On the Monotone Class Theorem

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The purpose of this note is to prove the Monotone Class Theorem, which is given as Theorem 1.17 in Karr: *Probability.* The proof is based on old lecture notes in MA8704 by Arvid Næss.

The setting is as in Chapter 1 of Karr, where Ω is the sample space and $\mathcal{F}, \mathcal{D}, \Pi$ etc. are sets of subsets of Ω.

Definition 1 *σ-algebra $\mathcal{F}$*

S1 $\Omega \in \mathcal{F}$

S2 $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

S3 $A_i \in \mathcal{F}, i = 1, 2, \ldots \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{F}$

Definition 2 *d-system $\mathcal{D}$*

D1 $\Omega \in \mathcal{D}$

D2 $A, B \in \mathcal{D}, A \subseteq B \Rightarrow B \setminus A \in \mathcal{D}$

D3 $A_i \in \mathcal{D}, i = 1, 2, \ldots; A_1 \subseteq A_2 \subseteq \cdots \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{D}$

Definition 3 *π-system Π*

P1 $A_i \in \Pi, i = 1, 2, \ldots, n \Rightarrow \cap_{i=1}^{n} A_i \in \Pi$, *which is equivalent to*

P1' $A, B \in \Pi \Rightarrow A \cap B \in \Pi$

Proposition 1 *If $\mathcal{B}$ is both a π-system and a d-system, then $\mathcal{B}$ is a σ-algebra.*

Proof: Suppose that $\mathcal{B}$ is both a π-system and a d-system. Then $\mathcal{B}$ satisfies D1, D2, D3, P1, and we need to prove that it also satisfies S1, S2 and S3.

S1: This holds trivially by D1.

S2: It follows by D2 that if $A \in \mathcal{B}$, we have $\Omega \setminus A = A^c \in \mathcal{B}$. Thus S2 holds.

S3: Let $A_i \in \mathcal{B}, i = 1, 2, \ldots$ and define

$$B_k = \cup_{i=1}^{k} A_i \text{ for } k = 1, 2, \ldots.$$

Then $B_k^c = \cap_{i=1}^k A_i^c$. Here, $A_i^c \in \mathcal{B}$ by D2, as we saw above, and hence $B_k^c \in \mathcal{B}$ by P1 and $B_k \in \mathcal{B}$ by another use of D2. Since, moreover, $B_1 \subseteq B_2 \subseteq \cdots$, it follows from D3 that $\cup_{i=1}^\infty B_i \in \mathcal{B}$. By the definition of B_k it is clear that

$$\cup_{i=1}^\infty B_i = \cup_{i=1}^\infty A_i$$

and hence also $\cup_{i=1}^\infty A_i \in \mathcal{B}$, which proves S3.

Theorem 1 (Monotone Class Theorem) *Let $\mathcal{S}$ be a π-system. Then*

$$\sigma(\mathcal{S}) = d(\mathcal{S})$$

Proof: First, since a σ-algebra is also a d-system, we have

$$d(\mathcal{S}) \subseteq \sigma(\mathcal{S}) \tag{1}$$

This is because $d(\mathcal{S})$ by definition is the *smallest* d-system that contains $\mathcal{S}$. Now suppose that we can prove that $d(\mathcal{S})$ is a π-system. Then $d(\mathcal{S})$ is a σ-algebra by the above Proposition. But then

$$\sigma(\mathcal{S}) \subseteq d(\mathcal{S}) \tag{2}$$

since $\sigma(\mathcal{S})$ is the smallest σ-algebra that contains $\mathcal{S}$. Hence, combining (1) and (2) we have $d(\mathcal{S}) = \sigma(\mathcal{S})$ and we would be done.

It thus remains to prove that $d(\mathcal{S})$ is a π-system.

Step 1: Let

$$\mathcal{D}_1 = \{B \in d(\mathcal{S}) : B \cap C \in d(\mathcal{S}) \text{ for all } C \in \mathcal{S}\}$$

Since $\mathcal{S}$ is a π-system, we have

$$\mathcal{S} \subseteq \mathcal{D}_1 \tag{3}$$

We now show that $\mathcal{D}_1$ is a d-system:

D1: Obviously $\Omega \in \mathcal{D}_1$.

D2: If $B_1, B_2 \in \mathcal{D}_1$ and $B_1 \subseteq B_2$, then for $C \in \mathcal{S}$,

$$(B_2 \setminus B_1) \cap C = (B_2 \cap C) \setminus (B_1 \cap C)$$

Here $B_1 \cap C$ and $B_2 \cap C$ are in $d(\mathcal{S})$ by assumption, and clearly $B_1 \cap C \subseteq B_2 \cap C$. Since $d(\mathcal{S})$ is a d-system, it follows that $(B_2 \cap C) \setminus (B_1 \cap C)$ and hence $(B_2 \setminus B_1) \cap C$ are in $d(\mathcal{S})$, using D2. But then $B_2 \setminus B_1 \in \mathcal{D}_1$ by definition of $\mathcal{D}_1$.

