Integrated Multimodel Control of Nonlinear Systems Based on Gap Metric and Stability Margin

Jingjing Du*†‡ and Tor Arne Johansen†

†Department of Engineering Cybernetics, Norwegian University of Science and Technology, NO 7491 Trondheim, Norway
‡School of Electrical Engineering and Automation, Henan Polytechnic University, Jiaozuo 454000, China

ABSTRACT: To avoid linear model redundancy and simplify the structure of a multimodel controller, two general integrated multimodel control design frameworks, which integrate the multimodel decomposition and the multimodel combination of a nonlinear system, are proposed based on the gap metric and stability margin criteria. One method uses the maximum stability margin (which is comparatively controller-independent) while the other uses the actual stability margin of a given controller design. For a prescribed linear control algorithm for local controller design, a smaller and better linear model bank that provides necessary information for multimodel controller design is obtained systematically without model redundancy. Besides, the local robust stability and performance of the system in each subregion can be achieved by the corresponding local controller. Many linear control techniques, such as PID, LQG, MPC, robust control, and others, can be used to design local controllers. A typical design scheme of the multimodel approach consists of two steps,7 decomposition and combination, as is shown in Figure 1. In the stage of decomposition, a nonlinear system is divided into a set of local linear systems according to certain rules/criteria; in the combination stage, local linear controllers are designed based on corresponding local models and subsequently combined into a global controller either by hard switching or soft switching.7–9 Generally, the typical design scheme has the following characteristics:

- The selection of model bank, i.e., the decomposition of a nonlinear system, is heavily dependent on experience and previous knowledge.2,7
- There is no direct connection between local linear models. The selection of previous linear local models has little decisive influence on the following models, i.e., the selection of \( P_i \) might not affect the selection of \( P_{ijp} \), \( j = 1, 2, 3 \ldots n_m - 1 \).
- The design of local controllers depends directly on the linear model bank, while the local controllers have little influence on the determination of linear model bank, since the models are usually determined before the controllers are designed.
- There is little relation among the design of local linear controllers. Namely, \( K_i \), \( i = 1, 2, 3 \ldots n_m \) are independent of each other. Local controllers can be designed based on corresponding models at the same time (in parallel).

In short, the typical multimodel control design procedure is not very systematic but rather problem dependent, and there is no close relationship between the determination of local linear models and the design of local linear controllers. Local models are handled in parallel as are the local controllers. Information flows unidirectionally from local models to local controllers. Unfortunately, this information asymmetry easily leads to linear model bank redundancy since designers tend to use more local linear models than needed in order to guarantee the global stability and robust performance. Furthermore, the model redundancy increases computational load and complicates the following multimodel controller structure.1,2 If the local model and controller can be handled together, and the model selection and local controller design can be connected with each other (i.e., the multimodel decomposition and combination are integrated), the disadvantages of the traditional design procedures may be avoided.

Efforts have been made to establish the connection between local model selection and local controller design. Tan et al.7 proposed a method to integrate the operating point selection and local controller design. Nevertheless, the operating points were selected from existing points or prediction points, which depend on a priori knowledge. Then later, Du et al.8 proposed an integrated multimodel control design procedure, which integrated the multimodel decomposition and the local controller design. However, it is effective only when the \( H^\infty \) loop shaping control technique is used to design local controllers.

1. INTRODUCTION

The multimodel control approach is popular in dealing with nonlinear control algorithms.1–17 The key point is to approximate a nonlinear system with a set of local linear systems. Then the well-known linear control techniques, such as PID, LQG, MPC, robust control, and others, can be used to design local controllers. A typical design scheme of the multimodel approach consists of two steps,7 decomposition and combination, as is shown in Figure 1. In the stage of decomposition, a nonlinear system is divided into a set of local linear systems according to certain rules/criteria; in the combination stage, local linear controllers are designed based on corresponding local models and subsequently combined into a global controller either by hard switching or soft switching.7–9 Generally, the typical design scheme has the following characteristics:

- The selection of model bank, i.e., the decomposition of a nonlinear system, is heavily dependent on experience and previous knowledge.2,7
- There is no direct connection between local linear models. The selection of previous linear local models has little decisive influence on the following models, i.e., the selection of \( P_i \) might not affect the selection of \( P_{ijp} \), \( j = 1, 2, 3 \ldots n_m - 1 \).
- The design of local controllers depends directly on the linear model bank, while the local controllers have little influence on the determination of linear model bank, since the models are usually determined before the controllers are designed.
- There is little relation among the design of local linear controllers. Namely, \( K_i \), \( i = 1, 2, 3 \ldots n_m \) are independent of each other. Local controllers can be designed based on corresponding models at the same time (in parallel).

