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Output feedback stabilization of constrained systems with nonlinear predictive control

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SUMMARY

We present an output feedback stabilization scheme for uniformly completely observable nonlinear MIMO systems combining nonlinear model predictive control (NMPC) and high-gain observers. The control signal is recalculated at discrete sampling instants by an NMPC controller using a system model for the predictions. The state information necessary for the prediction is provided by a continuous time high-gain observer. The resulting 'optimal' control signal is open-loop implemented until the next sampling instant. With the proposed scheme semi-global practical stability is achieved. That is, for initial conditions in any compact set contained in the region of attraction of the NMPC state feedback controller, the system states will enter any small set containing the origin, if the high-gain observers is sufficiently fast and the sampling time is small enough. In principle the proposed approach can be used for a variety of state feedback NMPC schemes. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: output feedback; nonlinear predictive control; NMPC; semi-global practical stability

1. INTRODUCTION

Model predictive control for systems described by nonlinear ODEs or difference equations has received considerable attention over the past years. Several schemes that guarantee stability in the state feedback case exist by now, see for example References [1–3] for recent reviews. Much fewer results are available in the case when not all states are directly measured. To overcome this problem, often a state observer together with a stabilizing state feedback NMPC controller is used. However, due to the lack of a general nonlinear separation principle, the stability of the resulting closed loop must be examined.

Observer-based output feedback NMPC has been considered by a number of researchers. In Reference [4] an optimization based moving horizon observer combined with the NMPC scheme proposed in Reference [5] is shown to lead to (semi-global) closed-loop stability. The approach in Reference [6] derives local uniform asymptotic stability of contractive NMPC in combination with a 'sampled' state estimator. In Reference [7, 8], see also Reference [9], asymptotic stability results for observer based discrete-time NMPC for 'weakly detectable' systems are given.

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For the approaches [7–9] it is in principle possible to estimate a (local) region of attraction of the resulting output feedback controller from Lipschitz constants of the system, controller and observer. However, it is in general not clear which parameters in the controller and observer should be changed to increase the region of attraction, or how to recover (in the limit) the region of attraction of the state feedback controller. This problem has been part-wise overcome in References [10] and [11]. In Reference [10] semi-global practical stability of instantaneous NMPC using high-gain observers has been established. These results are expanded in Reference [11] to sampled-data NMPC, and further expanded herein to MIMO uniformly completely observable systems.

With respect to the general output feedback stabilization problem for nonlinear systems, significant progress has been achieved recently. Based on Reference [12], different versions of the so-called nonlinear separation principle for a wide class of systems have been established [13–15]. All these approaches use a high-gain observer for state recovery. While the initial results cover control laws that are locally Lipschitz in the state, recently output feedback stabilization of systems that are not uniformly completely observable and that cannot be stabilized by continuous feedback have been achieved References [16, 17].

The results derived in this work are inspired by the 'general' nonlinear separation principles results as presented in References [13, 15], i.e. we propose to use continuous time high-gain observers in combination with NMPC. The main difference to the 'general' separation results [13, 15] is that we want to employ an NMPC controller that only recalculates the optimal input signal at sampling instants, as is customary in the NMPC literature. Between the sampling instants, the input signal is applied open-loop to the system. For uniformly completely observable nonlinear MIMO systems we achieve semi-global practical stability: For any desired subset of the region of attraction of the state feedback NMPC and any small region containing the origin, there exists a sampling time and an observer gain such that in the output feedback case, all states starting in the desired subset will converge in finite time to the small region containing the origin.

The results obtained are not focused on one specific NMPC approach. Instead, they are based on a series of assumptions that in principle can be satisfied by several NMPC schemes, such as quasi-infinite horizon NMPC [18], zero terminal constraint NMPC [19] and NMPC schemes utilizing control Lyapunov functions to obtain stability [20, 21].

The paper is structured as follows: In Section 2 we briefly state the considered system class and the assumed observability assumption. Section 3 introduces the proposed output feedback NMPC scheme, and the stability properties are established in Section 4. We conclude the paper in Section 5.

2. SYSTEM CLASS AND OBSERVABILITY ASSUMPTIONS

We consider nonlinear continuous time MIMO systems of the form:

$$\dot{x} = f(x,u), \quad y = h(x,u) \tag{1}$$

where the system state $x \in \mathscr{X} \subset \mathbb{R}^n$ is constrained to the set $\mathscr{X}$, and the measured output is $y \in \mathbb{R}^p$. The control input u is constrained to $u \in \mathscr{U} \subset \mathbb{R}^m$. We assume that the functions $f : \mathbb{R}^n \times \mathscr{U} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathscr{U} \to \mathbb{R}^p$ are smooth, and that $f(0,0) = 0$ and $h(0,0) = 0$. The control objective is to derive an output feedback control scheme that practically stabilizes the system while satisfying the constraints on the states and inputs. With respect to $\mathcal{X}$ and $\mathcal{U}$ we assume that

Assumption 1
$\mathcal{U} \subset \mathbb{R}^m$ is compact, $\mathcal{X} \subseteq \mathbb{R}^n$ is connected and $(0,0) \in \mathcal{X} \times \mathcal{U}$.

NMPC requires full state information for prediction. Since not all states are available via output measurements, we utilize a high-gain observer to recover the states. The assumed observability properties of the system (1) are characterized in terms of the observability map $\mathcal{H}$, which is defined via the successive differentiation of the output y:

$$\begin{aligned} Y &:= [y_1, \ldots, y_1^{(r_1)}, y_2, \ldots, y_p^{(r_p)}]^{\mathrm{T}} \\ &= [h_1(x,u), \ldots, \psi_{1,r_1}(x,u,\dot{u},\ldots,u^{(r_1)}), h_2(x,u), \ldots, \psi_{p,r_p}(x,u,\dot{u},\ldots,u^{(r_p)})]^{\mathrm{T}} \\ &=: \mathcal{H}(x,U) \end{aligned}$$

Here $\sum_{i=1}^{p}(r_i+1) = n$, and $U = [u_1, \dot{u}_1, \ldots, u_1^{(m_1)}, u_2, \dot{u}_2, \ldots, u_m, \dot{u}_m, \ldots, u_m^{(m_m)}]^{\mathrm{T}} \in \mathbb{R}^{m_U}$ where the m_i denote the number of really necessary derivatives of the input i with $m_U := \sum_{i=1}^{m}(m_i+1)$. The $\psi_{i,j}$'s are defined via the successive differentiation of y

$$\psi_{i,0}(x,u) = h_i(x,u), \quad i = 1,\ldots,p \tag{2a}$$

$$\psi_{i,j}(x,u,\ldots,u^{(j)}) = \frac{\partial \psi_{i,j-1}}{\partial x} f(x,u) + \sum_{k=1}^{j} \frac{\partial \psi_{i,j-1}}{\partial u^{(k-1)}} u^{(k)}, \quad i = 1,\ldots,p, j = 1,\ldots,r_p \tag{2b}$$

Note that in general, not all derivatives of the u_i up to order $\max\{r_1,\ldots,r_p\}$ appear in $\psi_{i,j}$. Given these definitions we can state the uniform complete observability property assumed, compare [15, 22].

Assumption 2 (Uniform complete observability)
The system (1) is uniformly completely observable in the sense that there exists a set of indices $\{r_1,\ldots,r_p\}$ such that the mapping defined by $Y = \mathcal{H}(x,U)$ is smooth with respect to x and its inverse from Y to x is smooth and onto for any U.