D3: If $B_i \in \mathcal{D}_1, i = 1, 2, \ldots$, where $B_1 \subseteq B_2 \subseteq \cdots$, then for $C \in \mathcal{S}$, we have

$$B_1 \cap C \subseteq B_2 \cap C \subseteq \cdots$$

Since $B_i \cap C \in d(\mathcal{S})$ (since $B_i \in \mathcal{D}_1$) it follows by D3 that

$$(\cup_{i=1}^\infty B_i) \cap C = \cup_{i=1}^\infty (B_i \cap C) \in d(\mathcal{S})$$

But then $\cup_{i=1}^\infty B_i \in \mathcal{D}_1$ and D3 holds for $\mathcal{D}_1$.

Thus we have shown that $\mathcal{D}_1$ is a d-system, which by (3) contains $\mathcal{S}$. Hence we have that $d(\mathcal{S}) \subseteq \mathcal{D}_1$, since $d(\mathcal{S})$ is the smallest d-system that contains $\mathcal{S}$.

On the other hand, by definition of $\mathcal{D}_1$, we have $\mathcal{D}_1 \subseteq d(\mathcal{S})$, so we must have $\mathcal{D}_1 = d(\mathcal{S})$.

Step 2: Let

$$\mathcal{D}_2 = \{A \in d(\mathcal{S}) : B \cap A \in d(\mathcal{S}) \text{ for all } B \in d(\mathcal{S})\}$$

By Step 1 we have

$$\mathcal{S} \subseteq \mathcal{D}_2 \tag{4}$$

To see this, let $C \in \mathcal{S}$. Then if $B \in d(\mathcal{S}) = \mathcal{D}_1$, we have $B \cap C \in d(\mathcal{S})$ by definition of $\mathcal{D}_1$, so $C \in \mathcal{D}_2$.

We now prove that $\mathcal{D}_2$ is a d-system. The arguments are similar to the ones used to prove that $\mathcal{D}_1$ is a d-system:

D1: Obviously $\Omega \in \mathcal{D}_2$.

D2: If $A_1, A_2 \in \mathcal{D}_2$ and $A_1 \subseteq A_2$, then for $B \in d(\mathcal{S})$,

$$(A_2 \setminus A_1) \cap B = (A_2 \cap B) \setminus (A_1 \cap B)$$

Here $A_1 \cap B$ and $A_2 \cap B$ are in $d(\mathcal{S})$ by assumption, so $(A_2 \setminus A_1) \cap B \in d(\mathcal{S})$ by D2 since $d(\mathcal{S})$ is a d-system. But then $A_2 \setminus A_1 \in \mathcal{D}_2$ by definition of $\mathcal{D}_2$.

D3: If $A_i \in \mathcal{D}_2, i = 1, 2, \ldots$, where $A_1 \subseteq A_2 \subseteq \cdots$, then for $B \in d(\mathcal{S})$, we have

$$A_1 \cap B \subseteq A_2 \cap B \subseteq \cdots$$

Since $A_i \cap B \in d(\mathcal{S})$ (since $A_i \in \mathcal{D}_2$) it follows by D3 that

$$(\cup_{i=1}^{\infty} A_i) \cap B = \cup_{i=1}^{\infty} (A_i \cap B) \in d(\mathcal{S})$$

But then $\cup_{i=1}^{\infty} A_i \in \mathcal{D}_2$ and D3 holds for $\mathcal{D}_2$.

Thus we have shown that $\mathcal{D}_2$ is a d-system, which by (4) contains $\mathcal{S}$. Hence we have that $d(\mathcal{S}) \subseteq \mathcal{D}_2$. On the other hand, by definition of $\mathcal{D}_2$, we have $\mathcal{D}_2 \subseteq d(\mathcal{S})$, so we must have $\mathcal{D}_2 = d(\mathcal{S})$.

The final step is then to conclude that $d(\mathcal{S})$ is a π-system. This follows in fact directly from the definition of $\mathcal{D}_2$ and the fact that $\mathcal{D}_2 = d(\mathcal{S})$. Indeed, suppose that $A, B \in d(\mathcal{S})$. Then since $A \in \mathcal{D}_2$ we must have $A \cap B \in d(\mathcal{S})$. Thus P1' holds and the proof of the theorem is complete.