In short, the typical multimodel control design procedure is not very systematic but rather problem dependent, and there is no close relationship between the determination of local linear models and the design of local linear controllers. Local models are handled in parallel as are the local controllers. Information flows unidirectionally from local models to local controllers. Unfortunately, this information asymmetry easily leads to linear model bank redundancy since designers tend to use more local linear models than needed in order to guarantee the global stability and robust performance. Furthermore, the model redundancy increases computational load and complicates the following multimodel controller structure.1,2 If the local model and controller can be handled together, and the model selection and local controller design can be connected with each other (i.e., the multimodel decomposition and combination are integrated), the disadvantages of the traditional design procedures may be avoided.

Efforts have been made to establish the connection between local model selection and local controller design. Tan et al.7 proposed a method to integrate the operating point selection and local controller design. Nevertheless, the operating points were selected from existing points or prediction points, which depend on a priori knowledge. Then later, Du et al.8 proposed an integrated multimodel control design procedure, which integrated the multimodel decomposition and the local controller design. However, it is effective only when the \( H^\infty \) loop shaping control technique is used to design local controllers.
In this work, we extend the method in ref 8 to propose two more general and systematic frameworks for integrated multimodel control, in which more than one linear control method can be used to design local controllers. One is called “control-relevant linear model bank selection”, in which the linear model bank selection is dependent on the local controllers. The other is “linear model bank selection using maximum stability margin”, in which the linear model bank selection is combined with the local controller design by a tuning parameter. The first one is relatively more systematic but the second one is simpler. In both of the proposed approaches, the local model selection and local controller design are carried out in series and are closely connected with each other in contrast to the traditional parallel design methods. The schematic diagram of the proposed systematic multimodel control is given in Figure 2. Compared with the traditional design procedures, our methods have the following advantages:

- They are more systematic and effective, as will be shown in this paper.
- Many linear control methods, e.g., $H_\infty$, PID, IMC, LQ, can be used to design local controllers.
- Only the necessary local models are added into the model bank. No redundant models exist, the online computational load is low, and correspondingly the multimodel controller structure is simplified.
- The local model selection is dependent on the local controllers, such that different local controllers may result in different linear model banks.
- The local robust stability and performance are achieved.

The paper is organized as follows. In Section 2, related background is shortly reviewed. In Section 3, the proposed general integrated design frameworks are detailed. Section 4 is about the multimodel combination. Section 5 presents some simulation results to illustrate the effectiveness of the proposed approaches, and comparisons have been made between them. Section 6 concludes the paper.

2. GAP METRIC AND STABILITY MARGIN

In this section, relevant theoretical background will be recalled briefly.

2.1. Gap Metric. The gap metric was introduced into the control literature by Zames and El-Sakkary as being appropriate for the study of uncertainty in feedback systems.\textsuperscript{18} The metric defines a notion of distance in the space of (possibly unstable) linear systems that does not assume that the plants have the same number of poles in the right half plane. El-Sakkary\textsuperscript{18} showed that gap metric was much more suitable to measure the distance between two linear systems than a metric based on norms. In the following, the relevant theory of the gap metric is briefly reviewed for completeness.

Let $P(s)$ be a rational transfer matrix with the following normalized right coprime factorization

$$P = NM^{-1}, \quad \text{with } \tilde{M} + \tilde{N} = I \quad (1)$$

where $(\tilde{\cdot})$ denotes complex conjugate, i.e., $\tilde{M}(s) = M^T(-s)$. The graph of $P$ is a closed subspace of $H_2$ (standard Hardy space) given by

$$G(P) = \begin{bmatrix} M \\ N \end{bmatrix} H_2 \quad (2)$$

It consists of all pairs $(u, y)$ such that $y = Pu$. The gap between two finite-dimensional linear systems $P_1$ and $P_2$ with the same number of inputs and outputs is defined by\textsuperscript{19}

$$\delta(P_1, P_2) = \| \Pi_{G(P_1)} - \Pi_{G(P_2)} \| \quad (3)$$

where $\Pi_{G(P)}$ denotes the orthogonal projection onto $G(P)$. It is shown by Georgiou that the gap metric defined by eq 3 can be computed as\textsuperscript{19}.
The relationship between the gap metric and stability margin is given by Proposition 1 and Corollary 1.  

