ELSEVIER
Journal of Process Control □ (□□□□) □–□
JOURNAL OF PROCESS CONTROL
www.elsevier.com/locate/jprocont

A note on stability, robustness and performance of output feedback nonlinear model predictive control

Lars Imsland[a,*], Rolf Findeisen[b], Eric Bullinger[b], Frank Allgower[b], Bjarne A. Foss[a]

[a]Department of Engineering Cybernetics, Norwegian University of Science and Technology, 7491 Trondheim, Norway
[b]Institute for Systems Theory in Engineering, University of Stuttgart, 70550 Stuttgart, Germany

Abstract

In recent years, nonlinear model predictive control (NMPC) schemes have been derived that guarantee stability of the closed loop under the assumption of full state information. However, only limited advances have been made with respect to output feedback in the framework of nonlinear predictive control. This paper combines stabilizing instantaneous state feedback NMPC schemes with high-gain observers to achieve output feedback stabilization. For a uniformly observable MIMO system class it is shown that the resulting closed loop is asymptotically stable. Furthermore, the output feedback NMPC scheme recovers the performance of the state feedback in the sense that the region of attraction and the trajectories of the state feedback scheme can be recovered to any degree of accuracy for large enough observer gains, thus leading to semi-regional results. Additionally, it is shown that the output feedback controller is robust with respect to static sector bounded nonlinear input uncertainties.

Keywords: Nonlinear model predictive control; Output feedback stabilization; High gain observers; Nonlinear separation principle

1. Introduction

Model predictive control (MPC), also referred to as moving horizon control or receding horizon control, has become an attractive feedback strategy, especially for linear or nonlinear systems subject to input and state constraints. In general, linear and nonlinear MPC are distinguished. Linear MPC refers to a family of MPC schemes in which linear models are used to predict the system dynamics, even though the dynamics of the closed loop system is nonlinear due to the presence of constraints. Linear MPC approaches have found successful applications, especially in the process industries [23]. By now, linear MPC theory is fairly mature. Important issues such as the online computations, the interplay between modeling, identification and control as well as system theoretic issues like stability are well addressed.

Linear models are widely and successfully used to solve control problems. However, many systems are inherently nonlinear. Higher product quality specifications, increasing productivity demands, tighter environmental regulations and demanding economical considerations require systems to be operated closer to the boundary of the admissible operating region. Often in these cases, linear models are not adequate to describe the process dynamics and nonlinear models must be used. This motivates the application of nonlinear model predictive control.

Model predictive control for nonlinear systems (NMPC) has received considerable attention over the past years. Many theoretical and practical issues have been addressed. Several existing schemes guarantee stability under full state information, see [1,7,19] for recent reviews. In practice, however, not all states are directly available by measurements. A common approach to output feedback NMPC is to employ a state feedback NMPC controller in combination with a state observer. If this approach is used, in general little can be said about the stability of the closed loop, since no universal separation principle for nonlinear systems exists.

Different approaches addressing the output feedback problem in NMPC exist. In [21] a moving horizon observer is presented, that together with the so called dual-mode NMPC scheme [20] lead to semi-regional

* Corresponding author.
E-mail address: lars.imsland@itk.ntnu.no (L. Imsland); findeise@ist.uni-stuttgart.de (R. Findeisen); bullinger@ist.uni-stuttgart.de (E. Bullinger); allgower@ist.uni-stuttgart.de (F. Allgöwer); bjarne.foss@itk.ntnu.no (B.A. Foss).

0959-1524/03/$ - see front matter
doi:10.1016/S0959-1524(03)00006-4

closed loop stability if no model–plant mismatch and disturbances are present. Semi-regional stability in this context means that for any subset of the region of attraction of the state feedback there exists a set of parameters (in [21] the sampling time and the enforced contraction rate of the observer error) such that this subset is contained in the region of attraction of the output feedback controller. However, for the results in [21] to hold it is required that a global (dynamic) optimization problem can be solved. In [16], see also [24], asymptotic stability for observer based discrete-time nonlinear MPC for "weakly detectable" systems is obtained. However, these results are of local nature. The stability is guaranteed only for a sufficiently small initial observer error. While the region of attraction of the resulting output feedback controller in principle can be estimated from Lipschitz constants of the system, observer and controller, it is not clear which parameters in the controller and observer must be changed to increase the region of attraction of the output feedback controller.

This article considers the use of high-gain observers in conjunction with instantaneous NMPC. In instantaneous NMPC it is assumed that the solution to the open loop optimal control problem is immediately available and instantaneously implemented on the process at all time instances. Hence, the optimal input is not employed in a "sampled" fashion, as is often done in NMPC. We show that for a special MIMO system class, the resulting output feedback NMPC scheme does allow performance recovery of the state feedback NMPC controller as the observer gain increases. Performance recovery in this context means that the region of attraction and the rate of convergence of the output feedback scheme approach that of the state feedback scheme. Furthermore, under additional technical conditions the resulting output feedback controller is robust with respect to static sector bounded nonlinear input uncertainties. The results are based on recently derived separation principles [2,9,27].

The presented approach can, in principle, be extended to the sampled-data case and to a more general system class. Preliminary results in this direction can be found in [10,11].

The paper is structured as follows: in Section 2 the class of systems is specified. Section 3 contains the description of the possible NMPC schemes for state feedback and presents the high-gain observer. In Section 4 the results on closed loop stability and performance for the nominal system are derived. Section 5 shows under additional technical assumptions that the output feedback scheme is robust with respect to static sector bounded nonlinear input uncertainties. Some of the properties and practical implications of the presented approaches are discussed in Section 6. In Section 7 the proposed output feedback controller is applied to the control of two example systems: a mixed-culture bioreactor with competition and external inhibition, and an inverted pendulum on a cart.

In the following, $\|\cdot\|$ denotes the Euclidean vector norm in $\mathbb{R}^n$ (where the dimension n follows from the context) or the associated induced matrix norm. The matrix blockdiag($A_1,\ldots,A_r$) denotes a block diagonal matrix with the matrices $A_1,\ldots,A_r$ on the "diagonal", while diag($\alpha_1,\ldots,\alpha_r$) denotes a diagonal matrix with the scalars $\alpha_1,\ldots,\alpha_r$ on the diagonal. Whenever a semicolon ";" occurs in a function argument, the subsequent arguments are additional parameters, i.e. $f(x;\gamma)$ means the value of the function f at x with the parameter set to γ.

2. System class

This paper considers the stabilization of nonlinear MIMO systems of the form

$$\dot{x}_1 = Ax_1 + B\phi(x,u) \tag{1a}$$

$$\dot{x}_2 = \psi(x,u) \tag{1b}$$

$$y = \begin{bmatrix} Cx_1 \\ x_2 \end{bmatrix} \tag{1c}$$

with $x^\top(t) = [x_1^\top(t), x_2^\top(t)]$. The system state consists of the vectors $x_1(t)\in\mathbb{R}^r$ and $x_2(t)\in\mathbb{R}^l$, and the vector $y(t)\in\mathbb{R}^{p+l}$ is the measured output. The control input is constrained, i.e. $u(t)\in\mathcal{U}\subset\mathbb{R}^m$, where:

Assumption 2.1. $\mathcal{U}\subset\mathbb{R}^m$ is compact and the origin is contained in the interior of $\mathcal{U}$.

