, n$ be a (possibly right censored/independent censoring) sample from the distribution. Observe that this is the same situation as considered in Exercise 1 of Slides 16, but now assuming that each cluster consists of exactly one item (individual).

b) Use the cited Exercise 1 to write down the likelihood and log likelihood for the present problem. Show that the log likelihood can be written

$$\ell(\lambda, \delta) = D_\bullet \log \lambda - \sum_{i=1}^{n} \left(D_i + \frac{1}{\delta} \right) \log \left(1 + \delta \lambda \tilde{T}_i \right)$$

where $D_\bullet = \sum_{i=1}^{n} D_i$ is the number of events.

c) Derive the score functions and explain how you can find the maximum likelihood estimates for λ and δ.

Exercise E5.2

An intuitive explanation of the fact that the hazard rates are "dragged down" with increasing δ on p. 18 of Slides 15, is that for large values of δ, there are many individuals with high hazard which hence fail first.

In this exercise we will show that the expression for $E(Z_i|H_i)$ on p. 11 of Slides 16 can be used to get some information on this effect of "failing first".

The calculation in Slides 16 is for a cluster of n_i individuals. We can, however, put $n_i = 1$.

a) Consider the population hazard $\mu(t)$ as calculated in Slides 15 on p. 17 and illustrated on p. 18. Explain why the formula on p. 11 of Slides 16 implies that

$$E(Z|T = t) = \frac{\delta + 1}{\delta A(t) + 1}$$

Describe in words what this equation tells us about the size of Z for an individual that fails at time t. Consider in particular $t = 0$.

b) Let now $\alpha(t) = t^2$ as on p. 18 of Slides 15. Calculate $E(Z|T = t)$ for some values of t and comment. For which value of t is $E(Z|T = t) = 1$?