**Proposition 1:** Suppose the feedback system with the pair \((P_0, K_0)\) is stable. Let \(P_{\varepsilon}: = \{P: \delta(P, P_0) < \varepsilon\}\) and \(K_{\varepsilon}: = \{K: \delta(K, K_0) < \varepsilon\}\). Then the feedback system with the pair \((P, K)\) is also stable for all \(P \in P_{\varepsilon}\) and \(K \in K_{\varepsilon}\) if and only if

\[
  b_{P,K} \geq \arcsin r_1 + \arcsin r_2
\]

where \(b_{P,K}\) is the gap metric. Then the feedback system with the pair \((P_0, K_0)\) is stable. Let \(P_{\varepsilon}: = \{P: \delta(P, P_0) < \varepsilon\}\) and \(K_{\varepsilon}: = \{K: \delta(K, K_0) < \varepsilon\}\). Then the feedback system with the pair \((P, K)\) is also stable for all \(P \in P_{\varepsilon}\) if and only if

\[
  b_{P,K} \geq \delta(P, P_0)
\]

Suppose we linearize a nonlinear system around all its operating points, then in a subregion we choose a local linear model from the linearized systems according to a certain criterion and design a stabilizing controller based on the local model. If the stability margin of the local model with its stabilizing controller is bigger than the biggest gap between the local model and the other linearized models in its subregion, then the controller can stabilize all of the linearized models in the subregion according to Corollary 1. So the local robust stability is guaranteed. If it is difficult to find such a stabilizing controller, we narrow the range of the subregion, reselect the local model, and repeat the above procedure. Thus, the local model selection and the local controller design are connected. This is the basic idea of our integrated multimodel control design frameworks. Note that the stability here denotes the local stability of the system around its equilibrium points. The global stability of the nonlinear system is not guaranteed, which is a limitation common to many multimodel control approaches applied to nonlinear systems. Besides, the stabilizing controller can be designed using many control algorithms and \(H_\infty\) control method is used in this paper to demonstrate the effectiveness of the proposed approaches.

### 3. General Integrated Multimodel Control Based on Gap Metric and Stability Margin

Consider a nonlinear process described by eq 9

\[
\begin{aligned}
  \dot{x} &= f(x, u) \\
  y &= g(x, u)
\end{aligned}
\]

where \(x \in \mathbb{R}^n\) is the state vector, \(u \in \mathbb{R}^r\) is the control input vector, \(y \in \mathbb{R}^m\) is the output vector, and \(f(\cdot)\) and \(g(\cdot)\) are differentiable functions.

For the nonlinear system eq 9, in order to get a simple and effective multimodel controller, we set up the following goals: (1) given a linear control algorithm (e.g., \(H_\infty\), PID, IMC, or LQ) for local linear controller design, we get the minimum number of local linear models with as little a priori knowledge as possible; (2) the local robust stability and performance of the closed-loop nonlinear system in all of the subregions should be guaranteed.

Let \(\theta\) be the scheduling variable of the system eq 9 and \(\Phi\) be the variation range of \(\theta, \theta \in \Phi\). \(\Phi\) is called the scheduling space of the system eq 9, and also the full operating space. We use the gap metric based dichotomy gridding method to grid the operating space \(\Phi\) and linearize the system around the gridding.
points to get a series of lineared models. Then we can use the gap metric as a measurement tool to divide the nonlinear system as in ref 8.

Du et al.\textsuperscript{8} proposed an improved gap metric based dividing algorithm. In the algorithm, the maximum stability margins of the local linear models are used as thresholds to decompose a nonlinear system, so a smaller linear model bank is obtained and dependency on a priori knowledge is largely reduced. It is a systematic method, and theoretically speaking, the improved dividing algorithm can be used regardless of the local control algorithm. However, it may be rather difficult to find a controller which satisfies both the stability margin and control performance requirements because the maximum stability margin is conservative. The method is effective in ref 8 because the $H_\infty$ loop shaping technique is used to design local controllers: the loop-shaping procedure guarantees that the local control performance is satisfactory; after loop-shaping, the use of maximum stability margins as threshold values guarantees the local robust stability. It is the connection of loop-shaping and maximum stability margin that makes the improved gap metric based dividing method effective.

However, for a common local control algorithm, such as the $H_\infty$ control algorithm, no loop-shaping is done before the controller is designed, and the control performance is not settled beforehand. If the maximum stability margin is used directly as threshold values to divide a nonlinear system and select the linear model bank, the closed-loop system may not have good performance, but only stability. In this paper, we are aiming to propose more general integrated multimodel control approaches in which many linear control algorithms can be used to design local controllers. In order to design the local systems with both a certain degree of robust stability and good control performance, we extend the method in ref 8:

(1) We replace the maximum stability margin with the actual stability margin so that a control-relevant linear model bank selection method is proposed.

(2) We revise the threshold value by introducing a tuning parameter $\epsilon$. The new threshold value is chosen as the minimum of $\epsilon$ and the maximum stability margin. Thus, we can tune $\epsilon$ to get a local controller with satisfactory performance and if a satisfactory controller is not easy to find, we adjust $\epsilon$. The tuning parameter $\epsilon$ gives back the controller design information to model bank selection. Thus, a more general integrated multimodel control method is proposed based on the maximum stability margin.