The $r\times r$ matrix A, $r\times p$ matrix B and the $p\times r$ matrix C have the following form

$$A = \text{blockdiag}[A_1, A_2, \ldots A_p],$$

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & 1 \\ 0 & \ldots & \ldots & \ldots & 0 \end{bmatrix}_{r_i\times r_i}$$

$$B = \text{blockdiag}[B_1, B_2, \ldots, B_p], \qquad B_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{r_i\times 1}$$

$$C = \text{blockdiag}[C_1, C_2, \ldots, C_p],$$

$$C_i = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}_{1\times r_i},$$

i.e. the x_1 dynamics consists of p integrator chains of length r_i, with $r = r_1 + \cdots + r_p$. Furthermore, the nonlinear functions ϕ and ψ satisfy:

Assumption 2.2. The functions $\phi : \mathbb{R}^{r+1} \times \mathcal{U} \rightarrow \mathbb{R}^r$ and $\psi : \mathbb{R}^{r+l} \times \mathcal{U} \rightarrow \mathbb{R}^l$ are locally Lipschitz in their arguments over the domain of interest with $\phi(0,0) = 0$ and $\psi(0,0) = 0$. Additionally ϕ is bounded as function of x_1.

Systems of this class are for example input affine nonlinear systems of the form

$$\zeta = f(\zeta) + g(\zeta)u, \qquad y = h(\zeta)$$

with full (vector) relative degree $(r_1, r_2, \ldots, r_p)$, that is, $\Sigma_{i=1}^{p} r_i = \dim\zeta$. For these systems it is always possible to find a coordinate transformation such that the system in the new coordinates fits the structure (1a) and (1c), see [14].

We do not need to state any observability and controllability assumption. The controllability assumption is implicitly, as usual in predictive control, contained in the assumption on the NMPC controller having a non trivial region of attraction. The observability of the system is guaranteed since the x_1 states can be recovered by the high-gain controller as shown in Section 3.2, and the x_2 states are assumed to be directly measured.

3. NMPC output feedback controller: setup

The proposed output feedback controller for the stabilization of the origin consists of a high-gain observer for estimating the states and an instantaneous state feedback NMPC controller.

3.1. State feedback NMPC

In the framework of predictive control, the value of the manipulated variable is given by the solution of an open loop optimal control problem. Herein, the open loop optimal control problem that defines the system input is given by

State feedback NMPC open loop optimal control problem:

Solve

$$\min_{\overline{u}(\cdot)} J\big(x(t), \overline{u}(\cdot); T_p\big) \tag{2}$$

subject to:

$$\dot{\overline{x}}_1 = A\overline{x}_1 + B\phi(\overline{x}, \overline{u}), \quad \overline{x}_1(0)x_1(t) \tag{3a}$$

$$\dot{\overline{x}}_2 = \psi(\overline{x}, \overline{u}), \quad \overline{x}_2(0) = x_2(t) \tag{3b}$$

$$\overline{u}(\tau) \in \mathcal{U}, \tau \in \big[0, T_p\big] \tag{3c}$$

$$\overline{x}\big(T_p\big) \in \Omega \tag{3d}$$

with the cost functional

$$J\big(x(t), \overline{u}(\cdot); T_p\big) := \int_0^{T_p} F(\overline{x}(\tau), \overline{u}(\tau))\mathrm{d}\tau + E\big(\overline{x}\big(T_p\big)\big). \tag{4}$$

The bar denotes internal controller variables and $\overline{x}(\cdot)$ is the solution of Eqs. (3a)–(3b) driven by the input $\overline{u}(\cdot) : [0,T_p] \rightarrow \mathcal{U}$ over the prediction horizon T_p with initial condition $x(t)$. The stage cost $F(\tilde{x}, \tilde{u})$ satisfies:

Assumption 3.1. $F : \mathbb{R}^{r+l} \times \mathcal{U} \rightarrow \mathbb{R}$ is continuous in all arguments with $F(0,0) = 0$ and $F(x,u) > 0 \;\forall (x,u) \neq (0,0)$.

The constraint (3d) in the NMPC open loop optimal control problem forces the final predicted state to lie in the *terminal region* denoted by Ω and is thus often called *terminal region constraint*. In the cost functional J, the deviation from the origin of the final predicted state is penalized by the *terminal state penalty term E*.

Notice that, for simplicity of exposition, only input constraints are considered (besides the terminal state constraint).

The optimal input signal resulting from the solution of the optimal control problem (2) is denoted by $\bar{u}^*(\cdot\,; x(t))$. The input applied to the system is given by

$$u(x(t)) := \bar{u}^*(\tau = 0\;; x(t)). \tag{5}$$

Note that the solution to the NMPC open loop optimal control problem must be available instantaneously at all times without delay. Such instantaneous NMPC formulations are often used for system theoretic investigations [18,19]. However, obtaining an instantaneous solution of the dynamic optimization problem (2) and (3) is often not possible in practice. Instead, a sampled-data NMPC approach is often employed. The open-loop optimal control problem is only solved at discrete sampling instants and the resulting input signal is applied open loop until the next sampling instant. If the sampling intervals are short compared to the system dynamics, the trajectories of the sampled-data implementation are often close to the instantaneous implementation.

If T_p, E, F are suitably chosen, the origin of the nominal state feedback closed loop system with the input (5) is asymptotically stable and the region of attraction $\mathcal{R} \subset \mathbb{R}^{r+l}$ contains the set of states for which the open loop optimal control problem has a solution. In the following it is assumed, that:

Assumption 3.2. The instantaneous state feedback $u(x)$ is locally Lipschitz in x and asymptotically stabilizes the system (1) with a region of attraction $\mathcal{R}$.

In principle this setup allows one to consider a whole variety of different NMPC schemes (e.g. [5,15] see also [19] for a review). In this sense, the results described in the next sections can be seen as a special "separation" principle for NMPC using high-gain observers. The main restriction is the requirement that the optimal input must be locally Lipschitz.

3.2. High-gain observer

The proposed (partial state) observer for the recovery of x_1 is a standard high-gain observer [28] of the following form

$$\dot{\hat{x}}_1 = A\hat{x}_1 + B\hat{\phi}((\hat{x}_1, x_2), u) + H\big(y_{x_1} - C\hat{x}_1\big),$$

where $H = \text{blockdiag}\big[H_1, \ldots, H_p\big]$ with

$$H_i^\top = \Big[\alpha_1^{(i)}/\varepsilon,\ \ \alpha_1^{(i)}/\varepsilon^2, \ldots, \alpha_{r_i}^{(i)}/ \in^{r_i}\Big]$$

and the $\alpha_j^{(i)}$s are such that the roots of

$$s^{r_i} + \alpha_1^{(i)} s^{r_i - 1} + \cdots + \alpha_{r_i-1}^{(i)} s + \alpha_{r_i}^{(i)} = 0, \quad i = 1, \ldots, p$$

are in the open left half plane. The vector y_{x_1} is the first part of the measurement vector related to the states x_1, i.e. $y_{x_1} = Cx_1$, and $\frac{1}{\varepsilon}$ is the high-gain parameter. A, B, C and ϕ are the same as in (1).

