EKSTRA MNFST 102 4.12.2001

Oppgave 1

a) H_0: Uavhengighet mellom leverandør og feilkategori

~~8 (8.64)~~	~~34~~	~~42~~	~~20~~	~~104~~
~~9~~	~~50~~	~~28~~	~~9~~	~~96~~
~~17~~	~~84~~	~~70~~	~~29~~	~~200~~

8 (8.84)	34 (43.68)	42 (36.40)	20 (15.08)	104
9 (8.16)	50 (40.32)	28 (33.60)	9 (13.92)	96
17	84	70	29	200

$$C = \frac{(8-8.84)^2}{8.84} + \frac{(34-43.68)^2}{43.68} + \cdots$$

$$= \underline{9.774}$$

Under H_0 er $C \sim \chi^2_{(2-1)(4-1)} = \chi^2_3$

(P-verdi 0.021 fra MINITAB)

Med 5% nivå skal vi forkaste hvis $C \geq 7.815$

altså: $\underline{\underline{\text{FORKAST } H_0}}$

a) $L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$

Ser at $\prod_{i=1}^{n} X_i$ er suffisient siden likelihooden avhenger av x-ene bare gjennom denne.

Andre: Enhver 1-1 funksjon av $\prod X_i$, f.eks $\sum \log X_i$, $e^{\prod_{i=1}^{n} X_i}$, ...

Men ogsaa f.eks. $(X_1, \dots, X_n)$, eller $(X_{(1)}, \dots, X_{(n)})$ (ordnede)

b) Se forst på $W = -2\theta \log X$ ~~ ~~

Da har vi

$F_W(w) = P(W \le w) = P(-2\theta \log X \le w)$

$= P\left(\log X \ge -\frac{w}{2\theta}\right) = P\left(X \ge e^{-\frac{w}{2\theta}}\right)$

$= 1 - F_X\left(e^{-\frac{w}{2\theta}}\right)$

Dermed $f_W(w) = f_X\left(e^{-\frac{w}{2\theta}}\right) \cdot e^{-\frac{w}{2\theta}} \cdot \frac{1}{2\theta}$

$= \theta e^{-\frac{w}{2\theta}(\theta-1)} \cdot e^{-\frac{w}{2\theta}} \cdot \frac{1}{2\theta}$

$= \frac{1}{2} e^{-\frac{w}{2}}$ som er tetthet i $\underline{\underline{\chi^2_2}}$

Dermed $Y = \sum_{i=1}^{n} W_i \sim \chi^2_{2n}$ Q.E.D.

c) Vet at $P\left(x_{\alpha/2, 2n} \leq -2\theta \sum \log X_i \leq x_{1-\alpha/2, 2n} \right) = 1-\alpha$

Dermed

$$P\left(\frac{x_{\alpha/2, 2n}}{-2 \sum \log X_i} \leq \theta \leq \frac{x_{1-\alpha/2, 2n}}{-2 \sum \log X_i} \right) = 1-\alpha$$

gir konfidensintervallet.
(da $-2 \sum \log X_i > 0$)

d) $\ell(\theta) = \log L(\theta) = n \log \theta + (\theta - 1) \sum_{i=1}^{n} \log X_i$

$$\ell'(\theta) = \frac{n}{\theta} + \sum_{i=1}^{n} \log X_i$$

Settes dette lik 0 får vi

$$\hat{\theta} = \frac{n}{-\sum \log X_i} \qquad \text{Q.E.D.}$$

e) Cramer-Rao:

$$\log f(x, \theta) = \log \theta + (\theta - 1) \log x$$

$$\frac{\partial \log f(x,\theta)}{\partial \theta} = \frac{1}{\theta} + \log x$$

$$\frac{\partial^2 \log f(x,\theta)}{\partial \theta^2} = -\frac{1}{\theta^2}$$

Dermed er CR.nedre.grense lik $\dfrac{1}{n \cdot \frac{1}{\theta^2}} = \underline{\dfrac{\theta^2}{n}}$

Dermed $\hat{\theta} \approx N(\theta, \frac{\theta^2}{n})$ for store n.

f) $\hat{\theta} = \frac{2n\theta}{Y}$

$$E(\hat{\theta}) = 2n\theta \cdot E\left(\frac{1}{Y}\right)$$

Trenger altså

$$E\left(\frac{1}{Y}\right) = \int_0^\infty \frac{1}{y} \cdot \frac{1}{2^n \Gamma(n)} y^{n-1} e^{-\frac{y}{2}} dy$$

$$= \frac{2^{\frac{2n-2}{2}}\Gamma\left(\frac{2n-2}{2}\right)}{2^n\Gamma(n)} \underbrace{\int_0^\infty \frac{1}{2^{\frac{2n-2}{2}}\Gamma\left(\frac{2n-2}{2}\right)} y^{\frac{2n-2}{2}-1} e^{-\frac{y}{2}} dy}_{=1}$$

$$= \frac{\Gamma(n-1)}{2\Gamma(n)} = \frac{\Gamma(n-1)}{2(n-1)\Gamma(n-1)} = \frac{1}{2(n-1)}$$

Dermed $E(\hat{\theta}) = 2n\theta \cdot \frac{1}{2(n-1)} = \underline{\underline{\frac{n}{n-1}\theta}}$