In the above modifications, the linear model bank selection is connected with the local controller design, which helps to avoid linear model redundancy and simplify the multimodel controller structure. The general integrated design frameworks are detailed as Algorithm 1 (using the actual stability margin) and Algorithm 2 (using the maximum stability margin).

**Algorithm 1.** Control-relevant linear model bank selection

Step 1: Grid the operating space of the considered nonlinear system using the gap metric based dichotomy gridding algorithm\textsuperscript{6} and linearize the nonlinear system around the gridding points. Suppose we get $n$ linearized models $P_i$ $(i = 1, ..., n)$.

Step 2: Calculate the gap-matrix\textsuperscript{10} between all pairs of the linearized models, and get their maximum stability margins according to

\begin{equation}
\delta_{\text{opt}}(P_i) = \sqrt{1 - \|b_i(N)M_iJ_i\|_1}^2
\end{equation}

where $P_i = M_i^{-1}N_i$ is the left normalized coprime factorization of the $i$th linearized model. Then an $n \times n$ matrix gap = $[\delta(P_i, P_j)]_{i,j=1}^{n \times n}$ and an $n \times 1$ vector $B_{\text{opt}} = [\delta_{\text{opt}}(P_i)]_{i=1}^n$ are acquired.

Step 3: Set $k = 1$.

Step 4: Set $l = k + 1$.

Step 5: Choose the best local linear model $P^*_k$ among the $k$th linearized model in the following sense:

\begin{equation}
P^*_k = \{P_k: \min_{k \leq m \leq l} \max_{k \leq i \leq l} \delta(P_k, P_i)\}
\end{equation}

Step 6: Compute the biggest gap between $P^*_k$ and the other linearized models, $\delta_{\text{max}}$:

\begin{equation}
\delta_{\text{max}} = \max_{k \leq i \leq l} \delta(P^*_k, P_i)
\end{equation}

Step 7: For the local linear model $P^*_k$, design a linear controller $K$. If $K$ satisfies both $b_{P^*_k} > \delta_{\text{max}}$ and the desired performance requirements, set $l = l + 1$ and go back to Step 5.

Step 8: If a controller with both $b_{P^*_k} > \delta_{\text{max}}$ and acceptable closed-loop performance is not found, set $l = l - 1$. Then the $l−k + 1$ successive linearized models are to be classified in one subregion, modeled by their best local linear model $P^*_k$ and stabilized by the corresponding controller $K$.

Step 9: Set $k = l$, and go back to Step 4. Repeat the above procedure until all the $n$ linearized models are classified.

Step 10: Suppose the nonlinear system is decomposed into $n_m$ subsystems, i.e., $n_m$ local linear models/controllers are designed after Step 9. The $n_m$ local linear controllers are combined into one global controller either by hard switching or by soft switching (weighting function), as discussed in Section 4.

**Algorithm 2.** Linear model bank selection using the maximum stability margin

S1: Grid the operating space of the considered nonlinear system using the gap metric based dichotomy gridding algorithm and linearize the nonlinear system around the gridding points. Suppose we get $n$ linearized models $P_i$ $(i = 1, ..., n)$.

S2: Calculate the gap-matrix between all pairs of the linearized models, and get their maximum stability margins according to eq 10. Then an $n \times n$ matrix gap = $[\delta(P_i, P_j)]_{i,j=1}^{n \times n}$ and an $n \times 1$ vector $B_{\text{opt}} = [\delta_{\text{opt}}(P_i)]_{i=1}^n$ are acquired.

S3: Set $k = 1$.

S4: Choose a threshold value:

\begin{equation}
\gamma_t = \min_{i} \delta_{\text{opt}}(P^*_i, \epsilon)
\end{equation}

where $\epsilon$ is a design parameter, which is typically chosen between 0.3 and 0.8.

S5: Set $l = k + 1$.

S6: Choose the best local linear model $P^*_k$ among the $k$th linearized model according to eq 11.

S7: Compute the biggest gap between $P^*_k$ and the other linearized models, $\delta_{\text{max}}$ according to eq 12.

S8: If $\delta_{\text{max}} < \gamma_t$, set $l = l + 1$, and go back to S6; otherwise, go to S9.

S9: Set $l = l - 1$. The $l−k + 1$ successive linearized models are to be classified in one subregion and modeled by the best local linear model $P^*_k$.

S10: For the linear local model $P^*_k$, design a linear controller $K$ satisfying $b_{P^*_k} > \delta_{\text{max}}$ and the desired performance requirements. If a controller with both $b_{P^*_k} > \delta_{\text{max}}$ and
acceptable closed-loop performance is not found, go back to S4 and adjust $\epsilon$. 

S11: Set $k = l$, and go back to S4. Repeat the above procedure until all the $n$ linearized models are classified.

S12: Suppose the nonlinear system is decomposed into $n_m$ subsystems. The $n_m$ local linear controllers are combined into one global controller either by hard switching or by soft switching, as discussed in Section 4.