Since the x_2 states are assumed to be directly measured it is only necessary to design an observer for the x_2 states.

Remark 3.1. Notice that the use of an observer makes it necessary to define a (bounded) input also for estimated states that are outside the feasibility region $\mathcal{R}$ of the controller. One possible choice is to fix the open loop input for $x \notin \mathcal{R}$ to an arbitrary value $u_f \in \mathcal{U}$: $u(x) = u_f$, $\forall x \notin \mathcal{R}$.

4. Nominal stability of output feedback NMPC using high-gain observers

In this section the nominal stability results for the proposed output feedback controller are derived, i.e. it is assumed that the plant and the model coincide $\big(\hat{\phi} = \phi\big)$. It is shown that the performance of the state feedback controller can be recovered to any precision (see Definition 4.1) and that asymptotic stability can be achieved for a sufficiently small value of ε in the observer.

Consider the closed loop system given by (1a)–(1c) and the control given as defined by the NMPC controller using the observed state $\hat{x}_1$ from the high-gain observer.

In the following, recovery of the performance of the state feedback controller by the output feedback controller for the nominal system and for sufficiently small ε is established. We distinguish between the state trajectory resulting from the application of the state feedback controller and the state trajectory resulting from the application of the output feedback controller using the high-gain observer. Specifically $x_{sf}(\cdot\,; x_0)$ denotes the trajectory resulting from the application of the state-feedback NMPC controller starting at $x_{sf}(0) = x_0$. The trajectory resulting from the application of the NMPC controlled based on the state estimates $\hat{x}_1$ starting from $x_\varepsilon(0) = x_0$ and initializing the observer with $\hat{x}_1(0) = \hat{x}_{10} \in \mathcal{Q}$ is denoted by $x_\varepsilon(\cdot\,; x_0, \hat{x}_{10})$. Here Q is an arbitrary but fixed compact set of possible observer initial conditions. The suffix ε indicates the dependence on the value of the high-gain parameter ε. Using this notation, the desired recovery of performance means:

Definition 4.1. [Performance recovery with respect to ε] Assume that $x_\varepsilon(t; x_0)$ and $x_{sf}(t; x_0, \hat{x}_{10})$ start from the same initial state x_0, i.e. $x_\varepsilon(0; x_0) = x_{sf}(0; x_0, \hat{x}_{10}) = x_0$. Then, recovery of performance with respect to ε means that for any $\delta > 0$ there exists an ε^* such that for all $0 < \varepsilon \leqslant \varepsilon^*$,

$$\big\| x_\varepsilon(t; x_0, \hat{x}_{10}) - x_{sf}(t; x_0) \big\| \leqslant \delta, \qquad \forall t > 0, \forall \hat{x}_{10} \in \mathcal{Q}.$$

Given this definition of performance recovery, the following theorem holds for the system controlled by the output feedback NMPC controller:

Theorem 4.1. *Assume that Assumptions 2.1–3.2 hold. Let S be any compact set contained in the interior of $\mathcal{R}$. Furthermore, the observer initial condition satisfies $\hat{x}_1(0) = \hat{x}_{10} \in \mathcal{Q}$ with $\mathcal{Q}$ arbitrary but fixed and compact. Then there exists a (sufficiently small) $\varepsilon^* > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon^*$ the closed loop system is asymptotically stable with a region of attraction of at least S. Further, the performance of the state feedback NMPC controller is recovered in the sense of Definition 4.1.*

Outline of Proof. The asymptotic stability follows from the proofs of Theorems 1, 2 and 4 in [2]. The application of these theorems is possible since the local Lipschitz property of the state feedback combined with the closed loop stability allow to use converse Lyapunov arguments to assure the existence of a Lyapunov function for the state feedback closed loop. Theorem 1 in [2] guarantees boundedness of solutions starting in S if $\varepsilon < \varepsilon_1^*$, with ε_1^* sufficiently small. Theorem 2 guarantees that the solutions starting in S will enter any ball around the origin in finite time if $\varepsilon < \varepsilon_2$,*[1] where ε_2^* is sufficiently small with $\varepsilon_2^* < \varepsilon_1^*$. Positioned in such a (small) ball, one can establish asymptotic stability for a $\varepsilon_3^* < \varepsilon_2^*$ as long as $\varepsilon < \varepsilon_3^*$, under the assumption $\phi_0 = \phi$. Furthermore, Theorem 3 in [2] shows that the trajectories of the controlled system using the observed state in the controller, converge uniformly to the trajectories of the controlled system using the true state in the controller, as

[1] Note, ε_2^* depends on the size of the ball.

$\varepsilon \to 0$. Hence, for ε small enough, the trajectories (and hence the performance) of the state feedback NMPC are recovered. □

The stability result derived is semi-regional, since for any compact subset S of R such a maximum value $\varepsilon^\star$ exists. In general the closer the set S approximates the set R the smaller ε is. Note that the performance recovery of Theorem 4.1 also implies recovery of the rate of convergence of the state feedback controller for sufficiently small ε and convergence of the state and output feedback trajectories.

Note that the satisfaction of the input constraints is guaranteed by the NMPC scheme and the boundedness of the input for $\hat{x} \notin \mathcal{R}$, see Remark 3.1.

In the next section, the result on performance recovery will be expanded to systems having unknown but sector bounded nonlinear static input uncertainties.

5. Robustness to input uncertainties

The results derived so far are only valid in the nominal case. In this section we show that the proposed output feedback controller is robustly stable with respect to unknown but sector bounded input nonlinearities. The result is based on the robustness result given in [2]. However, to utilize this result it is necessary that the state feedback controller robustly exponentially stabilizes the system. Thus, in a first step we show that the state feedback NMPC controller discussed in Section 3.1 leads to exponential stability even in the case of unknown static input uncertainties.

As uncertainty we consider that the input applied to the system is subject to a static (unknown) input uncertainty $u_\Delta = \Delta(u)$, as depicted in Fig. 1 . We will furthermore assume that $\Delta : \mathbb{R}^m \to \mathbb{R}^m$ has the following structure: $\Delta(u) = \mathrm{diag}(\delta_1(u_1), \ldots, \delta_m(u_m))$. To derive the result we furthermore limit the system class and strengthen the conditions on the state feedback NMPC controller used. For the purpose of this section we consider input affine systems of the form:

$$\dot{x}_1 = Ax_1 + B\tilde{\phi}(x)u \tag{6a}$$

$$\dot{x}_2 = \tilde{\psi}_1(x) + \tilde{\psi}_2(x)u. \tag{6b}$$

The matrices A and B have the same form as in Section 2, and, $\tilde{\phi}$, $\tilde{\psi}_1$ and $\tilde{\psi}_2$ have to satisfy similar assumptions as in the nominal case:

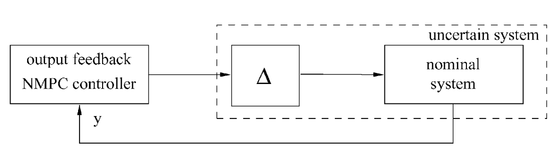

Fig. 1. Closed loop with unknown static input nonlinearity.