The inverse of $\mathcal{H}$ with respect to x is denoted by $\mathcal{H}^{-1}(Y,U)$, i.e. $x = \mathcal{H}^{-1}(Y,U)$.

No explicit stabilizability assumption is required to hold. The stabilizability is implicitly ensured by the assumption on the NMPC controller to have a non-trivial region of attraction.

3. OUTPUT FEEDBACK NMPC CONTROLLER

The output feedback control scheme consists of a state feedback NMPC controller and a high-gain observer for state recovery. While the optimal inputs are only recalculated at the sampling instants and are applied in-between open-loop, the high-gain observer operates continuously.

3.1. NMPC 'Open-loop' state feedback

In the frame of predictive control, the input is defined via the solution of an open-loop optimal control problem that is solved at the sampling instants. In between the sampling instants the optimal input is applied open-loop. For simplicity we denote the sampling instants by t_i, with $t_i - t_{i-1} = \delta$, δ being the sampling time. For a given t, t_i should be taken as the nearest sampling instant $t_i < t$. The open-loop optimal control problem solved in the considered NMPC set-up at any t_i is given by

$$\min_{\bar{u}(\cdot)} J(\bar{u}(\cdot); x(t_i)) \text{ subject to : } \quad \dot{\bar{x}}(\tau) = f(\bar{x}(\tau), \bar{u}(\tau)), \quad \bar{x}(\tau = 0) = x(t_i) \tag{3a}$$

$$\bar{u}(\tau) \in \mathcal{U}, \quad \bar{x}(\tau) \in \mathcal{X} \quad \tau \in [0, T_p] \tag{3b}$$

$$\bar{x}(T_p) \in \mathcal{E} \tag{3c}$$

The cost functional J is defined over the control horizon T_p by $J(\bar{u}(\cdot); x(t_i)) := \int_0^{T_p} F \times (\bar{x}(\tau), \bar{u}(\tau))\, \mathrm{d}\tau + E(\bar{x}(T_p))$. The bar denotes internal controller variables, $\bar{x}(\cdot)$ is the solution of (3a) driven by the input $\bar{u}(\cdot) : [0, T_p] \to \mathcal{U}$ with the initial condition $x(t_i)$. The solution of the optimal control problem is denoted by $\bar{u}^{\star}(\cdot; x(t_i))$. It is applied open-loop to the system until the next sampling time t_{i+1},

$$u(t; x(t_i)) = \bar{u}^{\star}(t - t_i; x(t_i)), \quad t \in [t_i, t_i + \delta) \tag{4}$$

The control $u(t; x(t_i))$ is a feedback, since it is recalculated at each sampling instant using new state measurements. Typically, the role of the end penalty E and the terminal region constraint $\mathcal{E}$ is to enforce stability of the state feedback closed loop. We do not go into any details about the different existing state feedback NMPC schemes that guarantee stability, see for example References [1, 3]. Instead we state the set of assumptions we require to achieve semi-global practical stability in the output feedback case.

Assumption 3
There exists a simply connected region $\mathcal{R} \subseteq \mathcal{X} \subseteq \mathbb{R}^n$ (region of attraction of the state feedback NMPC) with $0 \in \mathcal{R}$ such that:

1. *Stage cost is lower bounded by a $\mathcal{K}$ function*: The stage cost $F : \mathcal{R} \times \mathcal{U} \to \mathbb{R}$ is continuous, satisfies $F(0, 0) = 0$, and is lower bounded by a class $\mathcal{K}$ function* α_F: $\alpha_F(\|x\| + \|u\|) \leqslant F \times (x, u)$ $\forall (x, u) \in \mathcal{R} \times \mathcal{U}$.
2. *Optimal control is uniformly locally Lipschitz in terms of the initial state*: The optimal control $\bar{u}^{\star}(\tau; x)$ is piecewise continuous and locally Lipschitz in $x \in \mathcal{R}$, uniformly in τ. That is, for a given compact set $\Omega \subseteq \mathcal{R}$: $\|\bar{u}^{\star}(\tau; x_1) - \bar{u}^{\star}(\tau; x_2)\| \leqslant L_u \|x_1 - x_2\|$ $\forall \tau \in [0, T_p), x_1, x_2 \in \Omega$, where L_u denotes the Lipschitz constant of $\bar{u}^{\star}(\tau; x)$ in Ω.
3. *Value function is locally Lipschitz*: The value function, which is defined as the optimal value of the cost for every $x \in \mathcal{R}$: $V(x) := J(\bar{u}^{\star}(\cdot; x); x)$ is Lipschitz for all compact subsets of $\mathcal{R}$ and $V(0) = 0$, $V(x) > 0$ for all $x \in \mathcal{R}/\{0\}$.

*A continuous function $\alpha : [0, \infty) \to [0, \infty)$ is a $\mathcal{K}$ function, if it is strictly increasing and $\alpha(0) = 0$.

4. *Decrease of the value function along solution trajectories*: Along solution trajectories starting at a sampling instant t_i at $x(t_i) \in \mathcal{R}$, the value function satisfies

$$V(x(t_i + \tau)) - V(x(t_i)) \leqslant -\int_{t_i}^{t_i+\tau} F(x(s), u(s; x(s_i)))\,\mathrm{d}s, \quad 0 \leqslant \tau$$

Assumptions 3.1 and 3.4 are satisfied by various NMPC schemes, such as quasi-infinite horizon NMPC [18, 23], zero terminal constraint NMPC [19] and NMPC schemes utilizing control Lyapunov functions to achieve stability [20, 21]. Assumption 3.1 is a typical assumption in NMPC, often quadratic stage cost functions F are used. Assumption 3.4 implies that $\mathcal{R}$ is invariant under the state feedback NMPC for all trajectories starting at t_i in $\mathcal{R}$. It also implies convergence of the state to the origin for $t \to \infty$ [18, 24] and allows the use of suboptimal NMPC schemes [3, 25]. The strongest assumptions are Assumptions 3.2 and 3.3. For existing NMPC schemes Assumption 3.2 is often satisfied near the origin. In words, this (quite frequently made) assumption requires that two 'close' initial conditions must lead to 'close' optimal input trajectories. However, for example, it excludes systems that can only be stabilized by discontinuous feedback (as state feedback NMPC can stabilize [26]). Checking Assumptions 3.2 and 3.3 is in general difficult.

To establish the semi-global practical stability result in Section 4 it is necessary that for any compact subset $\mathcal{S} \subset \mathcal{R}$ we can find a compact outer approximation $\Omega_c(S)$ that contains $\mathcal{S}$ and is invariant under the NMPC state feedback.

Assumption 4
For all compact sets $\mathcal{S} \subset \mathcal{R}$ there is at least one compact set $\Omega_c(\mathcal{S}) = \{x \in \mathcal{R} | V(x) \leqslant c\}$ such that $\mathcal{S} \subset \Omega_c(\mathcal{S})$.

In general more than one such set exists, since c can be in the range $\sup_{x\in\mathcal{R}} V(x) > c > \max_{x\in\mathcal{S}} V(x)$. Assuming that such a set $\Omega_c(\mathcal{S})$ exists for all compact subsets $\mathcal{S}$ of $\mathcal{R}$ is strong. If it is not fulfilled, the results are limited to sets $\mathcal{S}$ that are contained in the largest level set $\Omega_c \subset \mathcal{R}$.

3.2. State recovery by high-gain observers

The NMPC state feedback controller requires full state information. In this paper we propose to recover the states from the output (and input) information via a high-gain observer. We briefly outline the basic structure of the high-gain observer used. Furthermore, we present two possibilities to avoid the need for analytic knowledge of the inverse of the observability map $\mathcal{H}^{-1}(Y, U)$, for which an analytic expression is often difficult to obtain. Since explicit knowledge and boundedness of the u derivatives that appear in the observability map is necessary, we also briefly comment on this issue at the end of this section.