Remark 1: Obviously, in Algorithm 1, the actual stability margin of the local controller is used as the threshold to select local model and controller banks. Different controllers may have different stability margins, and further result in different divisions and local models. Since in Algorithm 1 the local model selection is dependent on the local controller design, Algorithm 1 is called “control-relevant linear model bank selection method”. In Algorithm 2, the maximum stability margin, which depends only on the plant information, is used to formulate the threshold, but $\epsilon$ gives back the controller design information to model bank selection. So in Algorithm 2, the local model selection is also connected with the controller design, although not so directly dependent on the local controller design.

Remark 2: A controller with a value of stability margin greater than 0.3 generally has good robustness margins. However, it is not easy to design a controller with a bigger stability margin greater than 0.8, which also has a good control performance. Therefore, in S4 $\epsilon$ is typically chosen between 0.3 and 0.8. It should also be pointed out that $\epsilon$ can be different for different subregions.

Remark 3: In S10, it would be better if $h^{*}\Lambda > y_r$, but it is not necessary.

Remark 4: More than one linear control method, such as $H_\infty$, PID, IMC, LQ, and so on, can be used to design local controllers. In this work $H_\infty$ control is employed as an example to demonstrate the effectiveness of the proposed approaches.

Remark 5: In the proposed design procedures, local stability of the nonlinear system around its equilibrium points is guaranteed while the global stability of the nonlinear system is not guaranteed, e.g., when there are large transients, which is a limitation common to many multimodel control approaches applied to nonlinear systems.

Remark 6: The above design procedures are somewhat conservative because of the conservativeness of Proposition 1. Better performance may be obtained with local controllers that do not satisfy the robust stability requirement.

4. MULTIMODEL COMBINATION

After obtaining the local models/controllers of a nonlinear system, we need to combine these local linear controllers into a global one to act on the nonlinear system. For controller combination, basically, there are two methods:1 hard switching and soft switching. In hard switching, local controllers are scheduled according to certain switching condition. At one sample period, only one local linear controller is used in the feedback loop. The best local controller can be chosen to regulate the system, while the system output may oscillate during switching. In soft switching, weighting functions (functions of scheduling variables) are employed to combine local controllers into a global one. The output of global multimodel controller is a weighted average of the local controllers’ outputs. So the output of the system is relatively smooth during transition.

In this work, soft switching is chosen to avoid output oscillation. Trapezoidal functions are used to combine local controllers as trapezoidal functions are relatively simple in comparison with other weighting functions (e.g., Gaussian functions). Suppose the nonlinear system is decomposed into $n_m$ local linear systems. Let $u_i(t)$ be the output of the $i$th local controller at time $t$, then the output of the multimodel global controller is

$$u(t) = \sum_{i=1}^{n_m} \phi_i(\theta) u_i(t)$$

where $\theta$ is the scheduling variable of the system eq 9, and $\phi_i(\theta)$ denotes the trapezoidal function of the $i$th local controller.

5. CASE STUDY

In this section, the proposed integrated multimodel control approaches are applied to two nonlinear chemical processes for set-point tracking and disturbance rejection control. $H_\infty$ control method is employed as an example to design local linear controllers. For comparison, empirical local model banks from literature are also used to design multimodel $H_\infty$ controllers.

5.1. Case 1: An isothermal CSTR. Consider an isothermal continuous stirred tank reactor (CSTR) in which a first-order irreversible reaction takes place. The mass balance is

$$\frac{dC_A}{dt} = -kC_A(C_{A_i} - C_A)u$$

where $C_A$ (mol/L) is the reactant concentration, $u = q/V$ (min$^{-1}$) is the input, $C_{A_i}$ is the feed concentration, and $q$ (l/min) is the flow rate. $C_{A_i}$ is 1.0 mol/L, and the rate constant $k$ is 0.028 min$^{-1}$.

The static input–output curve of the system eq 15 is depicted in Figure 3. It is seen clearly that the slope angle of its

![Figure 3. Steady-state input-output map of isothermal CSTR with gridging points.](image)

static I/O curve decreases from nearly 90 degrees to nearly 0 degrees. According to the concept of included angle in ref 16, we know that this isothermal CSTR system has strong static nonlinearity.

$C_A$ is chosen as the scheduling variable since it characterizes the nonlinearity and the operating levels of the isothermal CSTR. The operating range is $\{C_A | C_A \in [0,1]\}$. Implementing
the gap metric based dichotomy gridding algorithm with $\gamma_1 = 0.15$, 19 static points are obtained to grid the operating range of CSTR system, as is displayed in Figure 3. Since the gaps between any two points are all smaller than 0.15, the more nonlinear an area is, the denser static points are, as shown in Figure 3. Clearly, the turning area of the curve is much more nonlinear than other areas.

