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A SIMPLE PATH-INDEPENDENT INTEGRAL FOR CALCULATING MIXED-MODE STRESS INTENSITY FACTORS

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Abstract—A path-independent integral is introduced for calculating stress intensity factors. The derivation of the integral is based on the application of the known Bueckner's fundamental field solution for a crack in an infinite body and on the reciprocal theorem. The method was applied to two-dimensional linear elastic mixed-mode crack problems. The key advantage of the present path-independent integral is that the stress intensity factor components for any irregular cracked geometry under any kind of loading can be easily obtained by a contour integral around the crack tip. The method is simple to implement and only the far field displacements and tractions along the contour must be known. The required stress analysis can be made by using any analytical or numerical method, e.g. the finite element method, without special consideration of the modelling of crack tip singularity. The application of this integral is also independent of the crack type, that is, there is no difference between an edge crack and an embedded crack, provided that the crack tip asymptotic behaviour exists.

INTRODUCTION

The basis for practical engineering application of linear elastic fracture mechanics in fracture and fatigue analysis of structures is the accurate and efficient calculation of stress intensity factors. This may be difficult, especially for complex geometries and loading cases, and for mixed-mode crack problems. Several methods for calculating stress intensity factors have appeared in the literature, see e.g. [1] and [2]. The most widely applied methods might be the virtual crack extension (VCE) method and the weight function method. The first one can be effectively implemented in the finite element method. But the VCE-method is not well suited for mixed-mode problems. On the other hand the weight function method can be extremely efficient in practical engineering applications. With the weight function method, the influence of geometry and load on the stress intensity factor can be separated. Once the weight function is known for a specific crack geometry in a given structure, the stress intensity factor under any loading case can be calculated by an integral of the crack-free tractions (tractions in the uncracked structure at the crack plane) multiplied by the weight function along the crack faces.

Weight function has two interpretations. One is the derivative of the displacement field with respect to crack length, known as Rice's weight function [3]. Another one is the fundamental field [1,4,5], which is generated by a pair of forces (F) applied at a distance (d) to the crack tip, when the distance d approaches zero and $F\sqrt{d}/\pi = B$ is kept constant at the same time. B is known as the strength of singularity of Bueckner's fundamental field. The fundamental field is regular everywhere except at the crack tip where the displacements and stresses are singular to, $d^{-1/2}$ and $d^{-3/2}$, respectively. It has been shown in Refs [6] and [7] that the boundary displacement of the

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fundamental field is identical to Rice's weight function [3]. The fundamental field has zero boundary tractions and no body force. Several fundamental fields have been derived for two- and three-dimensional infinite bodies [1,4], but it is very difficult to derive the fundamental field for a finite body due to the strong singularity at the crack tip. It is known that the fundamental field (weight function) depends on the geometry, and it is even more difficult to find the fundamental field for a finite body than to calculate the stress intensity factor directly. Also it is indicated by Sha and Yang [8] that, under mixed-mode loading, the weight function strongly depends on the boundary conditions.

The infinite body fundamental field can be interpreted as a special field for the finite body which consists of the infinite body fundamental field and a regular field due to the tractions on the free boundaries. This concept was used by Kuna *et al.* [5] for calculating the finite body fundamental field by solving the regular field associated with the boundary tractions. Although this is a very efficient method for calculating the weight function, it is still much easier to directly calculate the stress intensity factor, and in many cases, it is also not necessary to calculate the weight function.

In this paper it is shown how the already known solution of the infinite body fundamental field can be easily used for directly calculating the stress intensity factor without trying to obtain the real weight function.

Recently Liu and Zhang [9] introduced a semi-weight function concept. The so-called semi-weight function is the infinite body fundamental field. They applied the reciprocal theorem to the infinite body fundamental field and the stress–strain field caused by the actual loading. The method yields a path-independent integral which gives the stress intensity factor for the actual mode I loading. However, their derivation and the final formula were not very clear. In the present paper a very direct derivation of the path-independent integral for mode I, II, III and mixed-mode crack problems is given first. Then the accuracy of the method is demonstrated by many typical mode I crack problems. Finally the method is applied to two-dimensional mixed-mode crack problems. Very good accuracy was also obtained in all test examples.

The advantage of the path-independent integral is that, as long as the far field stresses and displacements along any contour around the crack tip are known, the calculation of the stress intensity factor can be easily performed. No special considerations have to be taken in modelling the crack tip singularity.

