A state feedback controller for a class of nonlinear positive systems applied to stabilization of gas-lifted oil wells

Lars Imsland [1], Gisle Otto Eikrem and Bjarne A. Foss [*]

Department of Engineering Cybernetics,
Norwegian University of Science and Technology,
7491 Trondheim, Norway
Tel. (+47) 73 59 43 76, Fax. (+47) 73 59 43 99

Abstract

A state-feedback controller is proposed for a certain class of multi-input nonlinear positive systems. The controller achieves closed loop convergence to a set, which in many cases implies convergence to an equilibrium. The controller is applied to the casing heading problem, an important instability problem occurring in the petroleum industry. Both simulations and a laboratory trial illustrate the merits of the controller.

Key words: nonlinear systems, positive systems, set stability, casing heading

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1 Introduction

When modeling systems for control based on first principles, one often obtains nonlinear ordinary differential equations where the state variables (mass, pressure, level, energy, etc.) are *positive*. In addition, the control input will also often be positive (valve openings, amount of inflow, heat input, etc.). Hence, the class of positive systems (systems with nonnegative states and inputs) is a natural class of systems to consider in a control setting.

[*] Corresponding author.
Email addresses: Lars.Imsland@sintef.no (Lars Imsland),
Gisle.Otto.Eikrem@itk.ntnu.no (Gisle Otto Eikrem),
Bjarne.Foss@itk.ntnu.no (Bjarne A. Foss).
[1] Presently at SINTEF ICT, Applied Cybernetics, 7465 Trondheim, Norway.

Since mass is an inherently positive quantity, systems modeled by mass balances [1] are perhaps the most natural example of positive systems. Another example is the widely studied class of compartmental systems [7,11], used in biomedicine, pharmacokinetics, ecology, etc. Compartmental systems, which are often derived from mass balances, are (nonlinear or linear) systems where the dynamics are subject to strong structural constraints. Each state is a measure of some material in a compartment, and the dynamics consists of the flow of material into (*inflow*) or out of (*outflow*) each compartment. If these flows fulfill certain criteria, the system is called compartmental.

Similar to compartmental systems, we will herein assume that each state can be interpreted as the "mass" (or measure of mass; concentration, level, pressure, etc.) of a compartment. However, we do not make the same strong assumptions on the structure on the flows between the compartments. Instead, we make other (strong) assumptions related to the system being controllable according to the control objective under input saturations. Furthermore, we assume that the compartments that constitute the state vector can be divided into groups of compartments, which we will call *phases*. Each phase will have a controlled inflow or outflow associated with it. The control objective will be to steer the mass of each phase (the sum of the compartment masses in that phase) to a constant, prespecified value.

For this model class we propose a state feedback controller, providing closed loop convergence of the sum of the states of each phase. The controller is inspired by [2], but the class of systems is larger, especially since the phase concept allows us to consider multiple input systems. Furthermore, we also allow saturated inputs and outflow-controlled systems. Similar to [2], we assume that the inputs are positive. The controller in [2] can be viewed as a special case of the controller herein. A related controller for a (similar) class of systems exhibiting *first integrals* is developed in [4]. Instead of controlling the sum of system states (the total mass), it is controlling (more general) first integrals to a specified value. The controller of [4] is different from the one considered herein, for instance the input can take on negative values. However, the stability properties of the closed loop are similar, in the sense that they both achieve convergence to a certain *set*.

The paper is structured as follows: In Section 2 the system class is presented. The controller and a convergence result from a general invariant domain of attraction are presented in Section 3. In Section 4, a simple example illustrates some issues related to the phase concept and stability of the closed loop. Finally, Section 5 provides a semi-realistic application to stabilization of gas-lifted oil wells, including simulations on a de-facto industry standard simulator and lab trials.

In the following, $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_+^n = \{x = [x_1, x_2, \dots, x_n]^\mathsf{T} \mid x_i \in \mathbb{R}_+\}$. Further, $\mathrm{blockdiag}(A_1, \dots, A_r)$ denotes a block diagonal matrix with the matrices $A_1, \dots, A_r$ on the "diagonal".

2 Model class

We consider nonlinear systems

$$\dot{x} = f(x, u), \tag{1a}$$

where the state is positive ($x \in \mathbb{R}_+^n$), and the input is positive and upper bounded, $u \in U := \{u \in \mathbb{R}_+^m \mid 0 \leq u_j \leq \bar{u}_j\}$. Each state can be interpreted as the "mass" (amount of material, or some measure of amount) in a compartment [2] . The controller we will propose exploits system structure, thus we assume the model equations to be on the following form:

$$f(x, u) = \Phi(x) + \Psi(x) + B(x)u. \tag{1b}$$

Loosely speaking, $\Phi(x)$ represents "interconnection structure" between compartments, $\Psi(x)$ represents *uncontrolled* external inflows to and outflows from compartments and $B(x)u$ represents *controlled* external inflows to and outflows from compartments.

Furthermore, we will assume that the state vector can be divided into m different parts, which will be denoted *phases*. Phase j will consist of r_j states, and have the control u_j associated with it, corresponding to *either* controlled inflow *or* outflow to compartments of that phase. The states in phase j will be denoted χ^j, such that $x = [(\chi^1)^\mathsf{T}, (\chi^2)^\mathsf{T}, \dots, (\chi^m)^\mathsf{T}]^\mathsf{T}$, and it follows that necessarily, $\sum_{j=1}^m r_j = n$. Corresponding to this structure, the vector functions $\Phi(x)$, $\Psi(x)$ and the matrix function $B(x)$ are on the form

$$\begin{aligned}
\Phi(x) &= \left[\phi^1(x)^\mathsf{T}, \phi^2(x)^\mathsf{T}, \dots, \phi^m(x)^\mathsf{T}\right]^\mathsf{T} \\
\Psi(x) &= \left[\psi^1(x)^\mathsf{T}, \psi^2(x)^\mathsf{T}, \dots, \psi^m(x)^\mathsf{T}\right]^\mathsf{T} \\
B(x) &= \operatorname{blockdiag}\left(b^1(x), b^2(x), \dots, b^m(x)\right).
\end{aligned}$$

Note that element j is (in general) a function of x, not (only) χ^j. Also note that the partitioning into phases need not be unique.

We will state the assumptions on these functions on the set $D \subseteq \mathbb{R}_+^n$. In the case of global results, $D = \mathbb{R}_+^n$.

[2] The word compartment *does not* imply that the system class we look at is compartmental [11]. However, it enjoys strong similarities with compartmental systems.

A1. **(Interconnection structure)** The function $\Phi : D \to \mathbb{R}^n$ is locally Lipschitz, $\phi_i^j(x) \geq 0$ for $\chi_i^j = 0$, and

$$\sum_{i=1}^{r_j} \phi_i^j(x) = 0, \; j = 1, \ldots, m.$$

A2. **(Controlled external flows)** The block diagonal matrix function $B(x) : D \to \mathbb{R}^{n \times m}$ is locally Lipschitz and satisfies:

a. Phase j has controlled inflow:

$$\begin{aligned} & b_i^j(x) \geq 0 \text{ for all } x \in D \\ & \sum_{i=1}^{r_j} b_i^j(x) > 0 \text{ for all } x \in D \end{aligned}$$

b. Phase j has controlled outflow:

$$\begin{aligned} & b_i^j(x) \leq 0 \text{ for all } x \in D \\ & \chi_i^j = 0 \Rightarrow b_i^j(x) = 0 \text{ (if } \{x \mid \chi_i^j = 0\} \cap D \neq \emptyset) \\ & \sum_{i=1}^{r_j} b_i^j(x) < 0 \text{ for all } x \in D \end{aligned}$$

These assumptions says that there is zero net contribution to phase mass from the "interconnection structure", and that the controlled flows really are inflow and outflow (and in the outflow-controlled case, that no mass can flow when a state is zero).