The gaps between the 19 linearized models are depicted in Figure 4. As is seen, the biggest gap is almost 1. The system exhibits strong open-loop nonlinearity according to the gap metric based nonlinearity measure. A single linear controller is not able to stabilize it in the whole operating range. We apply the control-relevant linear model bank selection method, Algorithm 1, to the system eq 15 to design a multimodel controller. The result is summarized in Table 1.

From Table 1, we can see that the system is divided into 2 subregions. The first subregion contains 15 linearized models (1 → 15), and the local linear model for the first subregion is the eighth linearized model $P_8^* = 0.2266/(s + 0.1236)$, with operating point $(C_A, u) = (0.7734, 0.0956)$. The biggest gap is $\delta_{\max}(P_8^*) = 0.6527$. The first subrange is $(C_A \leq 0 \leq C_X \leq 0.9)$. Therefore, for the first local linear model of the isothermal CSTR system, we should design a $H_\infty$ controller whose stability margin $b_{P,K}$ is bigger than $\delta_{\max}(P_8^*) = 0.6527$ in order to achieve the robust stability of the nonlinear system within the first subrange. Thus, we get an $H_\infty$ controller $K_1 = \left((9.541s + 1.179) / (s^2 + 11.27s + 0.009389)\right)$ with its stability margin $b_{P,K_1} = 0.7024$. The other column is interpreted in the same way, and Table 2 and Table 3 can be interpreted similarly.

Algorithm 2 is applied to the CSTR eq 15 with $\varepsilon = 0.65$, and the division result is summarized in Table 2.

The division results are similar. The isothermal CSTR eq 15 is decomposed into two subsystems whether using Algorithm 1 or Algorithm 2. The local models are “close”. For example, in Table 1 the first local model is the eighth one of the 19 linearized models and in Table 2 the first local model is the seventh one. And the ranges of the subregions are similar. We will investigate whether the multimodel controllers based on Tables 1 and 2 are also similar in the following simulations.

Trapezoidal functions shown in Figure 5 are used as weighting functions to combine the local $H_\infty$ controllers into global multimodel controllers. Then the multimodel $H_\infty$ controllers are employed for set-point tracking control over the entire operating range and disturbance rejection control. The closed-loop responses are displayed in Figures 6 and 7. $C_{A1}$ (solid line) is the output of the system eq 15 under our multimodel $H_\infty$ controller of Algorithm 1, and the input is $u_1$; $C_{A2}$ (dash-dot line) is the output under the multimodel $H_\infty$ controller of Algorithm 2, and the input is $u_\infty$. For comparison, an empirical linear model bank from ref 7, which is composed of three local linear models, is also used to design a multimodel $H_\infty$ controller with its output $C_{Ae}$ and input $u_\infty$ in Figures 6 and 7. In the following, the multimodel controllers based on empirical model banks are called empirical multimodel $H_\infty$ controllers for brevity.

From Figure 6, it is seen that all the outputs follow the reference signal quickly and accurately over the whole operating space. The control inputs vary accordingly in the feasible range. The transitions from one subregion to another one are also satisfactory: fast, smooth, and no chattering. Even around the boundary the outputs of the closed-loop system are still satisfactory: fast, smooth, and no chattering. Even around the boundary the outputs of the closed-loop system are still excellent. Carefully observed, it is seen that $C_{Ae}$ is slightly faster than $C_{A1}$ and $C_{A2}$, but $C_{A1}$ and $C_{A2}$ are more accurate and have smaller overshoots. The integrated absolute error (IAE) values of three multimodel $H_\infty$ controllers perform almost the same, no worse but even a little better than the empirical multimodel $H_\infty$ controller in set point tracking control.

Figure 7 shows the disturbance rejection control performance of the three multimodel $H_\infty$ controllers. In the first subregion, disturbance $\nu_1 = 0.05$ is added to the output at time = 90 and removed at time = 180. In the second subregion, disturbance $\nu_2 = 0.1$ is added to the output at time = 350 and removed at time = 430. In both subregions, $C_{A1}$ and $C_{A2}$ get back to the reference signal quickly and precisely whenever the disturbance is added or removed. Although $C_{Ae}$ is faster, it has much bigger overshoots in the second stage which may not be