Assumption 5.1. The functions $\tilde{\phi} : \mathbb{R}^{r+l} \to \mathbb{R}^{r\times m}$, $\tilde{\psi}_1 : \mathbb{R}^{r+l} \to \mathbb{R}^{l}$ and $\tilde{\psi}_2 : \mathbb{R}^{r+l} \to \mathbb{R}^{l+m}$ are locally Lipschitz in x over the domain of interest with $\tilde{\phi}(0) = 0$ and $\tilde{\psi}_1(0) = 0$. Additionally $\tilde{\phi}$ is bounded as function of x_1.

Note that we have to consider the system class (6) since we use the high-gain observer outlined in Section 3.2. The robust exponential stability result (to be derived) of the NMPC controller holds, however, for general input affine systems. To simplify notation we denote system (6) sometimes briefly by

$$\dot{x} = f(x) + g(x)u,$$

where $f(x) = [(Ax_1)^\top, (\tilde{\psi}_1(x))^\top]^\top$ and $g(x) = [(B\tilde{\phi}(x))^\top, (\tilde{\psi}_2(x))^\top]^\top$. With respect to the stage cost used in the NMPC controller we assume that:

Assumption 5.2. The stage cost in the NMPC controller is of the form

$$F(x, u) = l(x) + u^\top R(x)u, \tag{7}$$

where $l(x) + u^\top R(x)u > c_F\|x, u\|_2^2, \forall (x, u) \in \mathbb{R}^{r+l} \times \mathcal{U}$ with $c_F > 0$, and $R(x) = \mathrm{diag}(r_1(x), \ldots, r_m(x))$.

Nominal exponential stability is guaranteed (see e.g. [15]) by the following slightly strengthened assumption on the terminal region and terminal penalty term:

Assumption 5.3. Assume $E \in C^1$ is a proper F-compatible control Lyapunov function (CLF), i.e.

$$\frac{\partial E}{\partial x} f(x, k(x)) + l(x) + k(x)^\top R(x)k(x) \leqslant 0, \quad \forall x \in \Omega \tag{8}$$

for some locally Lipschitz control law $k(x)$, and

$$c_{1E}\|x\|^2 \leqslant E(x) \leqslant c_{2E}\|x\|^2, \quad \forall x \in \Omega. \tag{9}$$

with some $c_{2E} > c_{1E} > 0$.

This assumption can for example be satisfied using the quasi-infinite horizon NMPC scheme (QIH-NMPC) as described in [6]. As will be shown, this assumption is essential for robust exponential stability of the state feedback NMPC controller.

Since inverse optimality results are used to derive the robustness, it is additionally necessary [17] that the nominal open loop optimal control problem for the NMPC controller satisfies:

Assumption 5.4. The optimal control for the nominal system (6)

$$u(x(t)) := \bar{u}^\star(\tau = 0; x(t)).$$

is unconstrained in a (compact) region of interest. Further, the control is continuously differentiable, and the value function, defined by the optimal solution of the NMPC open loop optimal control problem

$$V(x;T_p) := J(x, \bar{u}^\star(\cdot; x(t)); T_p)$$

is twice continuously differentiable.

Conditions ensuring that the value function is C^2 for unconstrained NMPC-controllers can for example be found in [15].

5.1. Robust exponential stability of state feedback NMPC

As stated in the following lemma, the nominal state feedback NMPC controller robustly exponentially stabilizes the system if the input nonlinearity $\Delta(u)$ maps into the sector $\left(\frac{1}{2}, \infty\right)$ in the sense[2]

$$\frac{1}{2}s^\top s \leqslant s^\top \Delta(s) \leqslant \infty, \qquad \forall s \in \mathbb{R}^m. \tag{10}$$

That is, the system has a *sector margin* $\left(\frac{1}{2}, \infty\right)$ [25].

Lemma 5.1. *Assume that the assumptions of Theorem 4.1 and Assumptions 5.1–5.4 hold. If the input to the system is $\Delta(u^\star(\tau = 0, x))$, and if $\Delta(\cdot)$ satisfies (10), then the origin of system (6) under the state feedback controller is exponentially stable.*

Proof. The proof of the Lemma can be found in the Appendix. □

Note that the region of attraction $\tilde{\mathcal{R}}$ for the closed loop with the uncertainty $\Delta(\cdot)$ in general differs from the nominal region of attraction R of the state feedback. However, any compact level set $V(x) \leqslant c, c > 0$ contained in R is an inner estimate of $\tilde{\mathcal{R}}$ as shown in the proof of Lemma 5.1, since $\dot{V}(x) \leqslant 0$ for all $x \in \mathcal{R}$.

Remark 5.1. Strictly speaking Lemma 5.1 is only valid for the unconstrained case. However, in the presence of constraints the exponential convergence result at least holds locally. This follows from the fact that the local control law $k(x)$ corresponding to the choice of F and E renders the origin (locally) exponentially stable. Since the NMPC feedback leads to a lower cost than the local control law, local exponential stability of the NMPC controller follows.

Lemma 5.1 establishes the exponential stability of the closed loop in the case of input uncertainties using the state feedback NMPC controller. This result can be used to show closed loop robustness with respect to the considered input uncertainties in the output feedback case, which will be done in the remainder of the section.

5.2. Robustness of output feedback NMPC using high-gain observers

Using Lemma 5.1, and the robustness of the observer to modeling errors in ϕ [2],[3] one can adapt Theorem 5 in [2] to the present case:

Theorem 5.1. Assume that the assumptions of Theorem 4.1 and Assumptions 5.1–5.4 hold. Then for any compact subset $\mathcal{S} \subset \tilde{\mathcal{R}}$ and for any observer initial condition that satisfies $\hat{x}_1(0) = \hat{x}_{10} \in \mathcal{Q}$ with $\mathcal{Q}$ arbitrary but fixed and compact there exists an ε^* such that for $0 < \varepsilon \leqslant \varepsilon^*$ the system

$$\dot{x}_1 = Ax_1 + B\tilde{\phi}(x)\Delta(u)$$

$$\dot{x}_2 = \tilde{\psi}_1(x) + \tilde{\psi}_2(x)\Delta(u)$$

$$y^\top = \left[(Cx_1)^\top x_2^\top\right]$$

with

$$\frac{1}{2}s^\top s \leqslant s^\top \Delta(s) \leqslant \infty, \qquad \forall s \in \mathbb{R}^m,$$

controlled by the output feedback NMPC scheme using the model given by (6) in the controller and observer and the cost (7) is exponentially stable and has a region of attraction of at least $\mathcal{S}$. Further, the performance of the state feedback NMPC controller is recovered in the sense of Definition 4.1.

Proof. Utilizing Lemma 5.1 the proof follows from [2, Theorem 5]. □

6. Discussion of results

In the previous sections we outlined an output feedback NMPC scheme using a high-gain controller for state recovery. As shown, the scheme does lead to nominal stability. Moreover, based on a robust exponential stability result for state feedback NMPC, we showed that the output feedback controller is robustly stable for certain classes of (unknown) static input nonlinearities.

The results are based on the assumption that the NMPC controller is time continuous/instantaneous. In

[2] As the proof reveals, more general $R(x)$ and $\Delta(\cdot)$ satisfying $u^\top R(x)[\Delta(u) - \frac{1}{2}u] \geqslant 0$ can be tolarated. This is, however, not elaborated on in any further detail here.