The uncontrolled external flows must satisfy some "controllability" assumption in relation to the controlled flows. Before we define this, it is convenient to define the "mass" of each phase, being the sum of the compartment masses (the states) of that phase:

$$M_j(x) := \sum_{i=1}^{r_j} \chi_i^j.$$

Our control objective will be to control $M_j(x)$ to some prespecified (positive) desired mass of phase j, denoted M_j^*, from initial conditions in D. For the control problem to be meaningful, the intersection of the set where $M_j(x) = M_j^*$ and D should be nonempty.

A3. **(Uncontrolled external flows)** For given $M^* = [M_1^*, M_2^*, \ldots, M_m^*]^\mathsf{T}$, $\Psi(x) : D \to \mathbb{R}^n$ is locally Lipschitz and satisfies $\psi_i^j(x) \geq 0$ for $\chi_i^j = 0$ (if $\{x \mid \chi_i^j = 0\} \cap D \neq \emptyset$), and in addition, if:

a. Phase j has controlled inflow:

1. For $x \in \{x \in D \mid M_j(x) > M_j^*\}$, $\sum_{i=1}^{r_j} \psi_i^j(x) \leq 0$ and the set $\{x \in D \mid \sum_{i=1}^{r_j} \psi_i^j(x) = 0 \text{ and } M_j(x) > M_j^*\}$ does not contain an invariant set.

2. For $x \in \{x \in D \mid M_j(x) < M_j^*\}$, $-\sum_{i=1}^{r_j} \psi_i^j(x) < \sum_{i=1}^{r_j} b_i^j(x)\bar{u}_j$.

b. Phase j has controlled outflow:

1. For $x \in \{x \in D \mid M_j(x) < M_j^*\}$, $\sum_{i=1}^{r_j} \psi_i^j(x) \geq 0$ and the set $\{x \in D \mid \sum_{i=1}^{r_j} \psi_i^j(x) = 0 \text{ and } M_j(x) < M_j^*\}$ does not contain an invariant set.
2. For $x \in \{x \in D \mid M_j(x) > M_j^*\}$, $\sum_{i=1}^{r_j} \psi_i^j(x) < -\sum_{i=1}^{r_j} b_i^j(x)\bar{u}_j$.

Assumption A3.a.1 (A3.b.1) means that when the phase mass is large (small), the outflow of this phase is dominantly outflow (inflow). The "no invariant set" part plays the same role as the assumption of "zero state detectability" through (the equivalent of) $\sum_{i=1}^{r_j} \psi_i^j(x)$ in [2]. The assumption A3.a.2 (A3.b.2) means that when the phase mass is small (large) and the controller is saturating, the outflow (inflow) must be smaller than the (saturated) inflow (outflow)

Proposition 1 (Positivity) *For $x(0) \in \mathbb{R}_+^n$, the state of the system (1) fulfilling A1-A3 with $D = \mathbb{R}_+^n$, satisfies $x(t) \in \mathbb{R}_+^n$, $t > 0$.*

PROOF It suffices to notice that for $x_i = 0$, $\dot{x}_i \geq 0$. $\square$

3 Stabilizing state feedback controller

In this section, the state feedback controller is defined, and a convergence result is given for a general invariant set D that (is a subset of the set that) A1-A3 hold on. The set D could then be considered a region of attraction.

As mentioned in the previous section, our control objective is to control the total mass $M_j(x)$ of each phase to a prespecified value M_j^*. To this end, the following constrained, positive state feedback control law is proposed:

$$u_j(x) = \begin{cases} 0 & \text{if } \tilde{u}_j(x) < 0 \\ \tilde{u}_j(x) & \text{if } 0 \leq \tilde{u}_j(x) \leq \bar{u}_j \\ \bar{u}_j & \text{if } \tilde{u}_j(x) > \bar{u}_j \end{cases} \tag{2}$$

where

$$\tilde{u}_j(x) = \frac{1}{\sum_{i=1}^{r_j} b_i^j(x)} \left(-\sum_{i=1}^{r_j} \psi_i^j(x) + \lambda_j (M_j^* - M_j(x)) \right) \tag{3}$$

and λ_j is a positive constant. Assumption A2 ensures that $\sum_{i=1}^{r_j} b_i^j(x) \neq 0$ always, such that $\tilde{u}_j$ is always defined.

Define the set

$$\Omega = \{x \in \mathbb{R}_+^n \mid M_1(x) = M_1^*, \ldots, M_m(x) = M_m^*\}.$$

Assumption 1 *There exists a set D that is invariant for the dynamics (1) under the closed loop with control (2), and has a nonempty intersection with Ω.*
Assumption 2 *For $x \in \Omega \cap D$, $0 < \tilde{u}_j(x) < \bar{u}_j$.*

Under the given assumptions, the convergence properties of the controller are summarized as follows:

Theorem 1 *Under the given assumptions, the state of the system (1), controlled with (2) and starting from some initial condition $x(0) \in D$, stays bounded and converges to the set $\Omega \cap D$ which is positively invariant.*
Proof The set D is by Assumption 1 invariant, hence Assumptions A1-A3 hold along closed loop trajectories.

Define the positive semidefinite function

$$V(x) := \frac{1}{2} \sum_{j=1}^{m} \left(M_j(x) - M_j^* \right)^2, \tag{4}$$

with time derivative

$$\dot{V}(x) = \sum_{j=1}^{m} \left[M_j(x) - M_j^* \right] \left(\sum_{i=1}^{r_j} \psi_j^i(x) + \sum_{i=1}^{r_j} b_j^i(x) u_j(x) \right).$$

For $M_j(x) \neq M_j^*$, we have one of the following cases:

(1) If $0 \leq \tilde{u}_j \leq \bar{u}_j$, summand j is

$$\left[M_j(x) - M_j^* \right] \left(\sum_{i=1}^{r_j} \psi_j^i(x) + \sum_{i=1}^{r_j} b_j^i(x) u_j(x) \right) = -\lambda_j \left[M_j(x) - M_j^* \right]^2 < 0.$$

(2) If $\tilde{u}_j < 0$, then $u_j(x) = 0$ and summand j is

$$\left[M_j(x) - M_j^* \right] \sum_{i=1}^{r_j} \psi_j^i(x).$$

Assumption A3.a.1 and A3.b.1 ensures that this is negative for both inflow and outflow controlled phases.

(3) If $\tilde{u}_j \geq \bar{u}_j$, then $u_j(x) = \bar{u}_j$ and summand j is

$$\left[M_j(x) - M_j^* \right] \left(\sum_{i=1}^{r_j} \psi_j^i(x) + \sum_{i=1}^{r_j} b_j^i(x) \bar{u}_j \right).$$

Assumption A3.a.2 and A3.b.2 ensures that this is negative for both inflow and outflow controlled phases.

For the details of point 2 and 3, check [10]. We can conclude that $\Omega \cap D$ is invariant, since D is invariant, and by Assumption 2 and a continuity argument, case (1) (and thus $\dot{V}(x) < 0$) holds in the intersection between a neighborhood of $\Omega \cap D$ and D.