---

**Figure 4. Gaps between 19 linearized models of the isothermal CSTR.**

**Table 1. Control-relevant Linear Model Bank Selection of the Isothermal CSTR**

<table>
<thead>
<tr>
<th>subregion</th>
<th>1st</th>
<th>2nd</th>
</tr>
</thead>
<tbody>
<tr>
<td>linearized models included</td>
<td>1→15</td>
<td>16→19</td>
</tr>
<tr>
<td>operating point of the local linear model $(C_A, u)$</td>
<td>$18th (0.7734, 0.0956)$</td>
<td>$17th (0.9281, 0.3616)$</td>
</tr>
<tr>
<td>local linear model</td>
<td>$P_8^* = 0.2266/(s + 0.1236)$</td>
<td>$P_1^* = 0.07188/(s + 0.3896)$</td>
</tr>
<tr>
<td>$\delta_{\max}(P_8^*)$</td>
<td>0.6527</td>
<td>0.1785</td>
</tr>
<tr>
<td>subrange</td>
<td>$0 \leq C_A \leq 0.9$</td>
<td>$0.9 &lt; C_A \leq 1$</td>
</tr>
<tr>
<td>local linear controller</td>
<td>$K_1 = (9.541s + 1.179) / (s^2 + 11.27s + 0.009389)$</td>
<td>$K_1 = 2.695s + 1.05$</td>
</tr>
<tr>
<td>stability margin</td>
<td>$b_{P,K_1} = 0.7024$</td>
<td>$b_{P,K_1} = 0.1821$</td>
</tr>
</tbody>
</table>
acceptable. On the whole, our multimodel controllers are better than the empirical one in disturbance rejection control because our controllers are clearly superior in robust performance.

From the study of this example, we get that Algorithm 1 and Algorithm 2 can both work effectively. A smaller and better linear model bank can be obtained using our multimodel control design methods. The controllers based on our model banks have better robust control performance and simpler structures comparing with that using the empirical model bank. The worst case performance of our controllers is never bad, but the performance of the empirical controller is sometimes good but sometimes rather bad. Besides, the local stability is achieved in both subregions using our methods. Therefore, it is more important to get the right local linear models rather than to get more models, since more models will make the global controller more complex.

### 5.2. Case 2: An exothermal CSTR

Consider a benchmark exothermal continuous stirred tank reactor (CSTR) process with an irreversible, first-order reaction. The exothermal CSTR system is modeled by the following nonlinear differential equations:

\[
\begin{align*}
\dot{x}_1 &= -x_1 + D_a \times (1 - x_1) \times \exp\left(\frac{x_2}{1 + x_2/\gamma}\right) \\
\dot{x}_2 &= -x_2 + B \times D_2 \times (1 - x_1) \\
&\quad \times \exp\left(\frac{x_2}{1 + x_2/\gamma}\right) + \beta \times (u - x_2)
\end{align*}
\]

where \(x_1\) is the reagent conversion, \(x_2\) is the reactor temperature (output) and \(u\) is the coolant temperature (input). All variables are dimensionless. The nominal values for the constants are \(D_a = 0.072, \gamma = 20, B = 8\) and \(\beta = 0.3\), respectively. This exothermal CSTR system exhibits strong output multiplicity, as can be seen

![Figure 6. Set point tracking control of the isothermal CSTR.](image)

![Figure 7. Disturbance rejection control of the isothermal CSTR.](image)
from Figure 8. The output $y$ is chosen as the index variable for it characterizes the nonlinear behavior and marks the operating conditions of the exothermal CSTR process. The operating range is $\{y \in [0,6]\}$.

The gap metric based dichotomy gridding algorithm is applied with $\gamma_i = 0.15$, and 42 static points are obtained to grid the operating range of CSTR system, as is displayed in Figure 8. Since the gaps between any two points are all smaller than 0.15, the more nonlinear an area is, the denser static points are. Obviously, the two turning areas of the “S” curve are more nonlinear than other areas, as marked in Figure 8.

The gaps between the 42 linearized models are depicted in Figure 9. According to the gap metric based nonlinearity measure, the system exhibits strong open-loop nonlinearity, as the biggest gap is equal to 1. To be exact, $\delta(P_i, P_j)$ equals to 1 for $i = 17, 18, \ldots, 24$. According to the properties of the gap metric, the dynamical behaviors of the first and 17th (18th…24th) linearized models are far apart, so a division must be necessary between them to design a multimodel controller. We divide this exothermal CSTR system using the proposed approaches with $\varepsilon = 0.7$, and get the decomposition result in Table 3 (the decomposition results are the same for Algorithms 1 and 2). Table 3 can be interpreted as Table 1.

As shown in Table 3, the first and 17th linearized models are separated into two subregions, validating our previous analysis. The exothermal CSTR system is divided into 3 subsystems, and three local linear $H_{\infty}$ controllers are designed, satisfying both the local stability margin condition and the performance requirements. For controller combination, trapezoidal functions shown in Figure 10 are used as weighting functions. Then the three local $H_{\infty}$ controllers are combined into a global multimodel controller for set point tracking and disturbance rejection control. The closed-loop responses are displayed in Figures 11 and 12, where $y_{gs}$ is the output of the system eq 16 under our multimodel $H_{\infty}$ controller and $u_{gs}$ is the input. For comparison, the empirical linear model bank obtained at $u = 0$ from Galan et al. (composed of three local models) is used to design a multimodel $H_{\infty}$ controller, with its output $y_0$ and input $u_0$.