Basic high-gain observer structure: As is well known, application of the co-ordinate transformation $\zeta := \mathcal{H}(x, U)$ to the system (1) leads to the system in observability normal form in ζ co-ordinates

$$\dot{\zeta} = A\zeta + B\phi(\zeta, \tilde{U}), \quad y = C\zeta$$

Here A, B, C and ϕ are given by

$$A = \text{blockdiag}[A_1, \ldots, A_p], \quad B = \text{blockdiag}[B_1, \ldots, B_p]$$

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}_{r_i \times r_i} \qquad B_i = \begin{bmatrix} 0 \\ \cdots \\ 0 \\ 1 \end{bmatrix}_{1 \times r_i} \tag{5}$$

$$C = \text{blockdiag}[C_1, C_2, \ldots, C_p], \quad C_i = [1 \quad 0 \quad \cdots \quad 0]_{1 \times r_i}$$

$$\phi(\zeta, \tilde{U})^{\mathrm{T}} = [\psi_{1,r_1+1}(\mathscr{H}^{-1}(\zeta, U), u, \ldots, u^{(r_1+1)}), \ldots, \psi_{p,r_p+1}(\mathscr{H}^{-1}(\zeta, U), u, \ldots, u^{(r_p+1)})]$$

The functions ψ_{i,r_j+1}, $j = 1, \ldots, p$ are defined analogously to (2). The vector $\tilde{U}$ contains, similarly to U in the mapping $\mathscr{H}$, the input and all necessary derivatives. It is necessary to distinguish between $\tilde{U}$ and U, since, as can be seen from (5), $\tilde{U}$ might contain more u derivatives than U. Note that ϕ is locally Lipschitz in all arguments since f, h and $\mathscr{H}$ are smooth. The high-gain observer

$$\dot{\hat{\zeta}} = A\hat{\zeta} + H_\varepsilon(y - C\hat{\zeta}) + B\hat{\phi}(\hat{\zeta}, \tilde{U}) \tag{6}$$

allows recovery of the states ζ from $y(t)$ (and $\tilde{U}$) [13, 22]. The function $\hat{\phi}$ is the 'model' of ϕ used in the observer. The key assumption we need on $\hat{\phi}$ is that

Assumption 5
$\hat{\phi}$ is globally bounded.

Ideally one would like to use $\hat{\phi} = \phi$, if ϕ is bounded and known, since one can expect good observer performance in this case. If ϕ is not globally bounded one can generate at suitable $\hat{\phi}$ by bounding ϕ outside of a region of interest. In the extreme case, i.e. if ϕ is not or only very roughly known, Assumption 5 also allows to choose $\hat{\phi} = 0$.

The observer gain matrix H_ε is given by $H_\varepsilon = \text{blockdiag}[H_{\varepsilon,1}, \ldots, H_{\varepsilon,p}]$, with $H_{\varepsilon,i}^{\mathrm{T}} = [\alpha_1^{(i)}/\varepsilon, \alpha_2^{(i)}/\varepsilon^2, \ldots, \alpha_{r_i}^{(i)}/\varepsilon^{r_i}]$, where ε is the so-called high-gain parameter since $1/\varepsilon$ goes to infinity for $\varepsilon \to 0$. The $\alpha_j^{(i)}$s are design parameters and must be chosen such that the polynomials $s^{r_i} + \alpha_1^{(i)} s^{r_i - 1} + \cdots + \alpha_{r_i-1}^{(i)} s + \alpha_{r_i}^{(i)} = 0$, $i = 1, \ldots, p$ are Hurwitz. The state estimate used in the NMPC controller is obtained at the sampling instants t_i by

$$\hat{x}(t_i) := \mathscr{H}^{-1}(\hat{\zeta}(t_i^-), U(t_i; \hat{x}(t_{i-1}))) \tag{7}$$

Here $U(t_i; \hat{x}(t_{i-1}))$ contains the input and its derivatives obtained by the NMPC controller at time t_{i-1} for the time t_i. The variable t_i^- denotes the left limit of the corresponding trajectory for t_i. It is necessary to distinguish between the left limit t_i^- and the value at t_i since $\mathscr{H}$ depends on u and its derivatives leading to possible discontinuities in the state estimate, as discussed in some more detail in Section 3.3. Note that the high-gain observer allows to recover the full state information. However, the inverse mapping $\mathscr{H}^{-1}$ must be known explicitly. Furthermore the expanded input vector $\tilde{U}$ must always, not only at sampling instants, be known.

Avoiding the necessity to know $\mathscr{H}^{-1}$ analytically: One way to avoid the explicit knowledge of $\mathscr{H}^{-1}$ and $\tilde{U}$ is to set $\hat{\phi} = 0$ in (6). Then the inverse of the observability map $\mathscr{H}$, as well as information on U is only necessary at (just before) the sampling instant, and the equation $\hat{\zeta}(t_i^-) = \mathscr{H}(\hat{x}(t_i), U(t_i; \hat{x}(t_{i-1})))$ can be added to the dynamic optimization problem (3) that is solved in the NMPC controller at time t_i. This does not change the solution of (3), since the value of $\hat{x}$ is, due to the uniform complete observability assumption, uniquely defined. Another possibility to avoid explicit information on $\mathscr{H}^{-1}$ is to rewrite the observer equations in terms of the original co-ordinates as proposed in References [14, 27].

Obtaining the necessary u derivatives: To obtain a state estimate via the high-gain observer the applied input and the derivatives appearing in $\tilde{U}$ must be known. Furthermore, if u derivatives appear they must be bounded. Since the input is determined via an open-loop optimal control problem at the sampling instants, the NMPC set-up can be modified to provide the necessary information and guarantee the well behavedness of u and its derivatives. Different possibilities to achieve this exist: One can (a) augment the system model used in the NMPC state feedback by integrators at the input side, or (b) parameterize the input $u(t)$ in the optimization problem such that the input is sufficiently smooth with bounded derivatives. In the approach (a), adding the integrators leads to a set of new inputs and the NMPC controller must be designed to stabilize the expanded model. Furthermore, to guarantee boundedness of the inputs and its derivatives, constraints on the new inputs must be added. In the following we assume that the NMPC controller is designed such that it guarantees that the input is sufficiently often differentiable and that it provides the full $\tilde{U}$ vector.

Assumption 6

The input given by the NMPC controller is continuous over the first sampling interval, sufficiently often differentiable and bounded, i.e. the NMPC open-loop optimal control problem provides a continuous 'input' vector $U(\cdot; x(t_i))$ with $U(t_i + \tau; x(t_i)) \in \mathscr{U}_{\mathscr{H}}, \tau \in [0, \delta)$ with $\mathscr{U}_{\mathscr{H}} = \mathscr{U} \times \mathscr{U}_{\mathscr{H}d} \subset \mathbb{R}^{p+\tilde{m}_U}$, where $\mathscr{U}_{\mathscr{H}d} \subset \mathbb{R}^{\tilde{m}_U}$ is a compact set and $\tilde{m}_U$ is the number of derivatives in $\tilde{U}$.

This assumption does not exclude a piecewise constant (over the sampling interval) parameterization of the input, as often used in NMPC for the numerical solution of the open-loop optimal control problem (3) [28, 29]. Note that in the special case that the input and its derivatives do not appear in $\mathscr{H}$ no modification in the NMPC controller to achieve continuity of the input (and its derivatives) is necessary.