Moreover, since $\dot{V}(x) \leq 0$, $V(x(t)) \leq V(x(t_0))$ along system trajectories. From the construction of $V(x)$ it is rather easy [10] to see that for $x \in \mathbb{R}^n_+$, $\|x\| \to \infty$ if and only if $V(x) \to \infty$, hence $V(x(t))$ bounded implies that $\|x(t)\|$ is bounded. This allows us to conclude from LaSalle's invariance principle that $x(t)$ converges to the largest invariant set contained in $\{x \mid \dot{V}(x) = 0\} \cap D$. By the above and Assumption A3.a.1 and A3.b.1, there is no other invariant set for which $\dot{V}(x) = 0$ larger than $\Omega \cap D$. □

To use this theorem, we need to find invariant sets D. In some cases, the assumptions might hold globally in the sense that $D = \mathbb{R}^n_+$. However, in the general case, constructing a set D might be hard. Typically, one would look for sets of the shape $D = D_1$ or $D = D_2$, where

$$D_1 := \{x \in \mathbb{R}^n_+ | M_j^* - \underline{c}_j \leq M_j(x) \leq M_j^* + \overline{c}_j,\ j = 1, \ldots, m\}$$

that is, a "Lyapunov level set"-type region, and

$$\begin{aligned} D_2 := \{x \in \mathbb{R}^n_+ \mid \underline{\chi}_j^i \leq \chi_j^i \leq \overline{\chi}_j^i,\ i = 1, \ldots, r_j \text{ and} \\ M_j^* - \underline{c}_j \leq M_j(x) \leq M_j^* + \overline{c}_j,\ j = 1, \ldots, m\}. \end{aligned}$$

For further details and examples, we refer to [10].

Although this theorem merely shows convergence to the set Ω, it is possible to prove that Ω is asymptotically (set) stable [9]. In many applications, stability of equilibria is arguably more interesting. It is thus interesting to note that the controller (2) often (but not always, as the counterexample in [10] reveals) leads to a stable equilibrium. A sufficient condition for an asymptotically stable equilibrium can be found from the theory of semidefinite Lyapunov functions. Here, we state the following theorem which can be proved in a similar way as Theorem 5 in [4]:

Theorem 2 *Let the conditions of Theorem 1 hold. If the closed loop (1) has a single equilibrium in the interior of $\Omega \cap D$ that is asymptotically stable with respect to initial conditions in $\Omega \cap D$ and attractive for all initial conditions in $\Omega \cap D$, the equilibrium is asymptotically stable for the closed loop with a region attraction (of at least) D.*

We conclude this section with a brief remark about robustness: The proposed feedback scheme is independent of the interconnection structure and hence robust [3] to model uncertainties in $\Phi(x)$ (as long as Assumption A1 holds). This is the most important robustness property. As mentioned in [2], the interconnection terms are in practical examples often the terms that are hardest

[3] Robust in the sense that convergence to Ω still holds. Note that changes in $\Phi(x)$ will typically move the equilibria in Ω.

to model. However, the unconstrained controller also has some (weaker) robustness properties with respect to bounded uncertainties in $\Psi(x)$ and $B(x)$. For further details on this, we refer to [10].

4 Simple example: Tanks in series

This example illustrates the versatility of the phase concept, in addition to shedding light on how to compute a region of attraction and decide stability of equilibria. Further simulation studies, including a benchmark Van der Vusse reactor, can be found in [10].

Consider a system with three tanks in series,

$$\begin{aligned}\dot{x}_1 &= u_1 - \alpha_1\sqrt{x_1}\\ \dot{x}_2 &= \alpha_1\sqrt{x_1} - \alpha_2\sqrt{x_2}\\ \dot{x}_3 &= \alpha_2\sqrt{x_2} - \alpha_3\sqrt{x_3}u_2,\end{aligned}$$

where the states are the level (or mass, or pressure) in each tank. The inflow to the first tank and the outflow of the third thank can be controlled, and are bounded, $0 \leq u_i \leq \bar{u}_i$. The system is obviously positive.

According to the model structure in Section 2, there are several different control structures that can be chosen, depending on how we divide the state into phases, and which inputs we choose to control.

i) <u>One phase, inflow controlled:</u> Choosing u_1 as control, setting $u_2 = u_2^* > 0$ constant, and total mass $M(x) = x_1 + x_2 + x_3$ gives

$$\tilde{u}_1 = \alpha_3\sqrt{x_3}u_2^* + \lambda_1(M^* - M(x)).$$

The (single) phase is inflow controlled. Assumption A3.a.1 obviously holds globally, and Assumption A3.a.2 (which translates to $\alpha_3\sqrt{x_3}u_2^* < \bar{u}_1$ for $M(x) < M^*$) holds on $\mathbb{R}_+^3$ if M^* is chosen such that $\alpha_3\sqrt{M^*}u_2^* < \bar{u}_1$. Then, by Theorem 1 with $D = \mathbb{R}_+^3$, the state converges to $\Omega = \{x \mid M(x) = M^*\}$ from any (positive) initial condition.

ii) <u>One phase, outflow controlled:</u> Choosing u_2 as control, setting $u_1 = u_1^* > 0$ constant, and total mass $M(x) = x_1 + x_2 + x_3$ gives

$$\tilde{u}_2 = \frac{-1}{\alpha_3\sqrt{x_3}}\left[-u_1^* + \lambda_2(M^* - M(x))\right].$$

The single phase is outflow controlled. Assumption A3.b.1 holds globally, but Assumption A3.b.2 ($u_1^* < \alpha_3\sqrt{x_3}\bar{u}_2$ for $M(x) > M^*$) does not hold globally for any combination of M^* and $\bar{u}_2$ (consider e.g. an initial condition with $x_3(0) = 0$ and $M(x(0)) > M^*$).

Let $D_2 = \{x \in \mathbb{R}^3_+ \mid x_1 \geq a, x_2 \geq b, x_3 \geq c\}$ with a, b and c positive constants satisfying $a < \left(\frac{u_1^*}{\alpha_1}\right)^2$, $b < \left(\frac{\alpha_1}{\alpha_2}\right)^2 a$ and $c < \left(\frac{\alpha_2}{\alpha_3}\right)^2 b$. Assumption A3.b.2 hold on D_2 if $\bar{u}_2$ satisfies $u_1^* < \alpha_3\sqrt{c}\bar{u}_2$. Since D_2 is invariant, convergence to Ω (if the intersection between Ω and D_2 is nonempty) from initial conditions in D_2 follows from Theorem 1.

iii) Two phases, case 1: Phase 1 consist of x_1 and x_2 ($M_1(x) = x_1+x_2$), phase 2 of x_3 ($M_2(x) = x_3$):

$$
\begin{aligned}
\tilde{u}_1 &= \alpha_2\sqrt{x_2} + \lambda_1(M_1^* - M_1(x)),\\
\tilde{u}_2 &= \frac{-1}{\alpha_3\sqrt{x_3}}\left[-\alpha_2\sqrt{x_2} + \lambda_2(M_2^* - M_2(x))\right].
\end{aligned}
$$