DERIVATION OF THE PATH-INDEPENDENT INTEGRAL

A derivation of the path-independent integral for the calculation of stress intensity factors in a two-dimensional body follows. Let us consider a two-dimensional cracked structure under two mode I loading states N and S, see Fig. 1. State N is the real loading state with arbitrary tractions t, and displacements u on the boundary Γ and body force f, in the volume A. State S is the same structure cut from an infinite body fundamental field with non-zero tractions t^∞ and displacements u^∞ on boundary Γ. These tractions and displacements are known and they are given by the Bueckner's fundamental field. There are no tractions along the crack faces and no body force in the volume in state S due to the properties of the fundamental field.

Now let us consider a contour C enclosing the crack tip (which could include the whole boundary). The tractions and displacements on the contour C are t_c^s, u_c^s and t_c, u_c in state S and state N, respectively. Using the reciprocal theorem for the two loading states, we get

$$F^s u_N + \int_C t_c^s u_c \, \mathrm{d}s = \int_A f u^s \, \mathrm{d}A + \int_C t_c u_c^s \, \mathrm{d}s \tag{1}$$

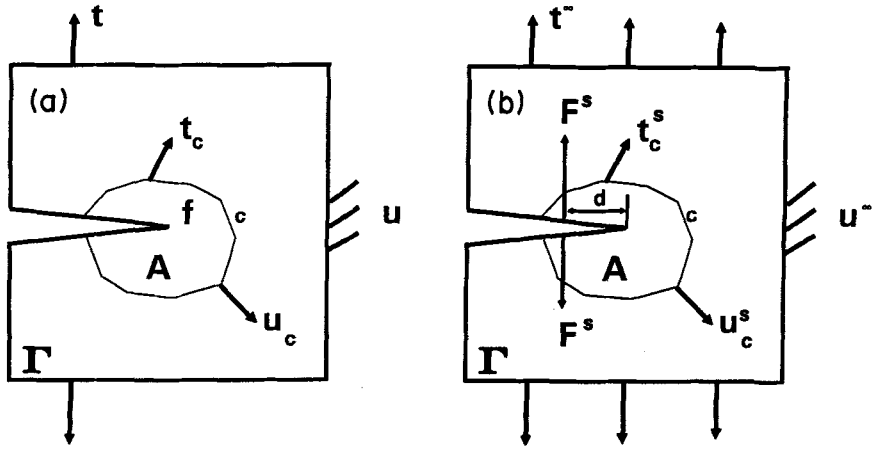

Fig. 1. Two loading states for the mode I problem (a) state N and (b) state S.

where u_N is the crack opening displacement in state N at the position where F^s was applied in state S, u^s is the displacement in state S within contour C. Let d approach zero compared to other dimensions, then u_N will be within the crack tip stress field in state N

$$u_N = 2v(r = d, \theta = \pi) = \frac{4\sqrt{2}}{\sqrt{\pi}H} K_I \sqrt{d} \tag{2}$$

where H is the generalized Young's modulus, $H = E$ for plane stress and $H = E/(1 - \nu^2)$ for plane strain, E the Young's modulus and ν the Poisson's ratio. Substituting equation (2) into equation (1) and rearranging gives

$$\frac{4\sqrt{2}}{\sqrt{\pi}H} F^s \sqrt{d} K_I = \int_A f u^s \, dA + \int_C t_c u_c^s \, ds - \int_C t_c^s u_c \, ds. \tag{3}$$

Choosing Bueckner's singularity strength as

$$B = F^s \sqrt{d}/\pi = \frac{H}{4\sqrt{2\pi}} \tag{4}$$

and substituting the last equation in equation (3) gives

$$K_I = \int_A f u^s \, dA + \int_C t_c u_c^s \, ds - \int_C t_c^s u_c \, ds \tag{5}$$

which can further be simplified to

$$K_I = \int_C t_c u_c^s \, ds - \int_C t_c^s u_c \, ds = \int_C (t_c u_c^s - t_c^s u_c) \, ds \tag{6}$$

in the case of zero body force in state N.

Note that the contour C is arbitrarily chosen, so the integral equations (5) and (6) are path-independent under linear-elastic isotropic conditions. It can be seen, that as long as the Bueckner's fundamental field in an infinite body, t^s and u^s, are known, the calculation of the stress intensity factor for mode I loading can be performed on any contour around the crack tip, provided

that the tractions and displacements along this contour can be computed. By using this path-independent integral, the complex analysis near the crack tip can be avoided.