[3] The modeling errors in ψ are not important for the estimation part, since the x_2-states are assumed to be measured.

practice, it is of course not possible to solve the nonlinear optimization problem instantaneously. Instead, typically, the open-loop optimal control problem will be solved only at certain sampling instants. The first part of the obtained control signal is then applied to the system, until the next sampling instant. Also some time is needed to compute the solution of the optimal control problem, thus the computed control is based to some degree on old information, introducing delay in the closed loop. In practice this requires that the dynamics of the process is slow compared to the NMPC sampling interval and to the time needed to solve the optimization problem. Note that some preliminary results with respect to the "standard" sampled NMPC setup can be found in [10,11].

One of the drawbacks of high-gain observers is that in a transient phase, due to the so called peaking phenomena [9,22], the observed state may be outside the region where the NMPC optimization problem has a feasible solution. As specified in Remark 3.1 in this case the input must be assigned some fall-back value. Under this condition the structure of the high-gain observer and the bounded inputs ensure [2] that ε can be chosen small enough so that the observer state converges to the true state before the true state leaves the region of attraction (and hence the feasibility area) of the NMPC controller.

It is assumed that the optimal control is Lipschitz in the initial state. In general, the solution of an optimal control problem (and hence, the state feedback defined in Assumption 3.2) can be non-Lipschitz in the initial values. In particular, it is known that NMPC can stabilize systems that are not stabilizable by continuous control [12].

7. Examples

In this section the derived results on the recovery of performance and the robustness of the output feedback control scheme with respect to sector bounded input uncertainties are verified considering two example systems: the control of a continuous mixed culture bioreactor and the control of an uncertain inverted pendulum on a cart.

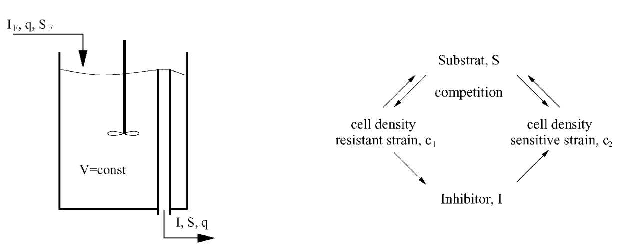

Fig. 2. Schematic diagram of the continuous mixed culture bioreactor and the strain/inhibitor interactions.

7.1. Control of a continuous mixed culture bioreactor

To demonstrate that the proposed output feedback NMPC scheme recovers the performance of the corresponding state feedback controller, the control of a continuous mixed culture bioreactor as presented in [13] is considered, see Fig. 2. The system consists of a culture of two cell strains, in the following called Species 1 and 2, that have different sensitivity to an external growth-inhibiting agent. The interactions of the two cell populations are illustrated in the right part of Fig. 2. The cell density of the inhibitor resistant strain is denoted by c_1, the cell density of the inhibitor sensitive strain is denoted by c_2, and the substrate and inhibitor concentrations in the reactor are denoted by S and I. Based on the full model described in [13] a reduced third order model of the following form can be obtained

$$\frac{\mathrm{d}c_1}{\mathrm{d}t} = \mu_1(S)c_1 - c_1u_1,$$

$$\frac{\mathrm{d}c_2}{\mathrm{d}t} = \mu_2(S,I)c_2 - c_2u_1,$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -pc_1I + u_2 - Iu_1.$$

The inputs are the dilution rate u_1 and the inhibitor addition rate u_2. The deactivation constant of the inhibitor for Species 2 is denoted by p. The specific growth rates $\mu_1(S)$ and $\mu_2(S,I)$ are given by

$$\mu_1(S) = \frac{\mu_{1,M}S}{K+S},\ \mu_2(S,I) = \frac{\mu_{2,M}S}{K+S}\frac{K_I}{K_I+I}$$

where K, K_I, $\mu_{1,M}$ and $\mu_{2,M}$ are constant parameters as specified in [13]. The substrate concentration is given by

$$S = S_f - \frac{c_1}{Y_1} - \frac{c_2}{Y_2}.$$

Here Y_1, Y_2 are the yields of the species and S_f is the substrate inlet concentration. The control objective is to stabilize the steady state $c_{1s} = 0.016$ g/l, $c_{2s} = 0.06$ g/l, $I_s = 0.005$ g/l. The outlined output feedback NMPC scheme is used to achieve this objective. The measured outputs are given by

$$y = \left[\ln\frac{c_1}{c_2}, c_1\right]^{\top}.$$

Performing the following coordinate transformation

$$z_1 = \ln\frac{c_1}{c_2}, \quad z_2 = \mu_1(S) - \mu_2(S,I), \quad z_3 = c_1,$$

the transformed system is of the model structure assumed in Section 2 ($x_1 = [z_1, z_2]^{\top}, x_2 = z_3$). As state feedback NMPC scheme, the quasi-infinite horizon NMPC strategy with the sampling time set to zero is used. The cost F weighs the quadratic deviation of the states and inputs in the new coordinates from their steady state values. For simplicity, unit weights on all states and inputs are considered. The horizon T_p is set to 20 h. A quadratic upper bound E on the infinite horizon cost and a terminal region Ω satisfying the assumptions of [5] are calculated using LMI/PLDI-techniques [3]. The piecewise linear differential inclusion (PLDI) representing the dynamics in a neighborhood of the origin is found using the methods described in [26]. The states z_1 and z_2 are estimated from the measurements y_1 and y_2 via a high-gain observer as described in Section 3.2. The parameters α_1 and α_2 in the observer are chosen to $\alpha_1 = \sqrt{2}$, $\alpha_2 = 1$. To show the recovery of performance different values of the high-gain parameter ε of the observer are compared. In all shown simulations the observer is initialized with the correct values for z_1 and z_3 (since they can be directly obtained from the measurements), whereas z_2 is assumed unknown and initialized with the steady state value. Fig. 3. exemplary shows closed loop system trajectories projected onto the c_1/c_2 phase plane for different observer gains $\frac{1}{\varepsilon}$ in comparison to the state feedback NMPC controller starting from the same initial condition. As can be seen, the larger the observer gain (the smaller ε), the closer the trajectories converge to the state feedback case. Fig. 4 shows the corresponding time behavior of the inhibitor concentration I (related to the unmeasured state z_2) and the inhibitor addition rate (input u_2) for different values of ε. Additionally, the real cost occurring, i.e. the integrated quadratic error between the steady state values for the states and inputs in transformed coordinates, is plotted. The cost of the output feedback controller approaches the cost of the state feedback controller for lower ε, which shows the recovery of performance. Notice that we use relatively low gains for the observer,

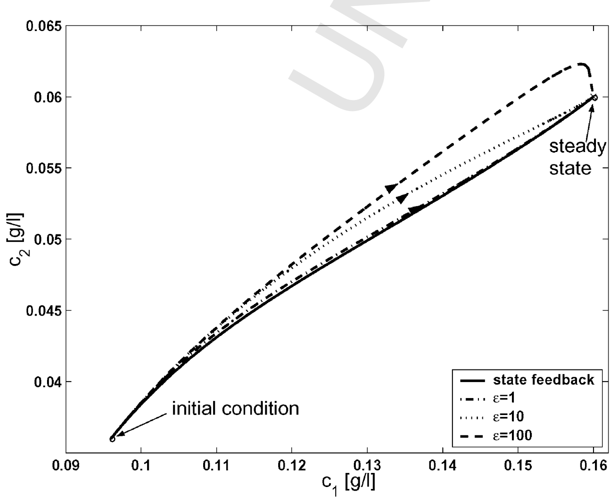

Fig. 3. Phase plot of c_1 and c_2.

