Unknown Input Observers and Fault Tolerant Control Allocation*

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Abstract

This research report focuses on the use of unknown input observers for detection and isolation of actuator and effector faults with control reconfiguration in overactuated systems. The proposed approach consists in tuning the observer parameters in order to make the filters decoupled from faults affecting selected groups of actuators or effectors. The control allocation actively uses input redundancy in order to make relevant faults observable. The case study of an overactuated marine vessel supports theoretical developments.

1 Introduction

The main objective of control allocation is to determine how to generate a specified virtual input from a redundant set of actuators and effectors. Control effectors are devices or surfaces producing forces and moments, such as thrusters, propellers, fins or rudders, while actuators are electromechanical devices responsible to tune the magnitude, position and orientation of single effectors. Due to input redundancy, several configurations leading to the same generalized force are admissible and for this reason the control allocation scheme commonly incorporates additional secondary objectives [3] [13] [17], such as power or fuel consumption minimization. On the other hand, usually there are also some limitation factors to be accounted for: actuators/effectors dynamics, input saturation and other physical or operational constraints. The mostly conventional domains in which control allocation is adopted are aerospace [18] and maritime [13] applications, as well as some automotive and mechatronic applications. One further advantage of actuator and effector redundancy is the possibility to reconfigure the control in order to cope with unexpected changes on the system dynamics, such as failures or malfunctions: in particular if the set of actuators and effectors is partially affected by faults, one can modify the control allocation scheme by preventing the use of inefficient devices in the generation of virtual input or compensating for the loss of efficiency. However, one key point for successfully re-allocating the control is the availability of adequate information about the faults that have occurred; indeed, some accurate fault estimation and/or a correct identification of the faulty actuators or effectors are necessary to address the reconfiguration. Recent results toward fault tolerant control allocation are based on sliding-mode techniques [1] [9] and adaptive control strategies [7] [23]. In particular in [1], assuming that sufficient knowledge of the occurred fault is provided by a suitable actuator effectiveness estimator, sliding-mode techniques are used to synthesize the virtual input by means of the safety effectors only; the approach proposed in [7] is focused instead on the development of adaptive control allocation schemes which incorporate a fault estimation error minimization problem, under the assumption that faults are piecewise constant functions having a slowly varying behavior. Further investigations on this topic, with a more application-oriented character, are proposed for reconfiguration in flight control [6], [27] and fault accommodation in automated underwater vehicles [20]. An interesting bibliographical survey about the general problem of fault tolerant control reconfiguration is provided in [28].

The aim of our research is to present the use of unknown input observers for fault detection/isolation and control reconfiguration in overactuated systems. Unknown input observers [8] are a very useful tool for generating robust detection filters, as they can be made insensitive to certain input space directions if some structural algebraic
conditions on the system are fulfilled. Adaptive approaches to fault detection and isolation based on unknown input observers have been recently proposed for second-order mechanical systems [10] and for aircraft control [25], while in [21] a distributed scheme is presented for diagnosis and identification of single and multiple faults in interconnected systems such as power networks or multi-robot formations.

Due to control redundancy, isolating faults affecting single actuators or effectors in overactuated systems can be a difficult task, as the same effects can be produced by faults occurring in different actuators or effectors: the family of filters needed to isolate the faults usually results to be larger compared to a control system framework with full-rank input matrix, and moreover there is an upper bound on the maximum number of simultaneously isolable faults. On the other hand, by constraining the inputs in prescribed configurations without altering system dynamics, control redundancy can be very helpful in separating the effects produced by multiple faults in order to identify which groups of effectors and actuators are losing effectiveness. A formalization of these ideas constitutes the main contribution of this report.

The document is structured as follows. In Section 2 the basic setup of control allocation is introduced and the general structure of unknown input observers (UIO) is defined; moreover some issues related to control reconfiguration are reported, such as control re-allocation in the presence of cost functions and input constraints. Section 3 is devoted to the presentation of the proposed method for designing families of detection/isolation filters based on UIO, namely zeroing fault input (ZFI) observers. The characteristic of ZFI observers is the robustness of the generated diagnosis signals, called residuals, to faults affecting particular actuators or effectors. Finally, in Section 4, the application of proposed theoretical results is extensively illustrated by the case study of an overactuated ship subject to thruster failures.

2 Control allocation setup

Let us consider the following linear system

\[
\begin{cases}
\dot{x}(t) = Ax(t) + B\tau(t) \\
y(t) = Cx(t)
\end{cases}
\]

with

\[
\tau(t) = Gu(t),
\]

where \(x \in \mathbb{R}^n\), \(\tau \in \mathbb{R}^k\), \(y \in \mathbb{R}^p\), \(u \in \mathbb{R}^m\), \(m > k\) and all matrices except \(A\) are assumed to be full-rank. The vector \(x\) is the state, which is assumed to be not accessible for direct measurements, while \(y\) is the measured output of the system. The vector \(u(t)\) represents the redundant control input and \(\tau(t)\) is the generalized control effect or virtual input. Without loss of generality, the desired control effect \(\tau_c(t)\) is assumed to be given by a suitable known function depending on the system output:

\[
\tau_c(t) = f(y(t)).
\] (1)

The above condition can also be generalized, assuming that the desired effect \(\tau_c\) and the measured output \(y(t)\) are related through a suitable dynamic law. A control allocation strategy is defined such that, whenever it is possible, the control \(u\) satisfies

\[
Gu(t) = \tau_c(t).
\] (2)