Phase 1 is inflow controlled. Assumption A3.a.1 holds, and Assumption A3.a.2 ($\alpha_2\sqrt{x_2} < \bar{u}_1$ for $M_1(x) < M_1^*$) holds globally if $\alpha_2\sqrt{M_1^*} < \bar{u}_1$. Phase 2 is outflow controlled. Assumption A3.b.1 holds globally, but Assumption A3.b.2 ($\alpha_2\sqrt{x_2} < \alpha_3\sqrt{x_3}\bar{u}_2$ for $M_2(x) > M_2^*$) does not hold globally. However, since $\alpha_2\sqrt{x_2} < \alpha_3\sqrt{x_3}\bar{u}_2$ for $M_2(x) > M_2^*$ holds when $x_2 < \left(\frac{\alpha_3}{\alpha_2}\right)^2 M_2^*\bar{u}_2$, we can have $D_1 = \{x \in \mathbb{R}^3_+ \mid 0 \leq x_1 + x_2 \leq \left(\frac{\alpha_3}{\alpha_2}\right)^2 M_2^*\bar{u}_2\}$ (obviously, with $M_1^* < \left(\frac{\alpha_3}{\alpha_2}\right)^2 M_2^*\bar{u}_2$) and convergence to Ω is guaranteed from initial conditions in D_1 from Theorem 1.

iv) Two phases, case 2: Phase 1 consist of x_1 ($M_1(x) = x_1$), phase 2 of x_2 and x_3 ($M_2(x) = x_2 + x_3$):

$$
\begin{aligned}
\tilde{u}_1 &= \alpha_1\sqrt{x_1} + \lambda_1(M_1^* - M_1(x)),\\
\tilde{u}_2 &= \frac{-1}{\alpha_3\sqrt{x_3}}\left[-\alpha_1\sqrt{x_1} + \lambda_2(M_2^* - M_2(x))\right].
\end{aligned}
$$

Phase 1 is inflow controlled. Assumption A3.a.1 holds globally, and Assumption A3.a.2 ($\alpha_1\sqrt{x_1} < \bar{u}_1$ for $M_1(x) < M_1^*$) holds if $\alpha_1\sqrt{M_1^*} < \bar{u}_1$. Phase 2 is outflow controlled. Assumption A3.b.1 holds globally, but Assumption A3.b.2 ($\alpha_1\sqrt{x_1} < \alpha_3\sqrt{x_3}\bar{u}_2$ for $M_2(x) > M_2^*$) does not hold globally.

Suppose we specify that initial conditions for x_1 should satisfy $\underline{a} \leq x_1(= M_1(x)) \leq \overline{a}$. Similarly as in ii), there exists b and c such that $D_2 = \{\underline{a} \leq x_1(= M_1(x)) \leq \overline{a}, x_2 \geq b, x_2 \geq c\}$ is (closed loop) invariant. Suppose that $\alpha_1\sqrt{\overline{a}} < \alpha_3\sqrt{c}\bar{u}_2$ holds, then Theorem 1 guarantees convergence to Ω from initial conditions in D_2.

Simulations indicate that the regions of attractions given in iii) and iv) are rather conservative. The reasons for this is that Theorem 1 requires the time derivative of $V_j(x)$ to be negative at all times. In both cases, if there is a large amount of mass in the first phase compared to the second phase, due to the saturation of the outflow, it is impossible to avoid the situation where the mass in the second phase increases. However, since the inflow to the first phase is also restricted, after a while the mass is distributed such that the masses in both phases decrease.

Let us analyze the dynamics in Ω in i) above (when the input $u_1 = \alpha_3\sqrt{x_3}u_2^*$). The dynamics in Ω can be parameterized by x_1 and x_2, with $x_3 = M^* - x_1 - x_2$:

$$\begin{aligned}\dot{x}_1 &= \alpha_3\sqrt{M^* - x_1 - x_2}u_2^* - \alpha_1\sqrt{x_1} =: f_1(x)\\ \dot{x}_2 &= \alpha_1\sqrt{x_1} - \alpha_2\sqrt{x_2} =: f_2(x).\end{aligned}$$

Note that there is a unique equilibrium in Ω, which by linearization is (asymptotically) stable. Furthermore, we see that

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -\frac{\alpha_3 u_2^*}{2\sqrt{M^* - x_1 - x_2}} - \frac{\alpha_1}{2\sqrt{x_1}} - \frac{\alpha_2}{2\sqrt{x_2}}$$

has the same sign for all x_1, x_2 such that $M^* - x_1 - x_2 > 0$ (and thus, in the interior of Ω). This allows us to use Bendixon's Criterion (see e.g. [13]) to conclude that no periodic orbits can exist in Ω. Moreover, the generalized Poincaré-Bendixson Theorem tells us that in the two-dimensional case, the only possibilities for trajectories confined to a compact set, are convergence to an equilibrium or a periodic orbit (or a graphic). This means that all trajectories in Ω converge to the (single) asymptotically stable equilibrium in Ω, and we may conclude from Theorem 2 that the equilibrium in Ω is globally (with respect to the positive orthant) asymptotically stable (for the total system).

A very similar analysis for case ii) above shows that D_1 is then a region of attraction for the unique equilibrium in Ω. In case iii) (and case iv)), the dynamics in Ω can be parameterized by $\dot{x}_2 = \alpha_1\sqrt{M_1^* - x_2} - \alpha_2\sqrt{x_2}$ ($\dot{x}_2 = \alpha_1\sqrt{M_1^*} - \alpha_2\sqrt{x_2}$) with unique equilibrium $x_2 = \alpha_1^2 M_1^*/(\alpha_1^2 + \alpha_2^2)$ ($x_2 = M_1^*\alpha_1^2/\alpha_2^2$). In both cases, $\dot{x}_2$ is strictly decreasing in x_2, and asymptotic stability of the equilibrium in Ω is immediate.

5 Stabilization of flow in gas-lifted oil wells

This section presents an application to an important instability problem in the petroleum industry. First, the problem and a simple mass-balance model with two manipulated variables, suitable for control design, is presented. A two input version of the controller is then applied to a stylistic (but realistic) well, analyzed for the simple model and assessed with simulations using the rigouros (de facto industry standard) multiphase flow simulator OLGA 2000. Thereafter, a one-input version (the second input is kept constant) is considered for a laboratory scale well.

5.1 *Gas-lifted oil wells*

The use of hydrocarbons is essential in modern every-day life. In nature, hydrocarbons are typically found in petroleum-bearing geological formations (*reservoirs*) situated under the earth's crust, and hydrocarbons from these reservoirs are produced by means of an *oil well*. An oil well is made by drilling a hole (wellbore) into the ground. A metal pipe (casing) is placed in the wellbore to secure the well, before "downhole well completion" is performed by running the production pipe (tubing), packing and possibly valves and sensors into the well and perforate the casing to make the reservoir fluid flow into the well. Detailed information on wells and well completion can e.g. be found in [8], see also Figure 1.

If the reservoir pressure is high enough to overcome the back pressure from the flowing fluid column in the well and the surface (topside) facilites, the reservoir fluid can flow to the surface. In some cases, the reservoir pressure is not high enough to make the fluid flow freely, at least not at the desired rate. A remedy is then to inject gas close to the bottom of the well, which will mix with the reservoir fluid, see Figure 1. The gas is transported from the topside through the gas-lift choke into the annulus (the space between the casing and the tubing), and enters the tubing through the injection valve close to the bottom of the well. The gas will help to "lift" the oil out of the tubing, through the production choke into the topside process equipment (separator). This is the type of oil well, called gas-lifted well, we will consider herein. A problem with these type of wells, is that they can become (open loop) unstable, characterized by highly oscillatory well flow, known as *casing heading*. The flow regime of the well (tubing) in this case is denoted *slug flow*. The two main factors that induce casing heading, is high compressibility of gas in the annulus, and gravity dominated pressure drop in the two-phase flow in the tubing.