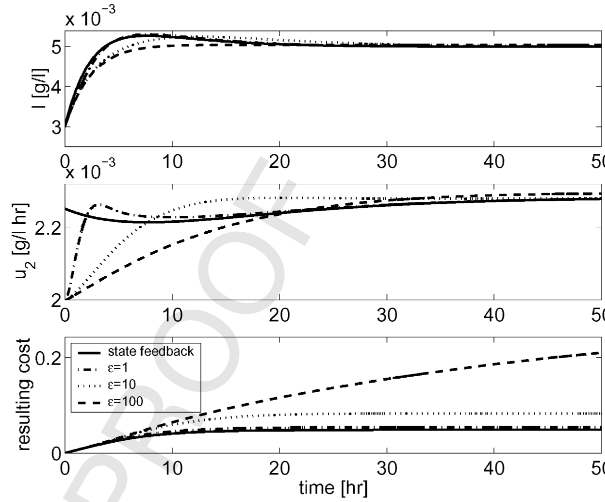

Fig. 4. Trajectories of I, u_2 and summed up cost.

meaning that ε is large. Higher observer gains can lead to problems in case of measurement noise. This is often considered as the main limitation using high-gain observers for state estimation.

This example verifies the stability of the closed loop and the recovery of performance for increasing values of the observer gain. In the next section, an unstable example system is considered to show the recovery of the region of attraction and the robustness to a sector bounded input uncertainty.

7.2. Control of an inverted pendulum

This section considers the control of an (unstable) inverted pendulum on a cart. The parameters and model equations of the cart-pendulum system are taken from [8]. Fig. 5 schematically shows the inverted pendulum on a cart system. The angle of the pendulum with the vertical axis is denoted by z_1. The input to the system is given by the force u which acts on the cart's translation and is limited to $-10\text{N} \leqslant u(t) \leqslant 10\text{N}$. The control objective is to stabilize the angle $z_1 = 0$ (upright position) while the cart's position is not limited (and thus not modeled and controlled). It is assumed that only the angle z_1 but not the angular velocity can be measured

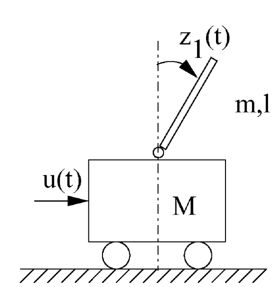

Fig. 5. Inverted pendulum on a cart.

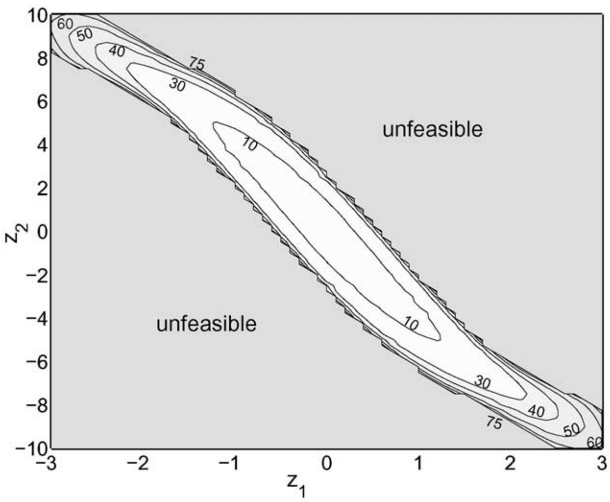

Fig. 6. Level sets of the quasi infinite horizon state feedback NMPC controller value function.

directly. The model of the system is given by the following equations:

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = \frac{ml\cos(z_1)\sin(z_1)z_2^2 - g(m+M)\sin(z_1) + \cos(z_1)\gamma u}{ml\cos^2(z_1) - \frac{4}{3}(m+M)l}$$

$$y = z_1$$

where z_2 is the angular velocity of the pendulum. The parameters $M = 1$ kg, $m = 0.2$ kg, $l = 0.6$ m and $g = 10\,\frac{\text{m}}{\text{s}^2}$ are constant. With respect to the "input gain" γ we consider that it is uncertain (but constant) and lies between $\gamma \in [1/2, 2]$. This uncertainty could for example result from an uncertainty in the motor constants of the motor that provides the necessary force (moment) on the cart. The nominal value of γ is 1. This model fits, besides the assumed input constraints, in the model class considered in Section 5 ($x_1 = [z_1, z_2]$, no x_2).

Similar to the first example, the stage cost is quadratic and the weights on the states and input are chosen as unit weights for simplicity, i.e. $F(z,u) = z^\top \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z + u^2$. As state feedback QIH-NMPC is used. The terminal penalty cost E and the terminal region Ω are obtained using the same techniques as for the continuous mixed culture bioreactor. The resulting terminal penalty cost E is given by: $E(z) = z^\top \begin{bmatrix} 311.31 & 66.20 \\ 66.20 & 34.99 \end{bmatrix} z$, and the terminal region Ω is given by $\Omega = \{z \in \mathbb{R}^2 | E(z) \leqslant 20\}$ The control horizon T_p is chosen to 0.5 s. In Fig. 6 the region of attraction and the contour lines of the value function of the state feedback NMPC controller are shown. These results are obtained solving the open loop state feedback NMPC problem for different initial conditions of z_1 and z_2. In the output feedback case, whenever the state estimate leaves the region of attraction of the state feedback QIH-NMPC scheme (i.e. there is no solution to the open loop optimization problem) the input is set to 0, compare Remark 3.1.

The states z_1 and z_2 are estimated from y using the described high-gain observer. The observer parameters α_1 and α_2 are chosen to $\alpha_1 = 2$ and $\alpha_2 = 1$. For all subsequent simulations the observer is started with zero initial conditions, i.e. $\hat{z}_1 = \hat{z}_2 = 0$.