Equation (6) can be extended to mode II and mode III loading cases similar to Paris *et al.* [7], if the mode II and mode III Bueckner's infinite body fundamental fields are known

$$K_\alpha = \int_C (t_c u^s_{c(\alpha)} - t^s_{c(\alpha)} u_c)\,\mathrm{d}s \quad \alpha = \mathrm{I, II, III} \tag{7}$$

where $u^s_{c(\alpha)}$ and $t^s_{c(\alpha)}$ are the mode α Bueckner's infinite body fundamental field.

Because the fundamental fields for a two-dimensional infinite body under all three loading modes are known, which are reproduced in the next section, it is possible to use the path-independent integral equation (7) in calculating stress intensity factors for two-dimensional crack problems, under modes I, II, III or mixed mode loading. Only simple analytical or numerical analysis is needed for solving the displacements and tractions along contour C. There is no extra difficulty in solving a mixed-mode loading case when compared to the pure mode case.

Similar forms to equation (7) can be found in Paris *et al.* [7], Sha and Yang [8] and Sham [10], such as

$$K_\alpha = \int_{S_T} t u^{s'}_{(\alpha)}\,\mathrm{d}s - \int_{S_u} t^{s'}_{(\alpha)} u\,\mathrm{d}s \quad \alpha = \mathrm{I, II\ III} \tag{8}$$

where S_T is the traction boundary and S_u is the displacement boundary. In equation (8), the weight function for a specific crack geometry, $u^{s'}$, must be found before this formula can be applied. $t^{s'}$ is the corresponding traction on the boundary due to the weight function. It can be seen that it is much easier to apply equation (7), because here t^s and u^s are already known.

KNOWN BUECKNER'S FUNDAMENTAL FIELDS

In this section we present the Bueckner's fundamental fields for an infinite body under the three loading modes I, II and III, which will be used in the following sections.

For the pure mode I problem, the Bueckner's fundamental field is solved by choosing Westergaard stress functions as follows

$$Z(z) = Bz^{-3/2}, \quad \bar{Z}(z) = -2Bz^{-1/2} \tag{9}$$

where $z = x + iy$. Making use of polar coordinates (r, θ) with the origin at the crack tip, the corresponding displacements and stresses can be obtained as

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = Br^{-3/2} \begin{bmatrix} \cos\frac{3}{2}\theta - \frac{3}{2}\sin\theta\sin\frac{5}{2}\theta \\ \cos\frac{3}{2}\theta + \frac{3}{2}\sin\theta\sin\frac{5}{2}\theta \\ \frac{3}{2}\sin\theta\cos\frac{5}{2}\theta \end{bmatrix} \tag{10}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{B}{2G} r^{-1/2} \begin{bmatrix} (1-\kappa)\cos\dfrac{\theta}{2} + \sin\theta\sin\dfrac{3}{2}\theta \\ (1+\kappa)\sin\dfrac{\theta}{2} - \sin\theta\cos\dfrac{3}{2}\theta \end{bmatrix} \tag{11}$$

where G is the shear modulus; $\kappa = 3\text{–}4\nu$ for plane strain, $\kappa = (3-\nu)/(1+\nu)$ for plane stress; and ν is the Poisson's ratio.

Following the same procedure as in the mode I problem, we can obtain mode II and mode III Bueckner's fundamental fields for cracks in an infinite body by taking mode II and mode III Westergaard stress functions as

$$Z(z) = B_{\text{II or III}} z^{-3/2}, \quad \bar{Z}(z) = -2B_{\text{II or III}} z^{-1/2}. \tag{12}$$

The corresponding stresses and displacements are for mode II

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix}_{\text{II}} = B_{\text{II}} r^{-3/2} \begin{bmatrix} -2\sin\frac{3}{2}\theta - \frac{3}{2}\sin\theta\cos\frac{5}{2}\theta \\ \frac{3}{2}\sin\theta\cos\frac{5}{2}\theta \\ \cos\frac{3}{2}\theta - \frac{3}{2}\sin\theta\sin\frac{5}{2}\theta \end{bmatrix} \tag{13}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_{\text{II}} = \frac{B_{\text{II}}}{2G} r^{-1/2} \begin{bmatrix} (\kappa+1)\sin\dfrac{\theta}{2} + \sin\theta\cos\dfrac{3}{2}\theta \\ (\kappa-1)\cos\dfrac{\theta}{2} + \sin\theta\sin\dfrac{3}{2}\theta \end{bmatrix} \tag{14}$$

and for mode III

$$\begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \end{pmatrix}_{\text{III}} = B_{\text{III}} r^{-3/2} \begin{pmatrix} -\sin\frac{3}{2}\theta \\ \cos\frac{3}{2}\theta \end{pmatrix} \tag{15}$$

$$(w)_{\text{III}} = \frac{B_{\text{III}}}{G} r^{-1/2} \left(2\sin\frac{\theta}{2}\right). \tag{16}$$

