Exercises from the text book

4.18

Coin number one comes up heads with probability 0.6 and coin number two with probability 0.5. A coin is continually flipped until it comes up tails, at which time that coin is put aside and we start flipping the other one.

Define state 0 to flip coin number one, and state 1 to flip coin number two. Transition matrix

\[
P = \begin{bmatrix}
0.6 & 0.4 \\
0.5 & 0.5
\end{bmatrix}
\]

a) Want to know what proportion of flips we use coin number one.

We find the stationary probabilities by solving

\[
\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, j \geq 0, \text{ where } \sum_{j=0}^{\infty} \pi_j = 1
\]

In this case we get

\[
\begin{align*}
\pi_0 &= 0.6\pi_1 + 0.5\pi_1 \\
\pi_1 &= 0.4\pi_0 + 0.5\pi_1 \\
\pi_0 + \pi_1 &= 1
\end{align*}
\]

By solving the system of equations, we get \(\pi_0 = \frac{5}{9}\) and \(\pi_1 = \frac{4}{9}\). This means that we will use coin number one in \(\frac{5}{9}\) of the flips.

b) If we start the process with coin number one, what is the probability that coin number two is used in the fifth flip?

We have to find \(P_{10}^4\).

\[
P^2 = \begin{bmatrix}
0.6 & 0.4 \\
0.5 & 0.5
\end{bmatrix} \begin{bmatrix}
0.6 & 0.4 \\
0.5 & 0.5
\end{bmatrix} = \begin{bmatrix}
0.56 & 0.44 \\
0.55 & 0.45
\end{bmatrix}
\]
\[
P^4 = p^2 p^2 \begin{bmatrix} 0.56 & 0.44 \\ 0.55 & 0.45 \end{bmatrix} \begin{bmatrix} 0.56 & 0.44 \\ 0.55 & 0.45 \end{bmatrix} = \begin{bmatrix} 0.5556 & 0.4444 \\ 0.5555 & 0.4445 \end{bmatrix}
\]

We see that the probability is \( P_{01}^4 \approx 0.444 \approx \frac{4}{9} \).

**4.20**

- \( P \) is doubly stochastic \( \iff \sum_i P_{ij} = 1 \ \forall j \)
- Let the chain be irreducible and aperiodic, and consist of \( M + 1 \) states \( 0, 1, \ldots, M \)
- Show that the limiting distributions are given by:

  \[
  \pi_j = \frac{1}{M+1} \quad j = 0, \ldots, M
  \]

(1)

- As the chain is aperiodic and irreducible, the limiting distributions are equal to the stationary distributions which fulfil:
  
  i) \quad \pi_j = \sum_i \pi_i P_{ij}
  
  ii) \quad \sum_i \pi_i = 1

- We see that by putting \( \pi_j = \frac{1}{M+1} \ \forall j \), we obtain:
  
  i) \quad \pi_j = \sum_i \pi_i P_{ij} = \sum_i \frac{1}{M+1} P_{ij} = \frac{1}{M+1} \sum_i P_{ij} = \frac{1}{M+1}
  
  ii) \quad \sum_i \pi_i = \sum_{i=0}^{M} \frac{1}{M+1} = \frac{1}{M+1} \cdot (M+1) = 1

- Therefore (1) is the limiting distributions of the Markov chain.
4.25

- Let $X_n$ be the number of pairs of shoes at the front door at time $n$. We have that:

$$P_{01} = P(X_n = 1 \mid X_{n-1} = 0) = P(\text{Departs from back door, enters at front door} \mid X_{n-1} = 0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P_{00} = 1 - P_{01} = \frac{3}{4} = P_{kk}$$

$$0 < i < k \begin{cases} P_{i-1,i} = P(\text{Departs from back door, enters at front door} \mid X_{n-1} = i - 1) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8} \\ P_{i,i-1} = P(\text{Departs from front door, enters at back door} \mid X_{n-1} = i - 1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \\ P_{ii} = 1 - \left(\frac{1}{4} + \frac{1}{8}\right) = \frac{1}{2} \end{cases}$$