The oil production for a typical oscillating well can be seen in Figure 2. This slug flow is undesirable since it creates operational problems for downstream processing equipment. Further, stabilizing the slug flow in the well leads to increased production, as illustrated in Figure 3. The casing heading problem is industrially important, as a considerable amount of such wells exhibit slug flow. This (or similar) control problems are considered in e.g. [12,5].

For simplicity, we will assume that the reservoir contains only oil, which is a good approximation if the fraction of gas and water is low. We assume realistic boundary conditions, that is, constant separator pressure (downstream the production choke), constant gas injection pressure (upstream the gas injection choke) and constant reservoir pressure (far from the well). The (vertical) well is 2km deep and the high fidelity model is modelled in OLGA 2000 dividing both the tubing and the annulus into 25 volumes.

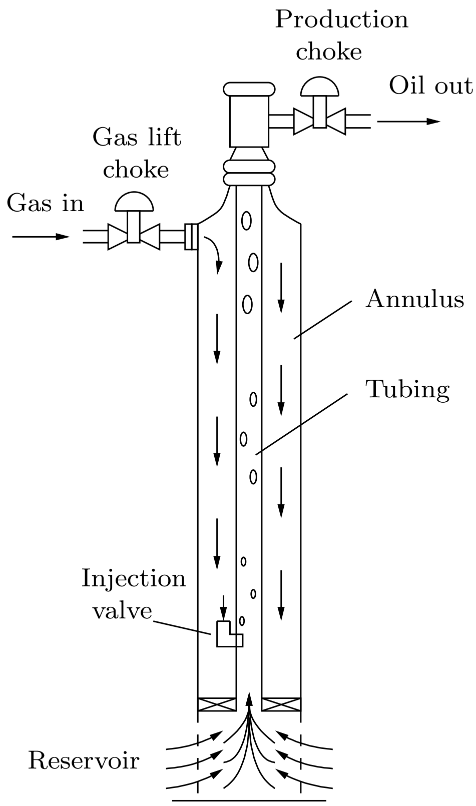


Fig. 1. A gas-lifted oil well.

5.2 A simple model of a gas-lifted oil well

As discussed above, the mechanisms that make the well produce in slugs, are related to the mass of gas in the annulus (compressibility) and the mass of fluid in the tubing (gravity). Consequently, it is reasonable to believe that an ODE based on mass balances will give a good description of the dynamic behavior of the well,

$$
\begin{aligned}
\dot{x}_1 &= -w_{iv}(x) + w_{gc}(x, u_1) && \text{mass of gas, annulus} \\
\dot{x}_2 &= w_{iv}(x) - w_{pg}(x, u_2) && \text{mass of gas, tubing} \\
\dot{x}_3 &= w_r(x) - w_{po}(x, u_2) && \text{mass of oil, tubing}
\end{aligned}
$$

where w_{gc} is the flow of gas through the gas injection choke, w_{iv} is the flow of gas through the injection valve, w_{pg} and w_{po} are the flow of gas and oil through the production choke and w_r is the inflow of oil from the reservoir. The challenge in making such a model, is to find the relation between the system masses (x) and the pressures in the system that determines the flows (w) based on valve-type equations. To keep the presentation short, we do not go into this, but refer to [10], and note that the three state model gives a reasonable approximation to the OLGA model as shown in Figure 2.

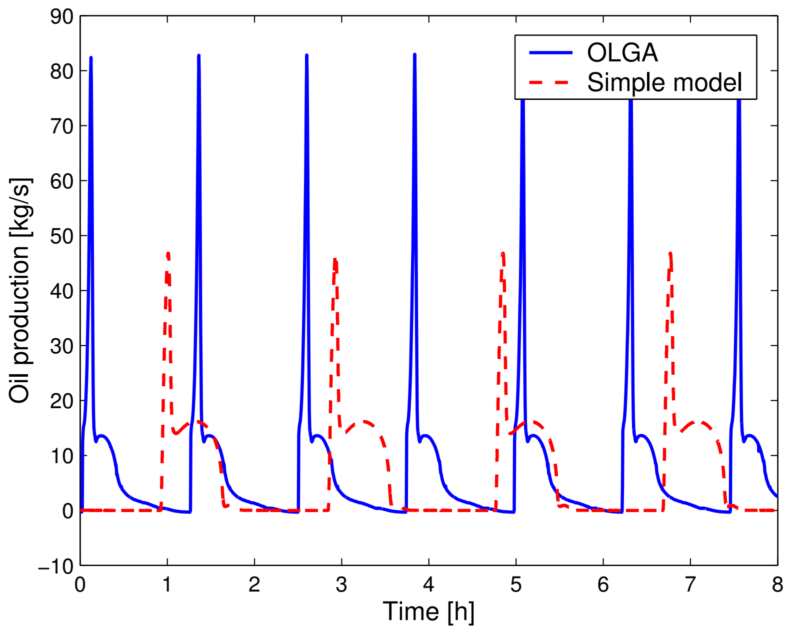


Fig. 2. Comparison of open loop (gas-lift choke is 50% open, production choke is 80% open) behavior between simple model and the rigouros multiphase flow simulator OLGA2000 [3,14].

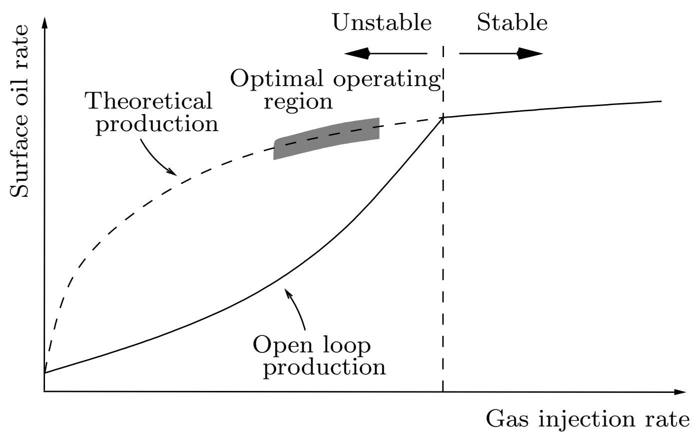


Fig. 3. Conceptual figure showing oil production as a function of gas injection rate. The dotted line is based on steady state calculations, while the solid line is based on dynamic simulations.

5.3 State feedback control

The system written as above can fulfill (a slightly modified) Assumption A2. However, as the expressions for the flows are rather inaccurate (especially for the multiphase flow through the production choke) we will assume that the flow of gas through the gas-lift choke and the flow of oil through the production choke are measured, and that fast control loops control these measured variables. The setpoints for these loops will be the new manipulated variables.

This will, in addition to being a more sensible “engineering approach”, simplify the equations. It also allows us to include rate saturations on the opening of the chokes in the simulations.

The dynamic model with the manipulated flows as inputs, is

$$\begin{aligned}\dot{x}_1 &= -w_{iv}(x) + v_1\\ \dot{x}_2 &= w_{iv}(x) - w_{pg}(x, u_2(x))\\ \dot{x}_3 &= w_r(x) - v_2\end{aligned}$$

We choose as phases the sum of gas in the tubing and annulus ($x_1 + x_2$, phase 1 being inflow controlled) and the oil in the tubing (x_3, phase 2 being outflow controlled). The upper saturations on both v_1 and v_2 (the maximum flows through the gas-lift choke and the production choke) depend on the state (through the pressures). Noting that the maximum flows are always obtained when the chokes are maximally open, Assumption A3 can be checked for these saturations. Denote the maximum flows as $\bar{v}_1(x)$ and $\bar{v}_2(x)$, which are given by inserting $u_1 = u_2 = 1$ into the expressions for $w_{iv}(x, u_1)$ and $w_{po}(x, u_2)$.