VERIFICATION OF THE PATH-INDEPENDENT INTEGRAL

Several mode I test cases were used to numerically verify the path-independent integral. The first two test examples were centre crack and edge crack problems in a plate. The third example was cracks emanating from a circular hole in a plate. In all the test examples the mode I fundamental field solution (10) and (11) for an infinite body was used. Plane strain condition was assumed in all cases. Young's modulus was taken to be 210,000 MPa and Poisson's ratio to be 0.3. The finite element method was used to calculate the tractions and displacements along the contour around the crack tip. Eight-noded isoparametric elements with reduced 2×2 integration were used, without modelling the singularity at the crack tip. In Figs 2–5, the crack position is shown by the line cc. All the analyses were performed using the ABAQUS program.

Single and double edge cracks in a plate

The first test case was an edge crack in a plate under remote uniform tension. The crack length to plate width ratio a/w was 0.5, and modelled plate height to width ratio L/w was 6. Two different meshes were used, see Fig. 2(a) and (b). In the fine mesh, there were four and in the coarse mesh two element rings around the crack tip. The contour C was chosen to go through the integration points in these element rings. Thus, eight and four different contours were used for the fine and coarse meshes, respectively. The resulting normalized stress intensity factors $K_{\text{I}}/\sigma\sqrt{\pi a}$ are shown in Table 1 (a) together with the solution of Tada *et al.* [11]. Contour 1 was nearest to the crack tip. Good agreement was obtained between the present results and those of Tada *et al.* for both meshes. The maximum difference was 3.7% at contour 2. It can be seen that as long as the contour was far from the crack tip, accurate results were obtained. The difference between the results of

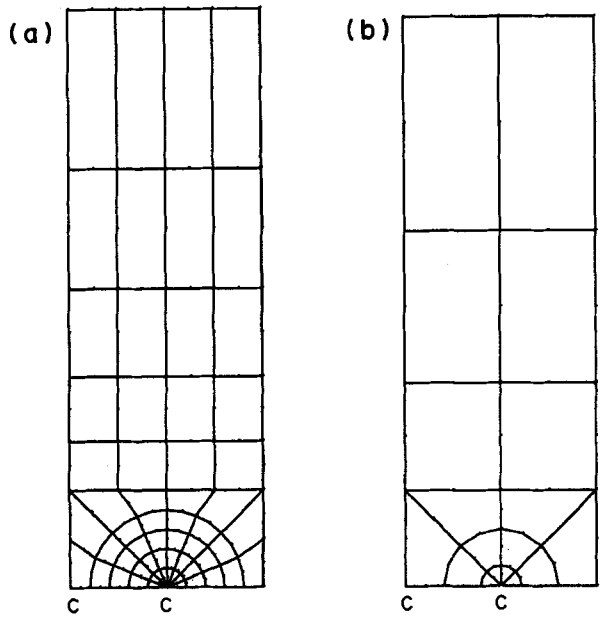

Fig. 2. Finite element meshes for edge and centre crack problems (a) fine mesh and (b) coarse mesh.

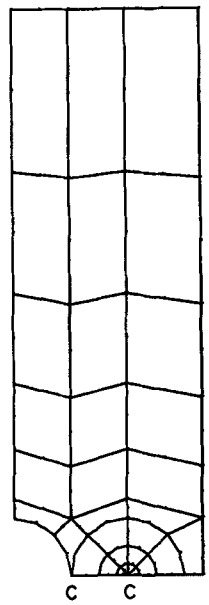

Fig. 3. Finite element mesh (1/4 model) for crack problems near a hole.

the two meshes was less than 0.6%, which shows that the accuracy of the computed stress intensity factors is not very sensitive to the element size around crack tip.

The stress intensity factors for double edge cracks in a plate under uniform tension were calculated using the same mesh, but with different boundary conditions. The normalized stress intensity factors are shown in Table 1(b) together with the solution of Tada *et al.* [11]. The difference between the present results and those of Tada *et al.* was even smaller for this case.