Transition matrix:

$$P = \begin{bmatrix}
\frac{3}{4} & \frac{1}{4} & 0 & \ldots & \ldots & \ldots & 0 \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{2} & 0 & \ldots & \ldots & 0 \\
0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \ldots & 0 \\
0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \ldots & \ldots & \frac{1}{4} & \frac{3}{4}
\end{bmatrix}$$

- The transition matrix is doubly stochastic (see exercise 4.20), hence $\pi_j = \frac{1}{k+1} \forall j$ ($P$ is both irreducible and aperiodic)

- We have:

$$P(\text{runs barefooted}) = P(X_n = 0 \mid \text{front door}) \cdot P(\text{front door}) + P(X_n = k \mid \text{back door}) \cdot P(\text{back door})$$

$$= \pi_0 \cdot \frac{1}{2} + \pi_k \cdot \frac{1}{2}$$

$$= \frac{1}{k+1} \cdot \frac{1}{2} + \frac{1}{k+1} \cdot \frac{1}{2}$$

$$= \frac{1}{k+1}$$

4.32

See solution in the book.
Exercises from exams

Eks. Jan. 98, oppg. 2

a)

When \( X_n = 0 \) we need to have \( X_{n+1} \sim \text{bin}(12, 0.06) \) which means
\[
P(X_{n+1} = 2 \mid X_n = 0) = \binom{12}{2} \cdot 0.06^2 \cdot 0.94^{10} = 0.128
\]
similarly
\[
P(X_{n+1} = 0 \mid X_n = 2) = \binom{10}{0} \cdot 0.06^0 \cdot 0.94^{10} \cdot \binom{2}{0} \cdot 0.5^0 \cdot 0.5^2 = 0.135
\]
Let \( A_n \) be the number of members who both have a car on day \( n \) and want a car on day \( n+1 \).
\[
P(X_{n+1} = 1 \mid X_n = 1) = P(X_{n+1} = 1 \mid X_n = 1, A_n = 0) \cdot P(A_n = 0 \mid X_n = 1)
+ P(X_{n+1} = 1 \mid X_n = 1, A_n = 1) \cdot P(A_n = 1 \mid X_n = 1)
= \binom{11}{1} \cdot 0.06^1 \cdot 0.94^{10} \cdot 0.5 + \binom{11}{0} \cdot 0.06^0 \cdot 0.94^{11} \cdot 0.5
= 0.431
\]
b)

\( W_n \): amount payed by the members on day \( n \).
\[
W_n = \begin{cases} 
X_n \cdot 125 & \text{if } X_n \leq 2 \\
2 \cdot 125 & \text{if } X_n = 3
\end{cases}
\]

In the long run:
\[
E[W_n] = 0 \cdot \pi_0 + 125 \cdot \pi_1 + 250 \cdot \pi_2 + 250 \cdot \pi_3
= 136.25
\]
Number of cars rented each day in the long run is:
\[
0 \cdot \pi_0 + 1 \cdot \pi_1 + 2 \cdot \pi_2 + 2 \cdot \pi_3 = 1.09
\]
Per wants to use his part of this, that is, the number of days Per uses a car is:

\[
P(\text{Per uses}) = \frac{1.09}{12} = 0.0908
\]

\[
P(\text{Per uses} \mid \text{Per wishes}) = \frac{P(\text{Per uses} \cap \text{Per wishes})}{P(\text{Per wishes})} = \frac{P(\text{Per uses})}{P(\text{Per wishes})}
\]

Have to find \(P(\text{Per wishes})\).