Then, for $j \in \{1, 2\}$, the controller is given by

$$v_j(x) = \begin{cases} 0 & \text{if } \tilde{v}_j(x) < 0\\ \tilde{v}_j(x) & \text{if } 0 \le \tilde{v}_j(x) \le \bar{v}_j(x)\\ \bar{v}_j(x) & \text{if } \tilde{v}_j(x) > \bar{v}_j(x)\end{cases}$$

where

$$\begin{aligned}\tilde{v}_1(x) &= w_{pg}(x, u_2(x)) + \lambda_1(M_g^* - x_1 - x_2)\\ \tilde{v}_2(x) &= w_r(x) - \lambda_2(M_o^* - x_3).\end{aligned}$$

For a detailed analysis of stability, and some notes on performance, we refer to [10]. Here, we briefly note that for the simple mass balance model of the oil well, asymptotic stability of an equilibrium follows from Theorem 1 and 2 for $M_g^* = 4400$ kg and $M_o^* = 4600$ kg, and with the set D chosen as

$$3640 \le x_1 \le 4240, \;\; 510 \le x_2 \le 590, \;\; 4550 \le x_3 \le 4650.$$

Simulations (on the simple model) show that the real region of attraction is considerably larger than the one found above, but not global. For instance, if the system is started in a “no production” state (tubing filled with oil – $x_2 = 0$), the system must be brought to a producing condition before the controller is turned on. This is due to the saturation of the chokes. If the tubing is filled with oil, the casing can be filled with enough gas such that $x_1 + x_2 = M_g^*$, without gas being inserted into the tubing. The “oil controller” tries to decrease the amount of oil, but is unsuccessful since the well cannot produce oil with no gas inserted. Increasing M_g^* (temporarily) might be a solution in this case.

5.4 OLGA simulations

Using the OSI [4] link between OLGA and Matlab, the controller, implemented in Matlab, was used on a well modeled in OLGA. The simulation results are shown in Figure 5 and 4. Note that these are state feedback simulation results, the masses and flows were assumed measured.

In the simulations, the well is operated in open loop the two first hours. In this period, the well is stabilized by using a high opening of the gas-lift choke ($u_1 = 0.7$) and a low opening of the production choke ($u_2 = 0.4$). Then, the controller (with $M_g^* = 3450$ kg and $M_o^* = 9400$ kg) is switched on, and remains on for three hours. We see that the controller stabilizes the well at a higher production, and with a significantly lower use of injection gas (in this case, the production increases approximately 2% while the use of injection gas is reduced with 40%). The controller is switched off after 5 hours, keeping the inputs constant. It is seen that the new operating point is open loop unstable. In Figure 5, we see that the controller does not quite reach the mass setpoints. This is mainly due to the flashing phenomena, meaning that there is mass leaving the oil phase entering the gas phase, which is not accounted for in the simple model (and hence the controller). This can be interpreted as errors in the external flows, which the controller has some robustness towards, as discussed in Section 3. The influence is more pronounced in the gas phase, since the total external flow in the oil phase is larger than in the gas phase. Simulations indicate that larger λ's ($\lambda_1 = \lambda_2 = 0.001\text{s}^{-1}$ was used in the simulations shown) reduce the steady state error. Choosing too high λ's leads to problems with saturations, and also numerical problems in Olga may occur. Another remedy for reducing this offset is including an estimate of the flashing in the equations.

5.5 Lab trial

A slightly different controller structure was tried on a lab setup at TU Delft in The Netherlands [5] . The laboratory setup is shown in Figure 6.

[4] OLGA Server Interface (OSI) toolbox, for use with Matlab, developed by ABB Corporate Research.

[5] The experimental setup is designed and implemented by Shell International Exploration and Production B.V., Rijswijk, and is now located in the Kramers Laboratorium voor Fysische Technologie, Faculty of Applied Sciences, Delft University of Technology.

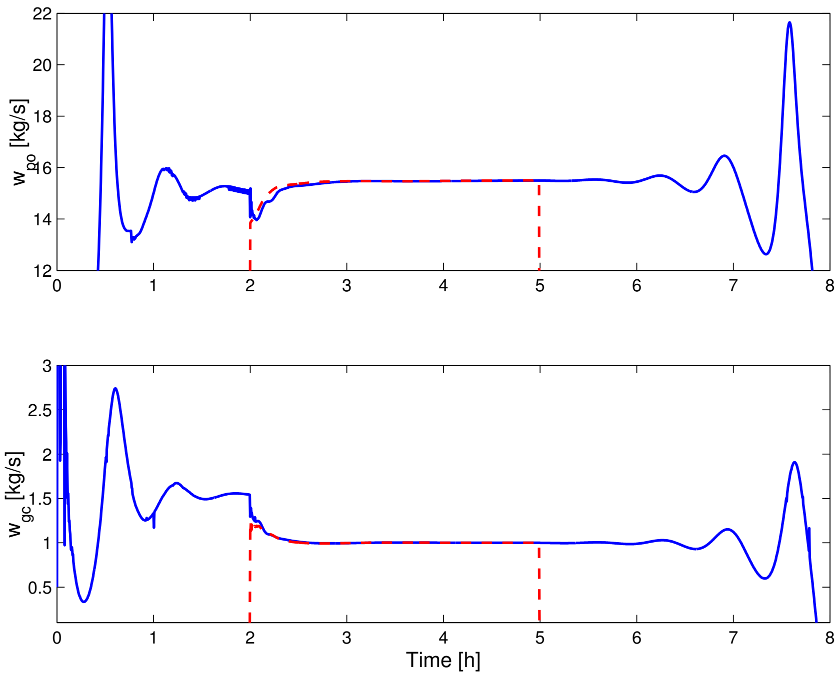


Fig. 4. Desired oil production and gas injection calculated by controller (- -) and the “real” values (-), OLGA simulation