3.3. Overall output feedback set-up

The overall output feedback control is given by the state feedback NMPC controller and a high-gain observer. The open-loop input is only calculated at the sampling instants using the state estimates of the observer. The observer state $\hat{\zeta}$ is initialized with $\hat{\zeta}_0$ which, transformed to the original co-ordinates, satisfies $\hat{x}_0 \in \mathscr{Q}$. The set $\mathscr{Q} \subset \mathbb{R}^n$ with $0 \in \mathscr{Q}$ is a compact subset of possible observer initial values. The closed-loop system with the observer specified in observability normal form can be described by

$$
\begin{aligned}
\text{system :}\quad & \dot{x}(t) = f(x(t), u(t; \hat{x}(t_i))), \quad x(0) = x_0 \\
& y(t) = h(x(t), u(t; \hat{x}(t_i))) \\
\text{observer :}\quad & \dot{\hat{\zeta}}(t) = A\hat{\zeta}(t) + B\hat{\phi}(\hat{\zeta}(t), \tilde{U}(t; \hat{x}(t_i))) + H_\varepsilon(y(t) - C\hat{\zeta}(t)) \\
& \text{with } \hat{\zeta}(t_i) = \begin{cases} \mathscr{H}(\hat{x}(t_i), U(t_i; \hat{x}(t_i))) & \text{if } \hat{x}(t_i) = \mathscr{H}^{-1}(\hat{\zeta}(t_i^-), U(t_i; \hat{x}(t_{i-1}))) \in \mathscr{Q} \\ \hat{\zeta}_0 & \text{if } \hat{x}(t_i) = \mathscr{H}^{-1}(\hat{\zeta}(t_i^-), U(t_i; \hat{x}(t_{i-1}))) \notin \mathscr{Q} \end{cases} \\
\text{NMPC :}\quad & \text{defined via (3), provides } u(t; \hat{x}(t_i)), U(t; \hat{x}(t_i)), \tilde{U}(t; \hat{x}(t_i)) \\
& \text{using } \hat{x}(t_i) = \mathscr{H}^{-1}(\hat{\zeta}(t_i^-), U(t_i; \hat{x}(t_{i-1}))) \text{ as state estimate}
\end{aligned}
\tag{8}
$$

Remark 3.1
While the observer itself operates continuously, it might be necessary to reinitialize the observer state ζ at the sampling instants, as defined in Equation (8). This is a consequence of the fact that $\mathscr{H}$ in general depends on u and its derivatives, which might be discontinuous at the sampling instants. While at a first glance the reinitialization seems to be unnecessary, it avoids the possibility that the observer 'initial' state $\hat{x}(t_i) = \mathscr{H}^{-1}(\hat{\zeta}(t_i^-), U(t_i; \hat{x}(t_{i-1})))$ at the sampling instant is, due to the possible discontinuity in u, outside of the compact set $\mathscr{Q}$. This is also the reason that one must differentiate between $\hat{\zeta}(t_i^-)$ (the left limit) and $\hat{\zeta}(t_i^-)$. A similar reinitialization is used in Reference [16].

Figure 1 shows the time sequence of the overall output feedback scheme. The arrows in Figure 1 pointing from the trajectories of y to $\hat{\zeta}$ illustrate that the high-gain observer is a continuous time system and continuously updated with the output measurements in between sampling instants. In contrast to the high-gain observer, the NMPC open-loop optimal control problem is solved only at the sampling instants t_i and the input is open-loop implemented in between.

Note that the observer estimate is not bounded to the feasibility region $\mathscr{R}$ of the NMPC controller. Since the open-loop optimal control problem will not have a solution outside $\mathscr{R}$, we define the input in this case to an arbitrary, bounded value.

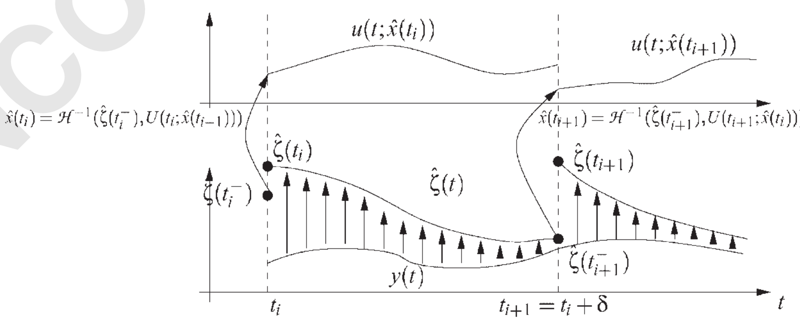

Figure 1. Time sequence of the overall output feedback control scheme.

4. PRACTICAL STABILITY

In this section the main result, semi-global practical stability of the closed-loop system state, is established. In the first step we show that for any compact subset of $\mathcal{R}$ for the system initial states and any compact set of initial conditions of the observer initial states, the closed-loop states stay bounded for small enough ε and δ. Furthermore, at least at the end of each sampling interval the observer error has converged to an arbitrarily small set. In a next step it is established that for a sufficiently small ε the closed loop system state trajectories converge in finite time to a (arbitrarily) small region containing the origin. In principle we use similar arguments as in Reference [13]. However, since we consider a sampled-data feedback employing open-loop input trajectories between the sampling instants, we cannot make use of standard Lyapunov and converse Lyapunov arguments. Instead we utilize the decrease-properties of the NMPC state feedback value function along solution trajectories.

In the following we suppress most of the time the (known) 'input' $U(t;x(t_i))$ and $\tilde{U}(t;x(t_i))$ in the notation, e.g. $\mathcal{H}(x)$ should be read as $\mathcal{H}(x,U)$. Furthermore, it is convenient to work in scaled observer error co-ordinates based on the observability normal form, i.e. we consider the scaled observer error η which is defined as $\eta = [\eta_{11},\dots,\eta_{1r_1},\eta_{pr_p}]$, with $\eta_{ij} = (\zeta_{ij} - \hat{\zeta}_{ij})/\varepsilon^{r_i-j}$. Hence we have that $\hat{\zeta} = \zeta - D_\varepsilon\eta$ with $D_\varepsilon = \mathrm{blockdiag}[D_{\varepsilon,1}, D_{\varepsilon,2},\dots,D_{\varepsilon,p}]$, $D_{\varepsilon,i} = \mathrm{diag}[\varepsilon^{r_i-1},\dots,1]$. Then the closed-loop system in between sampling instants is given by

$$\dot{x}(t) = f(x(t), u(t;\hat{x}(t_i)))$$

$$\varepsilon\dot{\eta}(t) = A_0\eta(t) + \varepsilon Bg(t, x(t), x(t_i), \eta(t), \eta(t_i^-))$$

where the matrix $A_0 = \varepsilon D_\varepsilon^{-1}(A - H_\varepsilon C)D_\varepsilon$ is independent of ε and where the function g is defined as the difference between $\hat{\phi}$ and ϕ,

$$g(t, x(t), x(t_i), \eta(t), \eta(t_i^-)) = \phi(\zeta(t), \tilde{U}(t;\hat{x}(t_i))) - \hat{\phi}(\hat{\zeta}(t), \tilde{U}(t;\hat{x}(t_i)))$$

Here the estimated system state $\hat{x}(t_i)$ and ζ, $\hat{\zeta}$ are given in terms of η, x and u by

$$\hat{x}(t_i) = \mathcal{H}^{-1}(\mathcal{H}(x(t_i)) - D_\varepsilon\eta(t_i^-)), \quad \zeta(t) = \mathcal{H}(x(t)), \quad \hat{\zeta}(t) = \mathcal{H}(x(t)) - D_\varepsilon\eta(t)$$

We often compare, over one sampling interval, the state trajectories of the output feedback closed loop with the trajectories resulting from the application of the state feedback NMPC controller starting at the same initial condition. The state feedback trajectories starting at $x(t_i)$ are denoted, with slight abuse of notation, by $\bar{x}(t;x(t_i))$: $\dot{\bar{x}}(t;x(t_i)) = f(\bar{x}(t;x(t_i)), u(t;x(t_i)))$, $\bar{x}(t_i; x(t_i)) = x(t_i)$, $t \in [t_i, t_i+\delta]$. For simplicity we furthermore assume, without loss of generality, that $0 < \varepsilon \leqslant 1$. This implies that $\|D_\varepsilon\| \leqslant 1$.