\[
P(\text{Per wishes on day } n)
= P(\text{Per wishes on day } n \mid \text{Per uses on day } n - 1) \cdot P(\text{Per uses on day } n - 1)
+ P(\text{Per wishes on day } n \mid \text{Per does not use on day } n - 1) \cdot P(\text{Per does not use on day } n - 1)
\]
\[
= 0.5 \cdot \frac{1.09}{12} + 0.06 \cdot (1 - \frac{1.09}{12})
\]
\[
= 0.100
\]
such that :

\[
P(\text{Per uses} \mid \text{Per wishes}) = \frac{0.0908}{0.100} = 0.909
\]

c) The behaviour of the members becomes dependent upon each other. Define:

\[
U_n = \begin{cases} 
1 & \text{if a certain member wishes to use a car} \\
0 & \text{otherwise}
\end{cases}
\]

This is a Markov chain with transition matrix

\[
P = \begin{bmatrix} 
0.94 & 0.06 \\
0.5 & 0.5
\end{bmatrix}
\]

The limiting distribution is given by:

\[
\begin{align*}
\pi_0 &= 0.94 \cdot \pi_0 + 0.5 \cdot \pi_1 \\
\pi_1 &= 0.06 \cdot \pi_0 + 0.5 \cdot \pi_1 \\
1 &= \pi_0 + \pi_1
\end{align*}
\]

\[
\Rightarrow \pi_0 = 0.8929 \quad \pi_1 = 0.1071
\]
Let $Y_n$ be the number of members who wish a car on day $n$. In the long run we get:

$$Y_n \sim \text{bin}(12, 0.1071)$$

such that:

- $P(Y_n = 0) = 0.2568$
- $P(Y_n = 1) = 0.3697$
- $P(Y_n = 2) = 0.2439$
- $P(Y_n \geq 3) = 0.1297$

Let:

- $V_n$: net income at day $n$, that is,
  
  $$V_n = \begin{cases} 
  125 \cdot Y_n & \text{if } Y_n \leq 2 \\
  125 \cdot Y_n - 250 \cdot (Y_n - 2) & \text{if } Y_n \geq 3 
  \end{cases}$$

In the long run:

$$E[V_n] = 125 \cdot P(Y_n = 1) + 250 \cdot P(Y_n = 2)$$

$$+ \sum_{i=3}^{12} (500 - 250 \cdot i) \cdot P(Y_n = i)$$

$$= 125 \cdot P(Y_n = 1) + 250 \cdot P(Y_n = 2)$$

$$+ 500 \cdot P(Y_n \geq 3) - 125 \cdot \sum_{i=3}^{12} i \cdot P(Y_n = i)$$

$$= E[Y_n] - P(Y_n = 1) - 2 \cdot P(Y_n = 2)$$

$$= 125 \cdot 0.3697 + 250 \cdot 0.2439 + 500 \cdot 0.1297 - 125 \cdot (12 \cdot 0.1071 - 0.3697 - 2 \cdot 0.2439)$$

$$= 118.58$$

**Eks. Aug. 98, oppg. 1**

a)

$$Pr\{X_2 = 1|X_0 = 0\} = Pr\{X_1 = 0 \land X_2 = 1|X_0 = 0\} = Pr\{X_1 = 0|X_0 = 0\} \cdot Pr\{X_2 = 1|X_1 = 0\}$$

$$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$
\[ Pr\{X_3 = 0|X_0 = 0\} = Pr\{X_1 = 0 \cap X_2 = 0 \cap X_3 = 0|X_0 = 0\} + Pr\{X_1 = 0 \cap X_2 = 1 \cap X_3 = 0|X_0 = 0\} + Pr\{X_1 = 1 \cap X_2 = 0 \cap X_3 = 0|X_0 = 0\} + Pr\{X_1 = 1 \cap X_2 = 2 \cap X_3 = 0|X_0 = 0\} \]
\[ = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} \]

\[ Pr\{X_4 = 3|X_0 = 0\} = Pr\{X_1 = 0 \cap X_2 = 1 \cap X_3 = 2 \cap X_4 = 3|X_0 = 0\} + Pr\{X_1 = 1 \cap X_2 = 2 \cap X_3 = 3 \cap X_4 = 3|X_0 = 0\} \]
\[ = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \frac{3}{16} = 0,1875 \]
b) Let $v_i = E[T | X_0 = i], T = \min_{n \geq 0} \{ n : X_n = 3 \}$.