In the analyses, ordinary eight-noded isoparametric elements were used at the crack tip. Proper modelling of the crack tip singularity could still improve the results, but it is felt that the few percent accuracy is sufficient in most cases.

Centre crack in a plate

The second example was a centre crack in a plate under remote uniform tension. The finite element meshes shown in Fig. 2(a) and (b) were used also for a centre-cracked plate, only the boundary conditions were changed. The resulting normalized stress intensity factors for the two meshes are shown in Table 1(c) together with the solution of Tada *et al.* [11]. Excellent agreement was obtained also for the centre-cracked problem when the first contour result is excluded. The large difference, up to 5%, for the result from the first contour was due to the use of ordinary elements at crack tip.

Centre and double edge cracks emanating from a hole

The third example was cracks near a circular hole in a plate subjected to uniform tension. The modelled plate height to width ratio L/w was 3.0, and crack length, net ligament length and the hole radius to plate width ratios were 0.3, 0.4 and 0.3, respectively. The finite element mesh is shown in Fig. 3, with only three circular element rings around the crack tip.

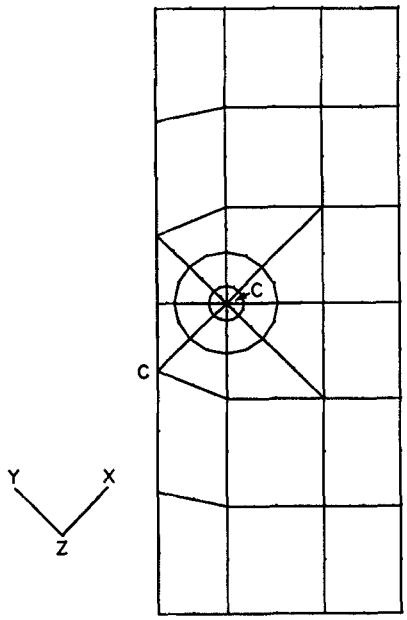

Fig. 4. Coordinate system and finite element mesh for a single edge inclined crack.

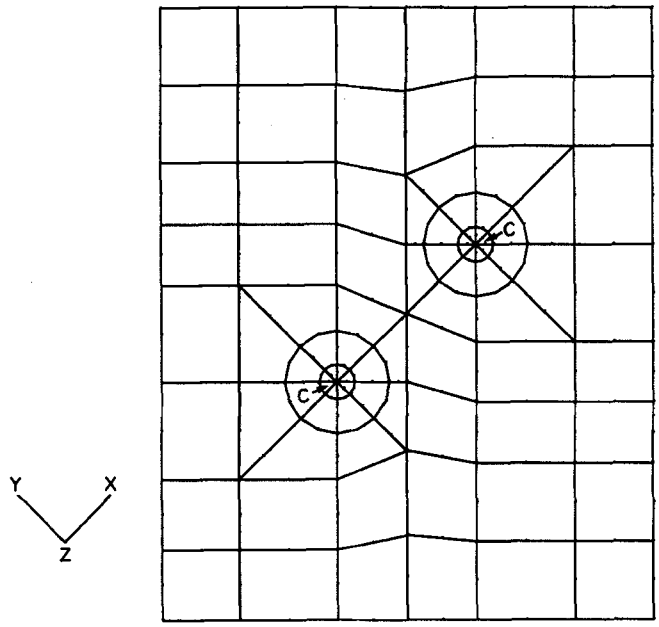

Fig. 5. Coordinate system and finite element mesh for an inclined centre crack.

The resulting normalized stress intensity factors for centre and double edge cracks emanating from a circular hole are shown in Table 2(a) and (b). Again the present results agree well with the results of Tada *et al.* [11]. A large difference was found only for the first element ring contour.

APPLICATION TO MIXED-MODE CRACK PROBLEMS

The path-independent integral was applied to two mixed-mode crack problems with an inclined edge crack and an inclined centre crack. The stress intensity factor components, K_{I} and K_{II}, were calculated with equation (7) using the mode I and mode II fundamental field solutions (10), (11) and (13), (14), respectively.