4.1. *Preliminaries*

Before we move to the practical stability and boundedness results we establish some properties of the observer and controller. In the following the set $\mathcal{Q} \subset \mathbb{R}^n$ is a fixed compact set for the observer initial state $\hat{x}_0$, whereas $\Gamma_\varepsilon := \{\eta \in \mathbb{R}^n | W(\eta) \leqslant \rho\varepsilon^2\}$ defines a set for the scaled observer error η that depends on ε. The constant ρ appearing in the following lemma is defined as $\rho := 16(\|P_0\|^4/\lambda_{\min}(P_0))k_g^2$ where k_g is an upper bound on g, and $W(\eta)$ is defined by $W(\eta) = \eta^{\mathrm{T}}P_0\eta$, where P_0 is the solution of the Lyapunov equation $P_0A_0 + A_0^{\mathrm{T}}P_0 = -I$. The following lemma is similar to a result obtained in Reference [13], and hence stated without proof.

Lemma 1 (Convergence of the scaled observer error)
Given any time $0 < T$ such that $\tilde{U}$ is continuous over $[0, T]$, two compact sets $\Omega_c \subset \mathscr{R}$ and $\mathscr{Q} \subset \mathbb{R}^n$, and let Assumptions 2 and 5 hold. Furthermore suppose that the system state satisfies $x(\tau) \in \Omega_c$, $0 \leqslant \tau \leqslant T$. Then there exists an $\varepsilon_1^\star$, a constant ρ, and a time $T_{\mathscr{Q}}(\varepsilon) \leqslant T$ such that for any $\hat{x}_0 \in \mathscr{Q}$ and for all $0 < \varepsilon \leqslant \varepsilon_1^\star$ the scaled observer states $\eta(\tau)$ are bounded for $\tau \in [0, T]$ and that $\eta(\tau) \in \Gamma_\varepsilon$, $\tau \in [T_{\mathscr{Q}}(\varepsilon), T]$. Furthermore, $T_{\mathscr{Q}}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Remark 4.1
Note that the size of the set Γ_ε and the time $T_{\mathscr{Q}}(\varepsilon)$ depend on ε. Decreasing ε leads to a shrinking Γ_ε while also shrinking the time $T_{\mathscr{Q}}(\varepsilon)$ needed to reach Γ_ε. Furthermore, note that an upper estimate of the error in the original co-ordinates for $\eta \in \Gamma_\varepsilon$ is given by

$$\|x - \hat{x}\| = \|\mathscr{H}^{-1}(\zeta, U) - \mathscr{H}^{-1}(\hat{\zeta}, U)\| \leqslant k_\Gamma \varepsilon \quad \text{for } \eta \in \Gamma_\varepsilon$$

where k_Γ is a constant that depends on the Lipschitz constant $L_{\mathscr{U}\mathscr{H}}$ of $\mathscr{H}^{-1}$ (and hence on $\mathscr{U}_{\mathscr{H}}$, Ω_c and $\mathscr{Q}$). Thus decreasing ε also decreases the observer error in the original co-ordinates after the time $T_{\mathscr{Q}}(\varepsilon)$. This, together with robustness properties of the state feedback NMPC controller are the key elements for proving the output feedback stability result.

The next lemma establishes a bound on the difference between the trajectories resulting from the NMPC controller with exact state information and the NMPC controller using an incorrect state estimate. From now on, Ω_c will denote level sets of $V(x)$ defined by $\Omega_c := \{x \in \mathscr{R} | V(x) \leqslant c\}$, and the set $\Omega_c(\mathscr{S})$ denotes a level set Ω_c that contains the (assumed compact) set $\mathscr{S} \subset \mathscr{R}$, i.e. $c > \max_{x \in \mathscr{S}} V(x)$.

Lemma 2 (Bounding state and output feedback trajectories)
Let Assumptions 1–4 hold. Given three compact sets $\mathscr{Q} \subset \mathbb{R}^n$, $\mathscr{S} \subset \mathscr{R}$ and $\Omega_c(\mathscr{S}) \subset \mathscr{R}$ with $\mathscr{S} \subset \Omega_c(\mathscr{S})$. Consider the system (1) driven by the NMPC open-loop control law (4) using the correct state x_0 (state feedback) and the state estimate $\hat{x}_0 \in \mathscr{Q}$ (output feedback)

$$\dot{x}(\tau) = f(x(\tau), u(\tau; \hat{x}_0)), \quad \dot{\bar{x}}(\tau) = f(\bar{x}(\tau), u(\tau; x_0)), \quad x(0) = \bar{x}(0) = x_0 \tag{9}$$

Then there exists a time $T_{\mathscr{S}} \leqslant T_p$ such that for all $\hat{x}_0 \in \mathscr{Q}$, $x_0 \in \mathscr{S}$, we have $x(\tau), \bar{x}(\tau) \in \Omega_c(\mathscr{S})$ and

$$\|x(\tau) - \bar{x}(\tau)\| \leqslant \frac{L_{fu} L_u}{L_{fx}} \|x_0 - \hat{x}_0\| (e^{L_{fx}\tau} - 1), \quad \tau \in [0, T_{\mathscr{S}}]$$

Here L_{fx} and L_{fu} are the Lipschitz constants of f in $\Omega_c(\mathscr{S}) \times \mathscr{U}$, and L_u is the 'Lipschitz constant' of u as defined in Assumption 3.2.

Proof
Since $\mathscr{S} \subset \Omega_c(\mathscr{S})$, u piecewise continuous and bounded there always exists a time $T_{\mathscr{S}} \leqslant T_p$ such that $x(\tau), \bar{x}(\tau) \in \Omega_c(\mathscr{S})$ for all $\tau \in [0, T_{\mathscr{S}}]$ and that the solutions are continuous. This follows from the fact that for $x(\cdot)$ in $\Omega_c(\mathscr{S})$ $\|x(\tau) - x_0\| \leqslant \int_0^t \|f(x(s), u(s))\| \, ds \leqslant k_\Omega \tau$, (and the same for $\bar{x}(\tau)$) where k_Ω is a constant depending on the Lipschitz constants of f and the bounds on u. The solutions to (9) for any $\tau \in [0, T_{\mathscr{S}}]$ can be written as: $x(\tau) = x_0 + \int_0^\tau f(x(s), u(s; \hat{x}_0)) \, ds$, $\bar{x}(\tau) = x_0 + \int_0^\tau f(\bar{x}(s), u(s; x_0)) \, ds$. Thus $\|x(\tau) - \bar{x}(\tau)\| \leqslant \int_0^\tau \|f(x(s), u(s; \hat{x}_0)) - f(\bar{x}(s), u(s; x_0))\| \, ds$. Since f is locally Lipschitz in $\mathscr{R}$ (and hence in $\Omega_c(\mathscr{S})$) and $u(\tau; x)$ is uniformly locally