First step analysis gives the equations

\[
\begin{align*}
v_0 &= \frac{1}{2}v_0 + \frac{1}{2}v_1 + 1 \\
v_1 &= \frac{1}{2}v_0 + \frac{1}{2}v_2 + 1 \\
v_2 &= \frac{1}{2}v_0 + \frac{1}{2}v_3 + 1 \\
v_3 &= 0
\end{align*}
\]

which give

\[
\begin{align*}
v_2 &= \frac{1}{2}v_0 + 1 \\
v_1 &= \frac{1}{2}v_0 + \frac{1}{2}(\frac{1}{2}v_0 + 1) + 1 = \frac{3}{4}v_0 + \frac{3}{2} \\
v_0 &= \frac{1}{2}v_0 + \frac{1}{2}(\frac{3}{4}v_0 + \frac{3}{2}) + 1 = \frac{7}{8}v_0 + \frac{7}{4}
\end{align*}
\]

\[\Rightarrow v_0 = \frac{7}{4} \cdot 8 = 14\]

c) Let $Y_n$ be the number of the last flips who coincides with the start of the sequence MKK and get the transition matrix

\[
\begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & 1/2 & 1/2 & 0 & 0 \\
1 & 0 & 1/2 & 1/2 & 0 \\
2 & 0 & 1/2 & 0 & 1/2 \\
3 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

First step analysis gives the equations:

\[
\begin{align*}
v_0 &= \frac{1}{2}v_0 + \frac{1}{2}v_1 + 1 \\
v_1 &= \frac{1}{2}v_1 + \frac{1}{2}v_2 + 1 \\
v_2 &= \frac{1}{2}v_1 + \frac{1}{2}v_3 + 1 \\
v_3 &= 0
\end{align*}
\]

which give
\[ v_2 = \frac{1}{2}v_1 + 1 \]
\[ v_1 = \frac{1}{2}v_1 + \frac{1}{2}(\frac{1}{2}v_1 + 1) + 1 = \frac{3}{4}v_1 + \frac{3}{2} \]
\[ \Rightarrow v_1 = \frac{3}{2} \cdot 4 = 6 \]
\[ v_0 = \frac{1}{2}v_0 + \frac{1}{2} \cdot 6 + 1 \]
\[ \Rightarrow v_0 = 4 \cdot 2 = 8 \]

It is reasonable that this solution is less than in b) as you return to state 0 if you get a "wrong flip" in \( X_n \), and as you only return to state 1 with a "wrong flip" in \( Y_n \).

d) The system of equations for the stationary distribution

\[
\begin{align*}
\pi_0 &= \frac{1}{2}\pi_0 + \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{2}\pi_3 \\
\pi_1 &= \frac{1}{2}\pi_0 \\
\pi_2 &= \frac{1}{2}\pi_1 \\
\pi_0 + \pi_1 + \pi_2 + \pi_3 &= 1
\end{align*}
\]

such that

\[
\pi_0 = \frac{1}{2}\pi_0 + \frac{1}{4}\pi_0 + \frac{1}{8}\pi_0 + \frac{1}{2}\pi_3 \\
\Rightarrow \frac{1}{8}\pi_0 = \frac{1}{2}\pi_3 \Rightarrow \pi_3 = \frac{1}{4}\pi_0
\]

and

\[
\pi_0 + \frac{1}{2}\pi_0 + \frac{1}{4}\pi_0 + \frac{1}{4}\pi_0 = 2\pi_0 = 1 \Rightarrow \pi_0 = \frac{1}{2}
\]

\[
\pi_0 = \frac{1}{2}, \pi_1 = \frac{1}{4}, \pi_2 = \frac{1}{8}, \pi_3 = \frac{1}{8}
\]

The chain has a limiting distribution because it is irreducible and aperiodic \((P_{00} > 0)\). The limiting distribution is equal to the stationary distribution given above.

In the long run we will obtain MMM as the three final flips after \( \frac{1}{8} \) of the flips.