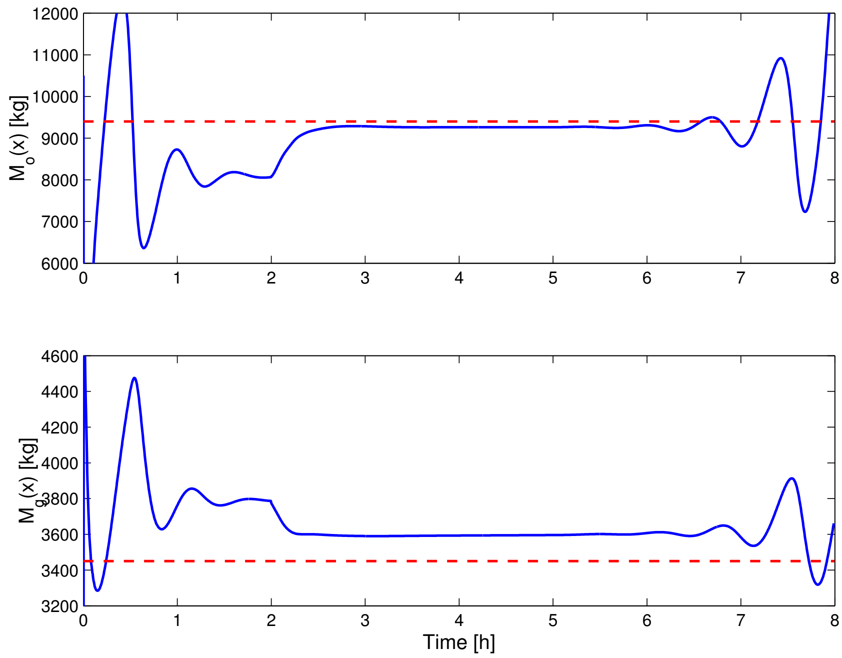
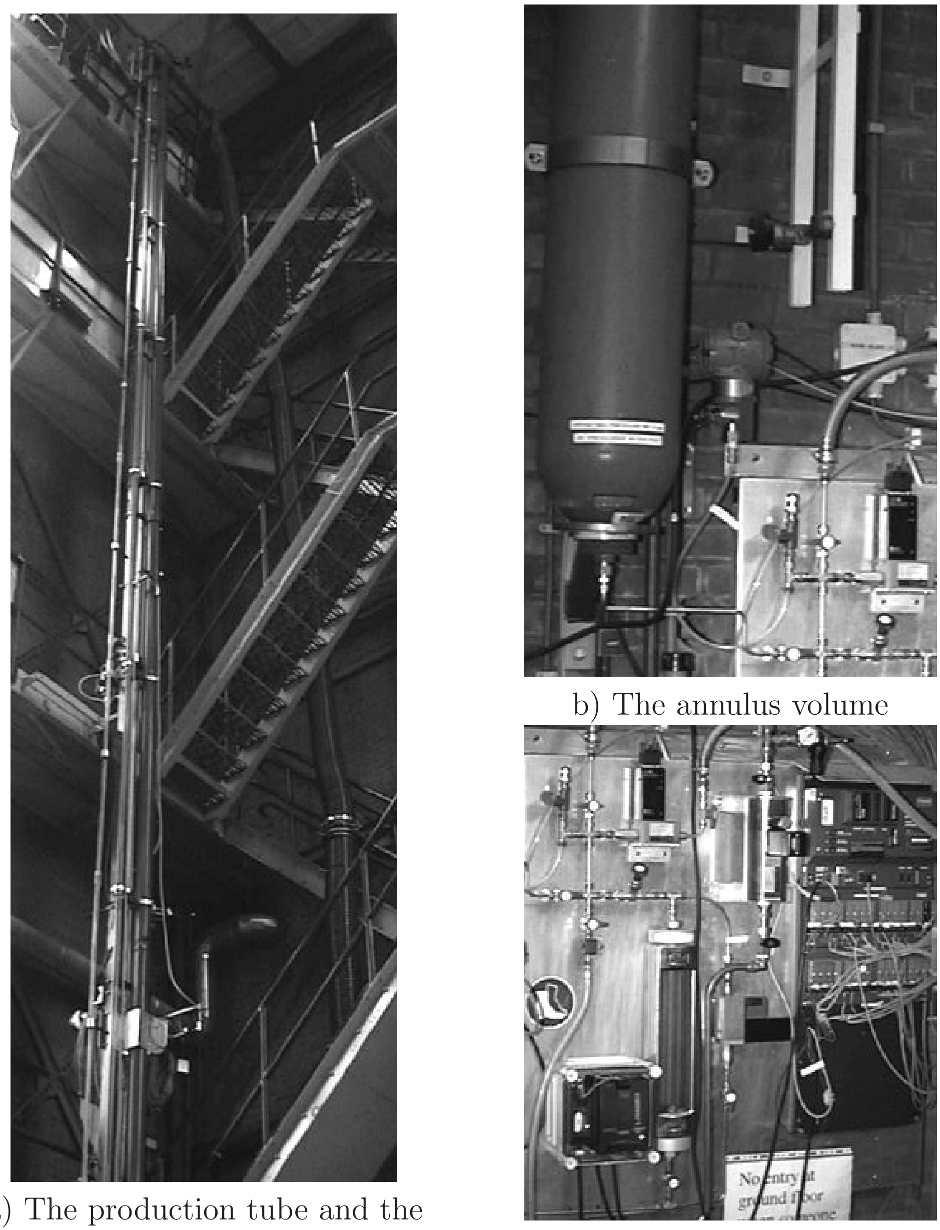


Fig. 5. Mass of phase 1 (gas) and 2 (oil) vs. setpoint, OLGA simulation

The laboratory installation represents a gas-lifted well, using compressed air as lift-gas and water as produced fluid. The production tube is transparent, facilitating visual inspection of the flow phenomena occurring as control is applied. The production tube measures 18m in height and with an inner diameter of 20mm, see Figure 6a. The fluid reservoir is represented by a tube of the same height and an inner diameter of 80mm. The reservoir pressure is given by the static height of the fluid in the reservoir tube. A 30 liter gas bottle represents the annulus, see Figure 6b, with the gas injection point located at the bottom of the production tube. In the experiments run in this study, gas is fed into the annulus from a constant pressure source, giving approximately a rate of 9 L/min (atmospheric conditions). Input and output signals to and from the installation are handled by a microcomputer system, see Figure 6c, to which a laptop computer is interfaced for running the control algorithm and presenting output.



a) The production tube and the reservoir tube.

b) The annulus volume

c) The microcomputer

Fig. 6. The gas-lift laboratory

Having only one control input available (the production choke), we treated the oil in the tubing and gas in annulus and tubing as one phase. This gives us less control freedom than in the simulations in the previous section, but can in many cases be more realistic - for instance there can be situations where the gas-lift choke is not available for control, for example if the amount of available lift gas topside is given by production constraints. An advantage of this structure is that the controller is independent (and hence robust) to mass transfer between the oil and gas phase in the tubing, and that tuning (in terms of total mass setpoint) should be significantly easier. The expected disadvantages are a smaller region of attraction, and that the achievable performance of the well (the oil production) is lower.

The input (flow through production choke) for the system with one phase is then given by (2) and

$$\tilde{v}_2(x) = w_r(x) + w_{gc}(x) - \lambda_2(M_o^* - x_1 - x_2 - x_3).$$

OLGA simulations of this controller (using all available measurements from OLGA) with a PI-controller ensuring the correct mass flow through the production choke, is shown in Figure 7.

The high frequency oscillations seen in Figure 7 are due either to errors in the way OLGA was setup for these simulations, or internal problems in OLGA, possibly due to the small physical size of this laboratory well. Since they are much faster than the interesting dynamics, they should not have an influence on the results and conclusions.

We see that the controller takes the system to a state with higher production. When the controller is turned off, the system starts to oscillate which shows that this operating point is open loop unstable.

Since we in this case have good knowledge of the in- and outflows, there is no steady-state error in the total mass. The fact that we do not know the flashing, does no longer give steady state error in total mass, since the flashing is now an “interconnection flow” (no longer an “external flow”) which the controller is robust against.

There is an interesting phenomena apparent in the OLGA simulations (Figure 7): After the (controlled) mass has converged, the inflow (and the choke opening) continues to move (drift) slowly, before a steady state is reached after around 10 minutes. An explanation for this in terms of the theory, is that there must be a slow dynamic mode in Ω, the set of masses where the total mass is constant. This slow dynamic mode was not observed with the simple model, but it was apparent in the lab trials, as we will see. Even though this mode should not have a direct impact on stability, it makes tuning hard and time consuming, and it obscures the relation between total mass setpoint and well production (production choke opening).