Lipschitz in x we obtain

$$\|x(\tau) - \bar{x}(\tau)\| \leqslant L_{fu} L_u \|x_0 - \hat{x}_0\| \tau + \int_0^\tau L_{fx} \|x(s) - \bar{x}(s)\| \, \mathrm{d}s$$

where L_{fu}, L_{fx} are the Lipschitz constants of f in $\Omega_c(\mathscr{S}) \times \mathscr{U}$, and L_u is the 'Lipschitz constant' of u as defined in Assumption 3.2. Using the Gronwall–Bellman inequality we obtain for all $\tau \in [0, T_{\mathscr{S}}]$ $\|x(\tau) - \bar{x}(\tau)\| \leqslant L_{fu} L_u / L_{fx} \|x_0 - \hat{x}_0\| (e^{L_{fx}\tau} - 1)$, which proves the lemma. □

In proving the main results, we make use of the following fact that gives a lower bound on the first 'piece' of the NMPC state feedback value function:

Fact 1
For any $c > \alpha > 0$, $T_p > \delta > 0$ the lower bound $V_{\min}(c, \alpha, \delta)$ on the value function exists and is non-trivial for all $x_0 \in \Omega_c / \Omega_\alpha$: $0 < V_{\min}(c, \alpha, \delta) := \min_{x_0 \in \Omega_c / \Omega_\alpha} \int_0^\delta F(\bar{x}(s; x_0), u(s; x_0)) \, \mathrm{d}s < \infty$.

4.2. Boundedness of the states

As a first result we establish that the closed-loop states are bounded for sufficiently small ε and δ.

Theorem 1 (Boundedness of the states, invariance of $\Omega_c(\mathscr{S})$)
Assume that the Assumptions 1–6 are fulfilled. Given arbitrary compact sets $\mathscr{Q}$, $\mathscr{S}$ and $\Omega_c(\mathscr{S})$ with $\mathscr{Q} \subset \mathbb{R}^n$ and $\mathscr{S} \subset \Omega_c(\mathscr{S}) \subset \mathscr{R}$. Then there exists $\delta_2^\star > 0$ such that for $\delta \leqslant \delta_2^\star$, $\delta > 0$, there exists an $\varepsilon_2^\star > 0$, such that for all $0 < \varepsilon \leqslant \varepsilon_2^\star$ and all initial conditions $(x_0, \hat{x}_0) \in \mathscr{S} \times \mathscr{Q}$, the observer error $\eta(\tau)$ stays bounded and converges at least at the end of every sampling interval to the set Γ_ε. Furthermore, $x(\tau) \in \Omega_c(\mathscr{S})$ $\forall \tau \geqslant 0$.

Proof
The proof is divided into two parts. In the first part it is shown that there exists a sufficiently small δ and a sufficiently small ε such that the observer error converges to the set Γ_ε at least at the end of the first sampling interval, starting with $\hat{x}(0) \in \mathscr{Q}$ and $x(0) \in \mathscr{S}$, and that $x(t)$ in the sampling time does not leave $\Omega_c(\mathscr{S})$. In a second step we establish that $x(t)$ remains in $\Omega_c(\mathscr{S})$ while $\eta(t)$ stays bounded and converges (at least) at the end of each sampling interval (t_i^-) to Γ_ε. Note that η might jump at the sampling instants t_i due to the discontinuities in U and the possible reinitialization (8). This is the reason why we cannot establish that η enters the set Γ_ε and stays there. However, this is not a problem for the control, since the state estimate is only needed at the end of each sampling interval. Figure 2 is an attempt to sketch the main ideas of the proof. We denote the smallest level set (Figure 3 clarifies some of the regions occurring in the proofs) of V that covers $\mathscr{S}$ by $\Omega_{c_1}(\mathscr{S})$, where the constant $c_1 < c$ is given by $c_1 = \max_{x \in \mathscr{S}} V(x)$.

First sampling interval, existence of ε, δ *such that* $\eta(t_1^-) \in \Gamma_\varepsilon$ *and* $x(\tau) \in \Omega_c(\mathscr{S})$, $\tau \in [0, t_1]$: Since $\mathscr{S}$ is strictly contained in $\Omega_c(\mathscr{S})$, there exist a time T_c such that trajectories starting in $\mathscr{S}$ do not leave $\Omega_c(\mathscr{S})$ on the interval $[t, t + T_c]$. The existence is guaranteed, since, similar to the proof of Lemma 2, as long as $x(t) \in \Omega_c(\mathscr{S})$, $\|x(t) - x_0\| \leqslant \int_0^t \|f(x(s), u(s))\| \, \mathrm{d}s \leqslant k_\Omega t$. We take T_c as the smallest possible (worst case) time to reach the boundary of $\Omega_c(\mathscr{S})$ from a point $x_0 \in \Omega_{c_1} \supset \mathscr{S}$, allowing $u(s)$ to take any value in $\mathscr{U}$. Due to the compactness of $\mathscr{Q}$ we know from Lemma 1 that $\eta(t) \in \Gamma_\varepsilon$ for $t \geqslant T_{\mathscr{Q}}(\varepsilon)$. Since $T_{\mathscr{Q}}(\varepsilon) \to 0$ as $\varepsilon \to 0$, there exists an ε_1 such that for all $0 < \varepsilon \leqslant \varepsilon_1$, $T_{\mathscr{Q}}(\varepsilon) \leqslant T_c/2$. Let $\delta_2^\star$ be such that for all $0 < \delta \leqslant \delta_2^\star$, the first sampling instant $t_1 = \delta$ satisfies

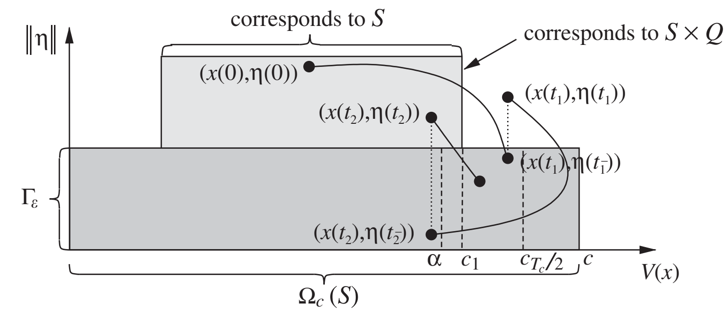

Figure 2. Sketch of the main ideas behind the proof of Theorem 1.

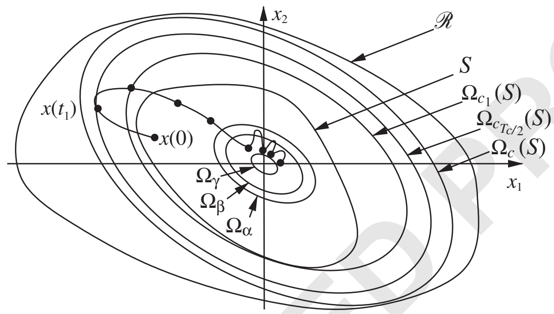

Figure 3. Regions involved in the practical stability proof.