Oppgave 3

a) $L = \sum_{i=1}^{n} (Y_i - \beta x_i)^2$

$$\frac{\partial L}{\partial \beta} = -\sum_{i=1}^{n} 2(Y_i - \beta x_i) x_i$$

Settes dette lik 0 får vi

$$\hat{\beta} = \frac{\sum x_i Y_i}{\sum x_i^2}$$

Da er $E(\hat{\beta}) = \frac{\sum x_i \cdot \beta x_i}{\sum x_i^2} = \beta$

$$Var(\hat{\beta}) = \frac{1}{(\sum x_i^2)^2} \cdot \sum x_i^2 \cdot x_i \sigma^2 = \frac{\sum x_i^3}{(\sum x_i^2)^2} \sigma^2$$

b) $V_i = \frac{Y_i}{\sqrt{x_i}}$. Da er $E(V_i) = \frac{\beta x_i}{\sqrt{x_i}} = \sqrt{x_i}\,\beta$

Skal altså minimere

$$\sum_{i=1}^{n} (V_i - \beta\sqrt{x_i})^2$$

Kan da bruke løsningen i a) med V_i for Y_i og $\sqrt{x_i}$ for x_i.

Dermed $\beta^* = \frac{\sum \sqrt{x_i}\, V_i}{\sum x_i} = \frac{\sum Y_i}{\sum x_i}$

Her er $E(\beta^*) = \dfrac{\sum \beta x_i}{\sum x_i} = \underline{\underline{\beta}}$

$$\mathrm{Var}(\beta^*) = \frac{1}{(\sum x_i)^2} \cdot \sum x_i \sigma^2 = \underline{\underline{\frac{\sigma^2}{\sum x_i}}}$$

c) Sammenligning av $\hat{\beta}$ og β^*:

1. Begge er forventningsrette. Må derfor sammenligne variansene.

Vi ser at

$$\mathrm{Var}\,\beta^* \leq \mathrm{Var}\,\hat{\beta}$$

$$\Updownarrow$$

$$\frac{1}{\sum x_i} \leq \frac{\sum x_i^3}{(\sum x_i^2)^2}$$

$$\Updownarrow$$

$$(\sum x_i^2)^2 \leq (\sum x_i^3)(\sum x_i)$$

Men den siste ulikheten gjelder alltid: Sett $a_i = x_i^{3/2}$ og $b_i = x_i^{1/2}$ i Cauchy-Schwarz' ulikhet

Likelihood:

$$\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{x_i}\,\sigma} e^{-\frac{1}{2x_i\sigma^2}(y_i-\beta x_i)^2}$$

$$= (2\pi)^{-\frac{n}{2}} \cdot \frac{1}{(\prod x_i)^{1/2}} \cdot \frac{1}{\sigma^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\frac{(y_i-\beta x_i)^2}{x_i}}$$

Ser at for hver fast σ vil dette maksimeres hvis vi minimerer

$$\sum_{i=1}^{n} \frac{(y_i-\beta x_i)^2}{x_i} = \sum_{i=1}^{n} (v_i - \beta\sqrt{x_i})^2$$

Men vi har sett at dette minimeres ved β^*. Altså er β^* MLE for β.

For å finne MLE for σ^2 ser vi på log likelihood

$$-\frac{n}{2}\log 2\pi - \frac{1}{2}\log\left(\prod x_i\right) - \frac{n}{2}\log(\sigma^2)$$

$$- \frac{1}{2\sigma^2}\sum_{i=1}^{n}\frac{(y_i-\beta^* x_i)^2}{x_i}$$

Deriver dette m.h.p. (σ^2) og sett lik 0. Vi får da

$$\hat{\sigma}^2_{ML} = \frac{1}{n}\sum_{i=1}^{n}\frac{(Y_i-\beta^* x_i)^2}{x_i}$$

Oppgave 4

a) $F_Y(y) = P(Y \le y) = P(aX \le y)$

$$= P\left(X \le \frac{y}{a}\right) = F_X\left(\frac{y}{a}\right)$$

Derved
$$f_Y(y) = f_X\left(\frac{y}{a}\right) \cdot \frac{1}{a}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\beta} \cdot \frac{a}{y} e^{-\frac{(\log y - \log a - \alpha)^2}{2\beta^2}} \cdot \frac{1}{a}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\beta} \cdot \frac{1}{y} e^{-\frac{(\log y - (\log a + \alpha))^2}{2\beta^2}}$$

dvs Lognormal$(\log a + \alpha, \beta)$.

~~b~~ $F_Z(z) = P(Z \le z) = P\left(\frac{1}{X} \le z\right) = P\left(X \ge \frac{1}{z}\right)$

$$= 1 - F_X\left(\frac{1}{z}\right)$$

Derved
$$f_Z(z) = f_X\left(\frac{1}{z}\right) \cdot \frac{1}{z^2}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\beta} \cdot z \cdot e^{-\frac{(-\log x - \alpha)^2}{2\beta^2}} \cdot \frac{1}{z^2}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\beta} \cdot \frac{1}{z} \cdot e^{-\frac{(\log x + \alpha)^2}{2\beta^2}}$$

dvs Lognormal$(-\alpha, \beta)$.

b) Vet at $X \sim \text{Lognormal}(\alpha, \beta)$

$$\Updownarrow$$

$$\log X \sim N(\alpha, \beta^2)$$

Dermed, for å vise at $W \sim$ Lognormal er det nok å vise at $\log W$ er normalfordelt.

Men $\log W = \underbrace{\log X_1}_{N(\alpha,\beta^2)} - \underbrace{\log X_2}_{N(\alpha,\beta^2)}$

Dermed $\log W \sim N(0, 2\beta^2)$

dvs $W \sim \text{Lognormal}(0, 2\beta^2)$