Fig. 7 shows the phase plot of the system states and the observer states of the closed loop system for different values of ε for $\gamma = 1$ (nominal system). As expected, for decreasing values of ε the trajectories of the state feedback control scheme are recovered. Comparing both plots one sees that for $\varepsilon = 0.1$, when the observer state and the real state are at the boundary of the region of attraction of the state feedback controller, a small estimation error does lead to infeasibility of the open loop problem and thus to divergence. For smaller values of ε the correct state is recovered faster and infeasibility/divergence are avoided. However, for smaller values of ε a bigger (but time-wise shorter) peaking of the observer

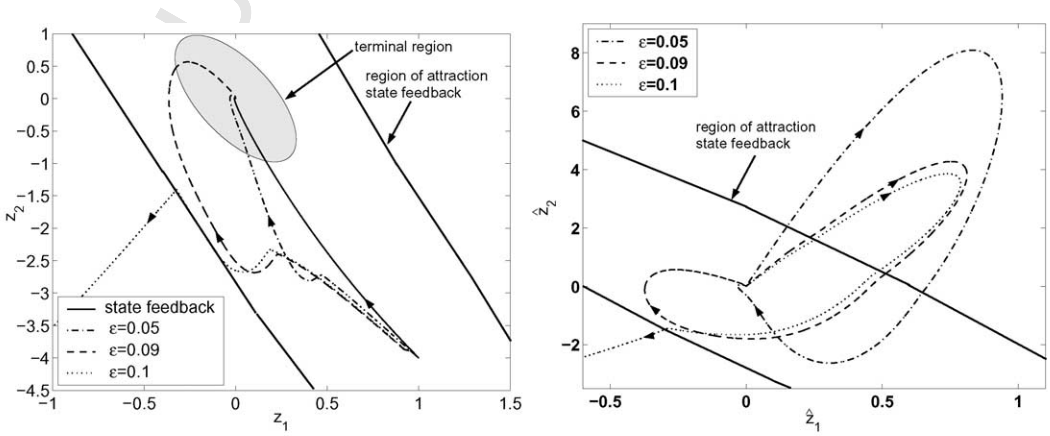

Fig. 7. Phase plot of the nominal system states (left) and the observer states (right) for $\gamma = 1$.

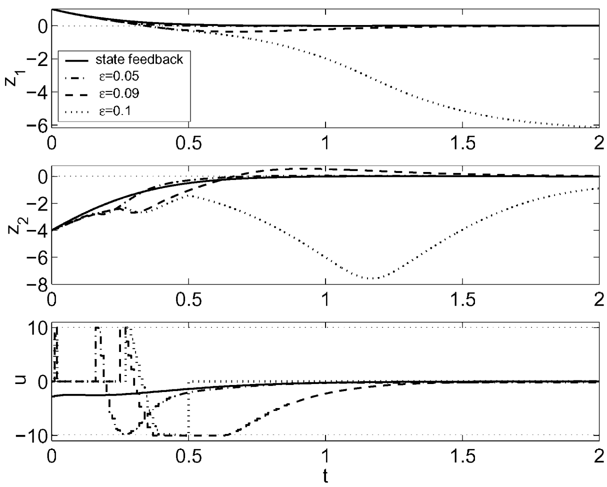

Fig. 8. Trajectories of z_1, z_2 and the input u.

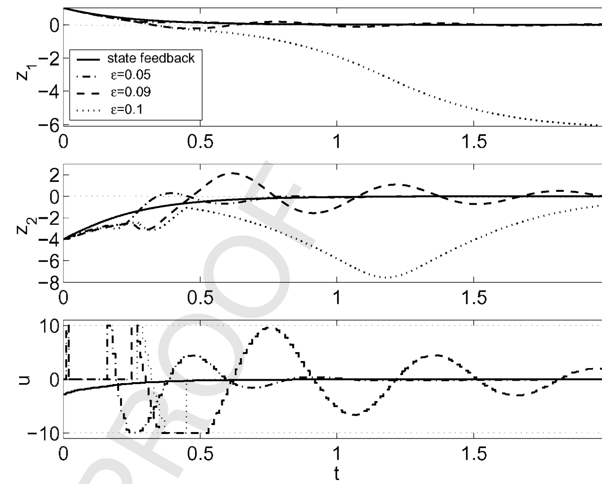

Fig. 9. Trajectories z_1, z_2 and u in case of input uncertainty ($\gamma = 2$).

error at the beginning occurs, see Fig. 7, right plot. This is also evident in the time plot of the states and inputs as shown in Fig. 8. Notice also that in the state feedback case for the initial conditions shown the input constraints are not hit, while for all output feedback cases the NMPC controller hits the input constraints.

To show the robustness with respect to sector bounded input nonlinearities, Fig. 9 shows the trajectories of the closed loop system for a value of $\gamma = 2$. In this case, the observer and NMPC controller use the nominal value of $\gamma_{\text{nom}} = 1$. The controller is still able to stabilize the system despite the gain uncertainty, however the performance degrades.

The given examples underpin the derived results on the recovery of the region of attraction and performance as well as the robustness for the high-gain observer based NMPC strategy. As shown, for a too slow high-gain observer the closed loop trajectories may diverge from a given initial condition. However, sufficiently small values of ε do lead to closed loop stability and satisfying recovery of performance.

8. Conclusions

Nonlinear model predictive control has received considerable attention during the past decades. However, no significant progress with respect to the output feedback case has been made. The existing solutions are either of local nature [16,24] or difficult to implement [21]. In this paper employing results from [2], an NMPC output feedback strategy is presented that achieves semi-regional stability and recovery of performance. The scheme consists of an NMPC state feedback controller and a high-gain observer. Besides nominal stability, the scheme possesses some robustness properties with respect to (unknown) sector bounded input nonlinearities. The main restrictions of the scheme are: (i) the special system structure assumed; (ii) that the NMPC controller is assumed to compute control solutions instantaneously; and (iii) that the optimal input of the NMPC controller must be locally Lipschitz as a function of the state. Results considering a sampled-data NMPC scheme instead of the instantaneous scheme, in addition to expanding the considered system class are suggested in [10,11].

From a practical perspective, one should additionally note the inherent problem of high gain observers with respect to measurement noise, which may restrict the applicability. As a consequence, the derived results should not be seen as directly applicable in practice. Instead, the results should mainly be regarded as an intermediate step towards a practically suitable output feedback NMPC scheme with guaranteed stability.

Appendix A

Proof of Lemma 5.1. For NMPC robust *asymptotic* stability results of this form have been derived in [17] and in [4]. Thus, we have to show that under the given assumptions also robust exponential stability is achieved.

Under essentially the same assumptions as used here, [4,17] show that the NMPC control law is inverse optimal, i.e. it is also optimal for a modified optimal control problem spanning over an infinite horizon with the cost function

$$\bar{J}(x, u(\cdot); \infty) = \int_0^\infty \bar{l}(x(\tau)) + u^\top(\tau) R(x(\tau)) u(\tau) \mathrm{d}\tau$$

where

$$\bar{l}(x) = l(x) - \frac{\partial}{\partial T_p} V(x; T_P).$$

Also the NMPC value function is the value function for the infinite horizon problem, i.e. $V(x;T_p) = \bar{V}(x;\infty)$ where $\bar{V}$ is the value function associated with the cost $\bar{J}$. Due to this inverse optimality in the nominal case the NMPC state feedback control scheme has the same (asymptotic) robustness properties (stability margins) as infinite horizon optimal control [17].