$T_{\mathcal{Q}}(\varepsilon) < t_1 < T_c/2$. Choose one such δ for the rest of the proof. Then $(x(t_1), \eta(t_1^-)) \in \Omega_c(\mathcal{S}) \times \Gamma_\varepsilon$, and the same hold for the next sampling instant, (since we used $T_c/2$ for choosing $\delta_2^\star$). We will in the following refer to the smallest level set covering all points that can be reached from points in $\Omega_{c_1}(\mathcal{S})$ in the time $T_c/2$ applying any admissible control by $\Omega_{c_{T_c/2}}(\mathcal{S})$. Note that by the arguments given above, $x(t_1) \in \Omega_{c_{T_c/2}}(\mathcal{S})$, with $c_{T_c/2} < c$.

Invariance of Ω_c *for* x *at sampling instants, convergence of* η *to* Γ_ε *for each* t_i^-: Consider a sampling instant t_i (e.g. t_1) for which we know that $x(t_i) \in \Omega_{c_{T_c/2}}(\mathcal{S})$ and that $x(t_i + \tau) \in \Omega_c(\mathcal{S})$ for $0 \leqslant \tau \leqslant \delta$ and $\eta(t_i^-) \in \Gamma_\varepsilon$. Note that we do not have to consider the case when $x(t_i + \tau) \in \Omega_{c_1}(\mathcal{S})$ for some $0 \leqslant \tau \leqslant \delta$, since the reasoning in the first part of the proof ensures in this case that the state will not leave the set $\Omega_{c_{T_c/2}}(\mathcal{S})$ in one sampling time (considering the worst case input). Hence we assume in the following that $x(t_i + \tau) \in \Omega_c(\mathcal{S})/\Omega_{c_1}(\mathcal{S})$.

Now consider the difference of the value function for the initial state $x(t_i)$ and the state $x(t_i + \tau)$,

$$\begin{aligned} & V(x(t_i + \tau)) - V(x(t_i)) \\ & \quad \leqslant V(\bar{x}(t_i + \tau; x(t_i))) - V(x(t_i)) + |V(x(t_i + \tau)) - V(\bar{x}(t_i + \tau; x(t_i)))| \\ & \quad \leqslant -\int_{t_i}^{t_i+\tau} F(\bar{x}(s, x(t_i)), u(s; x(t_i)))\, \mathrm{d}s + |V(x(t_i + \tau)) - V(\bar{x}(t_i + \tau; x(t_i)))| \end{aligned} \tag{10}$$

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Since V is Lipschitz in compact subsets of $\mathcal{R} \supset \Omega_c(\mathcal{S})$ we obtain:

$$V(x(t_i+\tau)) - V(x(t_i)) \leqslant -\int_{t_i}^{t_i+\tau} F(\bar{x}(s;x(t_i)), u(s;x(t_i)))\,\mathrm{d}s + L_V\|x(t_i+\tau) - \bar{x}(t_i+\tau;x(t_i))\|$$

where L_V is the Lipschitz constant of V in $\Omega_c(\mathcal{S})$. The integral contribution is only a function of the predicted open-loop trajectories of the NMPC state feedback controller. Fact 1 and Lemma 2 give:

$$V(x(t_i+\delta)) - V(x(t_i)) \leqslant -V_{\min}(c,\alpha,\delta) + L_V\frac{L_{fu}L_u}{L_{fx}}\|x(t_i) - \hat{x}(t_i)\|(e^{L_{fx}\delta} - 1) \tag{11}$$

for any fixed $\alpha < c_1$ and $x(t_i) \notin \Omega_\alpha$. From Remark 4.1 we know that there exists an $\varepsilon_2 \geqslant 0$ such that for all $0 < \varepsilon < \varepsilon_2$, $V(x(t_i+\delta)) - V(x(t_i)) \leqslant -\frac{1}{2}V_{\min}(c,\alpha,\delta) =: -\kappa_1$, where $\kappa_1 > 0$ is a constant given by the right-hand side of (11). Hence the state at the next sampling instant is at least within $\Omega_{c_{T_c/2}}(\mathcal{S})$ again, and thus also in $\Omega_c(\mathcal{S})$. Since $x(t_{i+1}) \in \Omega_{c_{T_c/2}}(\mathcal{S})$, it will by the reasoning in the first part not leave $\Omega_c(\mathcal{S})$ during the next sampling interval, and hence the arguments in the second part holds for this interval as well. By induction, the state will not leave $\Omega_c(\mathcal{S})$, and $x(t_{i+1}) \in \Omega_{c_{T_c/2}}(\mathcal{S})$. Setting $\varepsilon_2^\star := \min\{\varepsilon_1, \varepsilon_2\}$ concludes the proof. □

4.3. Semi-global practical stability of the systems states

In this section it is established that for any small ball around the origin, there exists an observer gain and a sampling time such that the state trajectory converges to the ball in finite time and stays inside the ball.

Theorem 2 (Practical stability)
Given arbitrary compact sets $\mathcal{Q}$, $\mathcal{S}$ and $\Omega_c(\mathcal{S})$ with $\mathcal{Q} \subset \mathbb{R}^n$ and $\mathcal{S} \subset \Omega_c(\mathcal{S}) \subset \mathcal{R}$. Furthermore, let the Assumptions 1–6 hold. Then, for any set Ω_α with $c > \alpha > 0$, there exists $\delta_3^\star > 0$ such that for $\delta \leqslant \delta_3^\star$, $\delta > 0$, there exists $\varepsilon_3^\star > 0$, such that for all $0 < \varepsilon \leqslant \varepsilon_3^\star$ and all $(x_0, \hat{x}_0) \in \mathcal{S} \times \mathcal{Q}$, the observer error $\eta(\tau)$ stays bounded and the state $x(\tau)$ converges in finite time to the set Ω_α and remains there.

Proof
First we show that there exists an ε sufficiently small, such that for any $0 < \beta < \alpha$, $\Omega_\beta \subset \Omega_\alpha$, trajectories originating in Ω_β at a sampling instant do not leave Ω_α (Figure 3 clarifies some of the regions occurring in the proof.) Then we establish that the states starting at $\hat{x}_0 \in \mathcal{Q}$ and $x_0 \in \mathcal{S}$ enter Ω_β in finite time. In the first part we consider any fixed $\delta \leqslant \delta_2^\star$.

'*Invariance*' *of* Ω_α *for* $x(t_i)$ *originating in* Ω_β: For $x(t_i) \in \Omega_\beta$ and $\tau \leqslant \delta$, by Lemma 2, the value functions of the state feedback and output feedback trajectories satisfy the bound

$$|V(x(t_i+\tau)) - V(\bar{x}(t_i+\tau;x(t_i)))| \leqslant L_V\frac{L_{fu}L_u}{L_{fx}}\|x(t_i) - \hat{x}(t_i)\|(e^{L_{fx}\tau} - 1). \tag{12}$$

Furthermore, the state feedback trajectory satisfies $\bar{x}(t_i+\tau;x(t_i)) \in \Omega_\beta$ for $\tau \in [0,\delta]$ by Assumption 3.4. So one can choose an ε_1 such that for $0 < \varepsilon \leqslant \varepsilon_1$, $V(x(t_i+\tau)) \leqslant \alpha$ for $\tau \in [0,\delta]$, for all $x(t_i) \in \Omega_\beta$. Thus the trajectory $x(t_i+\tau)$ does not leave the set Ω_α for $\tau \in [0,\delta]$. Now we define an additional level set Ω_γ inside of Ω_β given by $0<\gamma<\beta$. We proceed considering two cases, $x(t_i)\in\Omega_\gamma$ and $x(t_i)\in\Omega_\beta/\Omega_\gamma$.