On the whole, our multimodel $H_{\infty}$ controller outperforms the empirical multimodel controller in set point tracking control. Our multimodel $H_{\infty}$ controller has consistently good performance in the whole operating space: the output tracks the reference signal closely and the input varies accordingly within its feasible range. In the first subregion, the second subregion or the third subregion, or even on the boundaries $y = 2$ and $y = 4$, the tracking performance is satisfactory. The transition from one subregion (a boundary) to another is also good: smooth and accurate. However, the performance of the empirical multimodel $H_{\infty}$ controller is not consistent; it is quite bad in the last three periods, although it is a little faster: $y_0$ has quite big overshoots and static errors. Therefore, a better model bank is obtained by our integrated multimodel control design methods, and the corresponding global multimodel controller has better robust performance.

The disturbance rejection responses of the exothermal CSTR are shown in Figure 12. Disturbance $v_1 = 0.6$ is added at time = 20 and removed at time = 25; $v_2 = -0.9$ is added at time = 75 and removed at time = 100; $v_3 = 1.2$ is added at time = 130 and removed at time = 170. As is seen our multimodel controller performs satisfactorily in the above three cases for disturbance rejection; it can reject disturbances quickly and bring the output back to reference signal precisely in any operating level. Although the empirical multimodel controller is slightly quicker, $y_0$ has a quite big overshoot around time = 50 which is unacceptable.

From the above simulation experiments of the two CSTR systems, we get the following conclusions:

1. The proposed two integrated multimodel control frameworks work well. Compared to traditional methods, our methods can get a smaller and more effective linear model bank for multimodel controller design. Thus, linear model redundancy is avoided and the multimodel controller structure is simplified.

2. Compared to traditional methods, the proposed design frameworks are more systematic and effective. The linear model bank is selected more or less automatically and little previous knowledge is needed.

3. The control performance of our multimodel controllers is consistently good, not always as good as the empirical multimodel controllers, but never bad (even if the worst case
The performance of the empirical multimodel controllers is not that consistent with respect to performance, sometimes good but sometimes quite bad. Besides, local stability is achieved by the proposed methods. Overall, the proposed methods are better than traditional methods, especially in terms of robust control performance.

In Algorithm 1, the linear models are dependent on the local controllers, while in Algorithm 2 the linear models are affected by the local controllers through the tuning parameter $\varepsilon$. Algorithms 1 and 2 may have the same division results when $\varepsilon$ is properly tuned as Case 2.

Algorithm 1 is more complicated than Algorithm 2, because a linear controller needs to be designed and tested every time a linearized model is added into a subregion. For example, 19 $H_\infty$ controllers have to be tested for Case 1. But Algorithm 1 is more systematic regarding division since we do not need a tuning parameter $\varepsilon$.

Algorithm 1 is still simpler than empirical methods for three reasons. (a) We grid the system using the dichotomy method, so the number of linearized models is reduced greatly. (b) For a succession of linearized models it is usually very easy to design and test the controllers, because the parameters of the controllers are similar. Once a satisfactory controller is designed for a neighbor linearized model, only small changes are needed to get controllers for others. (c) All the design and test work is done offline; no more computational load is added in the online multimodel control implementation.

### 6. CONCLUSION

Integrated multimodel control of nonlinear systems is studied. Two fairly general design approaches are proposed, in which...
the concepts of gap metric and stability margin are used to determine the local linear models and meanwhile design the local controllers systematically. The local models and their controllers are bounded together and handled in series. Thus, the multimodel decomposition and the multimodel combination are integrated. Linear model redundancy can be avoided and the multimodel controller structure can be simplified. Two chemical processes are studied and comparisons have been made among the proposed approaches and traditional methods. Closed-loop simulations demonstrate that the proposed methods can significantly improve the worst case performance and are especially more effective in terms of robust performance.

Although $H_{\infty}$ control algorithm is employed to design local controllers in this work, we claim that the proposed design methods can be applicable to many other control algorithms such as IMC, PID and unconstrained MPC. The proposed methods aim at nonlinear systems with one scheduling variable. We will extend the methods to cover systems with multiple scheduling variables in future.

 ■ AUTHOR INFORMATION

**Corresponding Author**

*E-mail: hzedujing@163.com.

**Notes**

The authors declare no competing financial interest.

 ■ ACKNOWLEDGMENTS

This work was carried out during the tenure of an ERCIM “Alain Bensoussan” Fellowship Programme. The research leading to these results has received funding from the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement n$^\circ$ 246016, and this work was partly supported by the NSF (61104079) of China, and partly by the Doctors’ Funds (B2011-007) of Henan Polytechnic University.

 ■ REFERENCES