Single inclined edge crack in a plate

The first test case was a single inclined edge crack in a plate under uniform tension ($\sigma = 1$ MPa), see Fig. 4. The angle between the crack plane and vertical edge of the plate was 45°. The crack length, plate width and modelled plate height were 2, 5 and 12.5 mm, respectively. Because the plate geometry was not symmetrical with respect to the crack plane, the remote loading induced both K_{I} and K_{II} for the crack. Although the mode I and mode II stresses and displacements could have been separated from the total ones, as done by Sha and Yang [8], it was not necessary with the present method. When the finite element method is used, there is no need to use symmetrical modelling with respect to the crack plane. The computation was performed using the crack tip coordinate system in Fig. 4, with all quantities expressed in this coordinate system. The procedure for determining the mode I and mode II stress intensity factors was exactly the same as in the previous examples. Equation (7) was used separately for both mode I and mode II fundamental field

Table 1. Normalized stress intensity factors

Tada *et al.*		Contour 1	Contour 2	Contour 3	Contour 4	Contour 5	Contour 6	Contour 7	Contour 8
(a) Single edge crack ($a/w = 0.5$)									
2.8266	Mesh (a)	2.7491	2.9321	2.8967	2.9014	2.8991	2.9003	2.8994	2.8996
	Mesh (b)	2.7332	2.9307	2.8802	2.8915				
% Relative difference	Mesh (a)	−2.7	3.7	2.5	2.6	2.6	2.6	2.6	2.6
	Mesh (b)	−3.3	3.7	1.9	2.3				
(b) Double edge cracks ($a/w = 0.5$)									
1.1628	Mesh (a)	1.1132	1.1873	1.1729	1.1749	1.1739	1.1744	1.1741	1.1742
	Mesh (b)	1.1137	1.1932	1.1715	1.1777				
% Relative difference	Mesh (a)	−4.2	2.1	0.85	1.0	0.94	0.98	0.95	0.96
	Mesh (b)	−4.2	2.6	0.8	1.3				
(c) Centre crack ($a/w = 0.5$)									
1.1892	Mesh (a)	1.1306	1.2059	1.1913	1.1933	1.1923	1.1928	1.1926	1.1927
	Mesh (b)	1.1279	1.2090	1.1874	1.1933				
% Relative difference	Mesh (a)	4.9	1.4	0.18	0.35	0.26	0.30	0.28	0.29
	Mesh (b)	−5.2	1.7	−0.2	0.35				

solutions given in (10), (11) and (13), (14), respectively. The resulting stress intensity factors are shown in Table 3(a), together with the results of Rajiyah and Atluri [12], who used the alternating method. The agreement of the results is very good although a coarse mesh was used, see Fig. 4.

Inclined centre crack in a plate

The last example considered was an embedded inclined centre crack in a plate subjected to uniform tension ($\sigma = 1$ MPa). The inclined angle was 45°, and crack length ($2a$), plate width and modelled plate height were 4, 10 and 12.5 mm, respectively. The finite element mesh is shown in Fig. 5. The resulting stress intensity factors are given in Table 3(b). The results agree well with the results of Rajiyah and Atluri [12] who used the alternating method.

Table 2. Normalized stress intensity factors

Tada *et al.*	Contour 1	Contour 2	Contour 3	Contour 4	Contour 5	Contour 6
(a) Centre crack near a hole						
1.3122	1.2186	1.3007	1.2810	1.2856	1.2825	1.2837
% Relative	−7.1	−0.88	−2.4	−2.0	−2.3	−2.2
(b) Edge cracks near a hole						
1.0878	1.0515	1.1228	1.1059	1.1098	1.1068	1.1086
% Relative	−3.3	3.2	−1.7	2.0	1.7	1.9

Table 3. Stress intensity factors

	Alternating method	Contour 1	Contour 2	Contour 3	Contour 4
(a) Single edge inclined crack					
K_I	2.443	2.4659	2.6347	2.5868	2.6026
K_{II}	1.249	1.2084	1.2955	1.2728	1.2794
(b) Embedded inclined crack					
K_I	1.453	1.3702	1.4543	1.4519	1.4672
K_{II}	1.385	1.2849	1.3688	1.3540	1.3634

SUMMARY

A path-independent integral was presented and applied to two-dimensional mixed-mode crack problems. The path-independent integral provides a very effective and yet accurate method to determine the stress intensity factors for two-dimensional mixed-mode crack problems with arbitrary geometry. The procedure outlined is direct and simple, and may be extended to three-dimensional crack problems. In the calculation of stress intensity factors only stresses and displacements along a contour enclosing the crack tip are required. In the stress analysis of the cracked body, it is not necessary to model the crack tip singularity. The stress analysis can be performed by any convenient method.

The results of several test cases demonstrated the good accuracy of the method. Even though the accuracy could be improved by using finer mesh or special modelling of the crack tip singularity, it seems that it is not necessary in most practical cases. The advantages and generality of the method are thus obvious.

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