As noted in [4,17] the optimal control can be written as $u^{\star}(\tau = 0; x) = \gamma(x)$, where

$$\gamma(x) = -\tfrac{1}{2}R^{-1}(x)[V_x g[x]]^{\top},$$

with $V_x := \frac{\partial V(x;T_p)}{\partial x}(x;T_p)$. Furthermore, the nominal system satisfies

$$V_x f(x) + V_x g(x)\gamma(x) = -\bar{l}(x) - \gamma^{\top}(x)R(x)\gamma(x).$$

For the real system with the unknown static input nonlinearity, $\dot{V}$ is given by

$$\begin{aligned}
\dot{V}(x;T_p) &= V_x f(x) + V_x g(x)\Delta(\gamma(x)) \\
&= V_x f(x) + V_x \gamma(x) \\
&\quad + [V_x g(x)\Delta(\gamma(x)) - V_x g(x)\gamma(x)] \\
&= -\bar{l}(x) - \gamma^{\top}(x)R(x)\gamma(x) \\
&\quad + [V_x g(x)\Delta(\gamma(x)) - V_x g(x)\gamma(x)] \\
&= -\bar{l}(x) + \left[V_x g(x)\Delta(\gamma(x)) - \frac{1}{2}V_x g(x)\gamma(x)\right] \\
&= -\bar{l}(x) - 2\gamma^{\top}(x)R(x)\left[\Delta(\gamma(x)) - \frac{1}{2}\gamma(x)\right].
\end{aligned}$$

Since $R(x)$ and $\Delta(x)$ are diagonal it follows that

$$\dot{V}(x;T_p) \leqslant -\bar{l}(x) = -l(x) + \frac{\partial}{\partial T_p}V(x;T_P).$$

Additionally, we know [4,17] that $\frac{\partial}{\partial T_p}V(x;T_P) \leqslant 0$ 0. Thus, using Assumption 5.2 yields

$$\dot{V}(x;T_p) \leqslant -l(x) \leqslant -c_F\|x\|^2. \qquad (11)$$

Consequently, V is strictly decreasing along solution trajectories. Furthermore, for V to be a Lyapunov function showing exponential stability, it is required that V can be quadratically lower and upper bounded. Due to Assumptions 5.3 and 5.2 there exist constants $c_1 > 0, c_2 > 0, r > 0$ such that for all x with $\|x\| \leqslant r$

$$c_1\|x\|^2 \leqslant V(x;T_p) \leqslant c_2\|x\|^2,$$

This, together with (11) implies that $V(x;T_p)$ is a valid Lyapunov function showing exponential stability. □

References

[1] F. Allgöwer, T.A. Badgwell, J.S. Qin, J.B. Rawlings, S.J. Wright, Nonlinear predictive control and moving horizon estimation—an introductory overview, in: P.M. Frank (Eds.), Advances in Control, Highlights of ECC'99, Springer, 1999, pp. 391–449.

[2] A.N. Atassi, H.K. Khalil, A separation principle for the stabilization of a class of nonlinear systems, IEEE Trans. Automatic Control 44 (9) (1999) 1672–1687.

[3] S. Boyd, L. El Ghaoui, E. Feron, V. Balakishnan, Linear matrix inequalities in system and control theory, SIAM, Philadelphia, 1994.

[4] C.C. Chen, L. Shaw, On receding horizon feedback control, Automatica 18 (3) (1982) 349–352.

[5] F. Allgöwer, H. Chen, Nonlinear model predictive control schemes with guaranteed stability, in: R. Berber, C. Kravaris (Eds.), Nonlinear Model Based Process Control, Kluwer Academic, Dodrecht, 1998, pp. 465–494.

[6] H. Chen, F. Allgöwer, A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability, Automatica 34 (10) (1998) 1205–1218.

[7] L. Magni, G. De Nicolao, R. Scattolini, Stability and robustness of nonlinear receding horizon control, in: A. Zheng, F. Allgöwer. (Eds.), Nonlinear Predictive Control, Birkhäuser, 2000, pp. 3–23.

[8] S. Dussy, L.E. Ghaoui, Robust gain-scheduled control of a class of nonlinear parameter-dependent systems: application to an uncertain inverted pendulum, in: Proc. of the IEEE International Conference on Control Applications, pp. 516–521, 1996.

[9] F. Esfandiari, H.K. Khalil, Output feedback stabilization of fully linearizable systems, Int. J. Control 56 (5) (1992) 1007–1037.

[10] R. Findeisen, L. Imsland, F. Allgöwer, B.A. Foss, Output feedback nonlinear predictive control—a separation principle approach, in: Proceedings of 15th IFAC World Congress, 2002.

[11] R. Findeisen, L. Imsland, F. Allgöwer, B.A. Foss, Output feedback stabilization for constrained systems with nonlinear model predictive control. Submitted to International Journal on Robust and Nonlinear Control, 2002.

[12] F.A. Fontes, A general framework to design stabilizing nonlinear model predictive controllers, Syst. Contr. Lett. 42 (2) (2000) 127–143.

[13] K.A. Hoo, J.C. Kantor, Global linearization and control of a mixed culture bioreactor with competition and external inhibition, Math. Biosci. 82 (1986) 43–62.

[14] A. Isidori, Nonlinear control systems: an introduction, third ed., Springer Verlag, Berlin, 1995.

[15] A. Jadbabaie, J. Yu, J. Hauser, Unconstrained receding horizon control of nonlinear systems, IEEE Trans. Automat. Contr. 46 (5) (2001) 776–783.

[16] L. Magni, D. De Nicolao, R. Scattolini. Output feedback receding-horizon control of discrete-time nonlinear systems, in: Preprints of the 4th Nonlinear Control Systems Design Symposium 1998-NOLCOS'98. IFAC, July 1998, pp. 422–427.

[17] L. Magni, R. Sepulchre, Stability margins of nonlinear receding–horizon control via inverse optimality, Syst. Contr. Lett. 32 (4) (1997) 241–245.

[18] D.Q. Mayne, H. Michalska, Receding horizon control of nonlinear systems, IEEE Trans. Automat. Contr. 35 (7) (1990) 814–824.

[19] D.Q. Mayne, J.B. Rawlings, C.V. Rao, P.O.M. Scokaert, Constrained model predictive control: stability and optimality, Automatica 26 (6) (2000) 789–814.

[20] H. Michalska, D.Q. Mayne, Robust receding horizon control of constrained nonlinear systems, IEEE Trans. Automat. Contr. AC-38 (11) (1993) 1623–1633.

[21] H. Michalska, D.Q. Mayne, Moving horizon observers and observer-based control, IEEE Trans. Automat. Contr. 40 (6) (1995) 995–1006.

[22] V.N. Polotskii, On the maximal errors of an asymptotic state identifier, Automat. Remote Contr. 11 (1979) 1116–1121.

[23] T.A. Badgwell, S.J. Qin, An overview of industrial model predictive control technology, in: C.E. Garcia, J.C. Kantor, B. Carnahan (Eds.), Fifth International Conference on Chemical Process Control—CPC V, American Institute of Chemical Engineers, 1996, pp. 232–256.

[24] P.O.M. Scokaert, J.B. Rawlings, E.S. Meadows, Discrete-time stability with perturbations: application to model predictive control, Automatica 33 (3) (1997) 463–470.

[25] M. Janković, M. Sepulchre, P.V. Kokotović, Constructive nonlinear control, Springer-Verlag, Berlin, 1997.

[26] O. Slupphaug, L. Imsland, B. Foss, Uncertainty modelling and robust output feedback control of nonlinear discrete systems: a mathematical programming approach, Int. J. Robust Nonlinear Contr. 10 (13) (2000) 1129–1152.

[27] A. Teel, L. Praly, Global stabilizability and observability imply semi-global stabilizability by output feedback, Syst. Contr. Lett. 22 (4) (1994) 313–325.

[28] A. Tornambè, High-gain observers for non-linear systems, Int. J. Syst. Sci. 23 (9) (1992) 1475–1489.