$x(t_i)\in\Omega_\beta/\Omega_\gamma$: Similar to Equation (11) in the proof of Theorem 1, we can show for $x(t_i)\notin\Omega_\gamma$,

$$V(x(t_i+\delta))-V(x(t_i))\leqslant -V_{\min}(c,\gamma,\delta)+L_V\frac{L_{fu}L_u}{L_{fx}}\|x(t_i)-\hat{x}(t_i)\|(e^{L_{fx}\delta}-1)$$

Choose ε_2 such that for $0<\varepsilon<\varepsilon_2$,

$$V(x(t_i+\delta))-V(x(t_i))\leqslant -\frac{1}{2}V_{\min}(c,\gamma,\delta)=:-\kappa_2 \tag{13}$$

Hence $x(t_i+\delta)\in\Omega_\beta$ for $x(t_i)\in\Omega_\beta/\Omega_\gamma$. Additionally, we know from the first part of the proof that also the states between the sampling instants t_i and $t_i+\delta$ do not leave Ω_α. The bound (13) implies that $x(t_i)$ reaches the set Ω_γ in finite time, for which (13) is not valid anymore.

$x(t_i)\in\Omega_\gamma$: To show that we can find an ε such that $x(t_i+\tau)\in\Omega_\beta$, we use again Equation (12) and note that the state feedback trajectory satisfies $\bar{x}(t_i+\tau;x(t_i))\in\Omega_\gamma$ for $\tau\in[0,\delta]$. Choosing an $\varepsilon_3\leqslant\min\{\varepsilon_1,\varepsilon_2\}$ sufficiently small, we know that for all $0<\varepsilon\leqslant\varepsilon_3$, $V(x(t_i+\tau))\leqslant\beta$ for $\tau\in[0,\delta]$, and for all $x(t_i)\in\Omega_\gamma$. From the given arguments it follows that $x(t_i+\delta)\in\Omega_\beta$ for all $x(t_i)\in\Omega_\beta$ and $x(t_i+\tau)\in\Omega_\alpha$ for all $\tau\in[0,\delta]$. Thus it is clear that once $x(t_i)$ enters the set Ω_β, the trajectories stay for all times in $\Omega_\alpha\supset\Omega_\beta$.

Finite time convergence to Ω_β: It remains to show that for any $(x_0,\eta_0)\in\mathcal{S}\times\mathcal{Q}$, there exists a (finite) sampling instant t_m with $x(t_m)\in\Omega_\beta$. We know from Theorem 1 that for sufficiently small δ and ε, $x(\tau)\in\Omega_c(\mathcal{S})\ \forall\tau>0$. Set $\delta_3^\star=\delta_2^\star$, and choose a $\delta<\delta_3^\star$. Set $\varepsilon_3^\star=\min\{\varepsilon_2^\star,\varepsilon_3\}$, where $\varepsilon_2^\star$, depending on δ, is specified as in Theorem 1. Furthermore, note that Theorem 1 guarantees boundedness of $\eta(\tau)\ \forall\tau>0$, and that η is at least at the end of all sampling intervals inside of Γ_ε. Hence, to show convergence to Ω_β, note that (13) is valid for all $x(t_i)\in\Omega_c/\Omega_\gamma$. Therefore, for any initial state in $\mathcal{S}$ the state enters Ω_β in a finite time less than or equal to $((c-\beta)/\kappa_2)\delta$. □

Theorem 2 implies practical stability of the system state $x(t)$. Choosing α and ε small enough, we can guarantee that x converges to any set containing a neighborhood of the origin. Thus the closed-loop system state is semi-globally practically stable with respect to the set $\mathcal{R}$, in the sense that for any $\mathcal{S}\subset\mathcal{R}$ and any ball around the origin there exists an observer gain and a sampling time, such that the system state reaches the ball from any point in $\mathcal{S}$ in finite time and stays therein afterwards.

4.4. Discussion

The derived results are mainly based on the fact that NMPC is to some extent robust to measurement errors. This robustness is restricted by the integral contribution on the right-hand side of Equation (10). Utilizing this robustness in the output feedback case has some direct consequences. For example the level sets of the value function, which are invariant in the state feedback case, are in general no longer invariant in the output feedback case. Furthermore, the following points are important to note:

Satisfaction of constraints: The satisfaction of the input constraints is guaranteed by the NMPC scheme and the boundedness of the input for $\hat{x}\notin\mathcal{R}$. The state constraints are satisfied since $\mathcal{S}\subset\mathcal{R}\subseteq\mathcal{X}$, and since a sufficiently high observer gain and a sufficiently small sampling time is chosen, such that even initially the state does not leave the set $\mathcal{R}\subseteq\mathcal{X}$.

Limited required sampling time: The sampling time must only be small enough to guarantee that the trajectory during the initial phase, for which the observer error is often significant, does not leave the desired region of 'attraction' $\Omega_c(\mathcal{S})$. Beyond this, only ε must be decreased, while δ can be kept constant. This follows from the use of a predictive control scheme that uses a system model. The open-loop input signal applied to the system (which is not fixed to a constant value in general) corresponds to the predicted behavior. In comparison, the general output feedback approach presented in References [16, 17] requires a sufficient decrease in the sampling time to achieve practical stability, since the input during the sampling time is fixed to the constant value $u(t) = k(x(t_i))$ given by a state feedback controller $k(x)$.

Need to reinitialize the observer: We only establish that the observer error stays bounded and converges to Γ_ε at the end of each sampling interval. This is a toll for allowing a general observability map that can depend on u and its derivatives. Since in NMPC the input is often discontinuous at the sampling instants, this means the scaled observer error η in general also is discontinuous at these points. To avoid that η might 'jump' out of Γ_ε at the sampling time, we have to enforce $\hat{x}(t_i)$ inside of $\mathcal{Q}$ to guarantee that η converges again to the set Γ_ε for t_{i+1}^-. In the case that $\mathcal{H}$ is independent of u or that the inputs are continuous, e.g. due to added input integrators, the reinitialization is not necessary. In this case it is easy to show that the observer error converges in one sampling time to Γ_ε and stays there indefinitely, thus the whole closed loop including the observer states is practically stable.

5. CONCLUSIONS

It is a widespread intuition that NMPC, which inherently is a state feedback approach, can be applied to systems where only output measurements are available, if an observer with 'good enough' convergence properties is used. In this paper a new output feedback NMPC scheme for the class of uniformly completely observable systems is derived, using a high-gain observer to obtain 'fast enough' estimates of the states. It is shown that under certain assumptions on the NMPC controller, the approach confirms the intuition. To be more precise the derived output feedback scheme achieves semi-global practical stability, i.e. for a fast enough sampling frequency and fast enough observer, it recovers up to any desired accuracy the NMPC state feedback region of attraction (semi-global) and steers the state to any (small) compact set containing the origin (practical stability). The semi-global stability result obtained is the key difference to previous output feedback NMPC schemes, delivering direct tuning knobs to increase the resulting region of attraction of the closed loop. No specific state feedback NMPC scheme is considered during the derivations. Instead a set of assumptions are stated that the NMPC scheme must fulfil. In principle these assumptions can be satisfied by a series of NMPC schemes [18–21]. If the input does not appear in the observability map, the NMPC controller need not be modified to guarantee that the input is sufficiently smooth. Thus, in this case the derived results can be seen as a special separation principle for NMPC. Still, several open questions for output feedback NMPC remain. For e.g., one of the key assumptions on the NMPC controller is that the applied optimal open-loop input is locally Lipschitz in terms of the state. This assumption is in general hard to verify. Thus future research will focus on either relaxing this condition, or to derive conditions under which an NMPC scheme does satisfy this assumption.

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