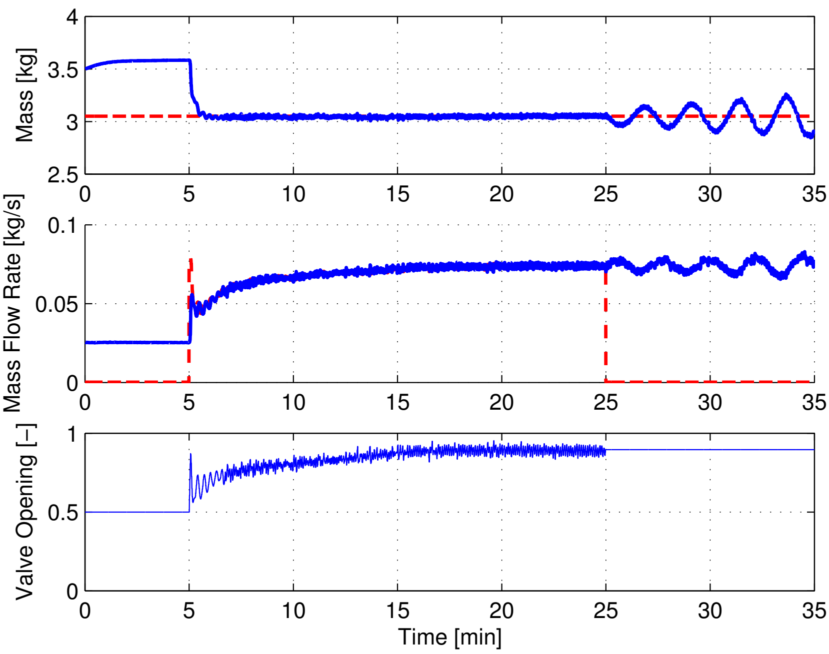


Fig. 7. Total mass vs setpoint, flow through production choke vs setpoint calculated by mass controller, and production choke opening. The controller is turned on after 5 minutes, and turned off after 25 minutes.

The same controller was implemented on the lab. Some limitations in the lab setup posed challenges to the trial. First, there were no available (multiphase) flow measurements of the flow through the production choke. Thus, instead of letting the controller compute the flow through the production choke, we used a simple correlation to compute the desired differential pressure (dp) over the production choke. A (noisy) dp measurement was then used to obtain the desired dp. Second, the phase masses in the lab setup were not measured, necessitating a state observer. The development of a state observer is reported in [6].

The controller proved hard to tune to satisfactory (or even stable) operation, compared with the corresponding simulations on the OLGA simulator model of the lab. The reason for this we believe is partly the bad performance of the inner control loop (due to both the noisy dp measurement and errors in the correlation between the dp measurement and the corresponding multiphase flow). In addition (and probably equally important) comes the slow dynamic mode, as explained above for the OLGA simulations.

In Figure 8 we see the results of a trial, where the well is started rather close to the total mass setpoint. The first transient is due to the convergence of the observer, and then the controller is turned on after two minutes. The controller keeps the mass (and hence the production) stable, until the controller is turned off after 30 minutes. Thereafter, we see that the well starts oscillating. The drift in production choke opening from the controller is turned on to about 25 minutes is clearly seen in the lowermost plot.

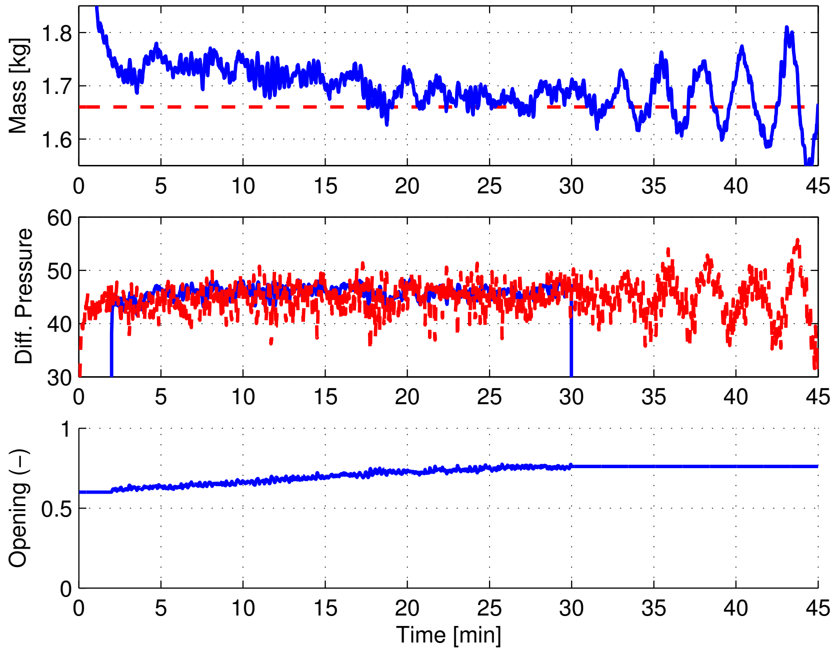


Fig. 8. Estimated mass vs setpoint, differential pressure measurement vs setpoint calculated by mass controller, and production choke opening. The controller is turned on after 2 minutes, and turned off after 30 minutes.

6 Concluding remarks

A controller for a class of positive systems is proposed, leading to closed loop convergence to (or, stability of) a set. The system class could potentially have high applicability, as illustrated by the range of examples studied herein and in [10]. The main restrictive assumption is Assumption A3, ensuring that the "Lyapunov function" used in the proof of the main result is decreasing when the input saturates. The way these assumptions are used in the proof of Theorem 1, along with experience from the examples, indicate that it should be possible to get less conservative conditions, at least for a specific system.

The closed loop system has some robustness properties, most importantly robustness towards unmodeled interconnection terms. These are often the terms that are hardest to model, as for instance in the gas-lift case in Section 5. The closed loop convergence holds independently of the rate-of-convergence parameter λ_j. This means that this parameter (at least nominally) can be used to shape the closed loop performance in terms of the convergence of the mass of each phase, without affecting stability.

The suggested approach leads to stability of a compact set. The approach does not give any guarantees pertaining to the behavior *on* this set, apart from boundedness. However, if the set contains an equilibrium that is asymptotically stable with respect to the set, the equilibrium is also asymptotically stable in the original state space.

The controller was applied to stabilization of gas-lifted wells. Both analysis on the simple model, simulations using the multiphase flow simulator OLGA, and the lab trials confirm that the developed controller was able to stabilize the flow in the gas-lifted well at an open-loop unstable operating point with increased oil production and reduced use of lift gas.

An important feature of the controller is that the developed state feedback controller is independent of the flow through the injection valve, $w_{iv}(x)$, and hence is robust to modeling errors in this flow. This is in contrast to the fact that the system can be open loop stabilized (or destabilized) by the characteristics of this valve. In some cases this valve is designed to always be in a critical flow condition, effectively decoupling the annulus dynamics from the tubing dynamics. Even though this takes care of the instability problem, operational degrees of freedom are lost compared to the approach herein since it implies a constant, given at the design stage, gas injection into the tubing.

Experience from both OLGA simulations and lab trials have shown us that the controller is hard to tune to satisfactory performance - especially for the case where only production choke was used as control. The are several reasons for this - most importantly, perhaps, the difficulty in finding a good relation between mass setpoints and resulting well production. Contributing to this are the slow mode apparent in Ω, and also (and probably related) that for some wells (including the lab setup) the fact that large variations in production gives only rather small variations in mass.

Acknowledgments

The Gas Technology Center NTNU-SINTEF and the NFR project Petronics are acknowledged for financial support. We thank Scandpower Petroleum Technology AS for providing us with the OLGA 2000 simulator. Shell International Exploration and Production B.V, Rijswijk and TU Delft are gratefully acknowledged for letting us use their laboratory facilities.

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