Although the above linear equation always admits (uncountable) exact solutions when \(rk(G) = k\), there are possible constraints or bounds to be met and this may lead to the existence of approximate solutions only:

\[
\begin{cases}
u \in U \\
Gu(t) = \tau_c(t) \neq \tau_e(t)
\end{cases}
\] (3)

We point out that the approximate effect \(\tilde{\tau}_e(t)\) may differ from the desired one \(\tau_e(t)\) but it is a known quantity, as it can be computed exploiting the input constraints given by \(U\) which is an assigned set. In the unconstrained case, a simple solution can be obtained using the right pseudo-inverse matrix [15]:

\[
u(t) = G^{-R}\tau_e(t), \quad G^{-R} := G^T(GG^T)^{-1}.
\] (4)
In the case of constrained control $u \in U$, several methods for control allocation are available in the literature. One of the simplest approaches is to compute the exact solution in the unconstrained case and then to saturate it according to the bounds on the control inputs; on the other hand, the control effect provided by such saturated solution may be significantly different from the desired one (see [11] for instance). Other sophisticated and possibly more precise methods are based on redistributed pseudo-inverses [22] [24], Daisy-chaining [5], dynamic allocation [16] [26] and error minimization by linear or quadratic programming [3] [14] [19].

In this paper we consider the class of faults acting on effectors and actuators efficiency by decreasing their effectiveness: these can be modeled by a multiplicative term $\Delta(t)$:

$$\tau(t) = G\Delta(t)u(t), \Delta(t) = diag[\delta_1(t), ..., \delta_m(t)], \delta_i(t) \in [0, a], a \geq 1.$$  

It follows that, whenever $\delta_i(t) \equiv 1 \forall i = 1, ..., m$, the controller operates with nominal conditions and hence 

$$\tau(t) = Gu(t) = \hat{\tau}_c(t) (= \tau_c(t) \text{ if no input constraint is considered})$$

On the other hand if one of the actuators is subject to a loss of effectiveness or complete failure, i.e. if $\delta_i(t) \neq 1$ for some $i$, the designed control law will no longer be able to ensure the desired effect, this meaning that, in the case of fault presence, one may have 

$$\tau(t) \neq \hat{\tau}_c(t)$$

with a consequent deterioration of system performances. Such problems can be avoided by accommodating the fault effects if a suitable control reconfiguration policy is considered. Defining a set of diagnosis signals, usually called residuals, one can detect and isolate the faults; then, performing the correct reconfiguration of the control input, one can track the (approximate) desired effect $\hat{\tau}_c$ properly again. The approach presented in this paper is based on Unknown Input Observers UIO (see for instance [8]); the main advantage of such observers is that, if some structural conditions are met, the parameters can be designed such that the resulting estimation error is independent of some inputs of the systems, even if these are not measured directly. The general structure of an UIO is the following:

$$\begin{cases}
\dot{z}(t) = Fz(t) + RBv(t) + Ky(t) \\
\dot{x}(t) = z(t) + Hy(t)
\end{cases}$$

where the matrices $F, R, K$ and $H$ are design parameters and $v(t), y(t)$ are, respectively, a known reference input signal and the measured output of the system to be estimated through the observer; the signal $v(t)$ is usually set equal to the nominal and unperturbed input that is commanded to the system. It is worth to note that, in order to achieve a correct asymptotic state estimation, the matrix $F$ has to be Hurwitz.

As it will be shown in the following, unknown input observers are a useful tool for the task of isolating faults in overactuated systems. Moreover, thanks to input redundancy, the control can be re-allocated in order to limit or avoid the use of faulty effectors/actuators once these have been isolated. On the other hand, this is not the only advantage of input redundancy in the considered framework: indeed, control allocation can be combined together with the fault isolation scheme in order to enlarge the family of identifiable faulty events.

The reconfiguration can be performed by different methods, depending on several factors such as actuator dynamics, bounds on energy consumption, limited control inputs rates or other control constraints. It is worth to note that, due to the negative effects of faults, also the desired control effect $\tau_c(t)$ might be requested to change with respect to the original one in order to recover the deteriorated system performances, this corresponding to update the relation between $\tau_c(t)$ and the output signal $y(t)$ given by (1).

In the simplest case of unconstrained inputs, the nominal control allocation law is given by (4)) and therefore, if the actuators $i_1, ..., i_q$ are faulty and $q \leq m - k$, to get the desired effect $\tau_c(t)$ it is sufficient to re-allocate the control action setting

$$u_{i_1} = u_{i_2} = \cdots = u_{i_q} \equiv 0$$

and assigning the other components of $u(t)$, which are grouped for convenience in a vector $\hat{u} \in \mathbb{R}^{m-q}$, according to

$$\hat{u}(t) = \tilde{G}^{-R} \tau_c(t),$$

(5)

where the matrix $\tilde{G} \in \mathbb{R}^{k \times (m-q)}$ is obtained from $G$ by neglecting the columns $i_1, ..., i_q$. 

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On the other hand, since control reconfiguration can be regarded as a reduced-order control allocation problem in which some of the inputs are neglected, the use of the aforementioned techniques for handling input constraints can be straightforward extended. However, by turning off the input signals corresponding to faulty actuators, the redundancy of control inputs is reduced and the error between the desired control effect and the control effect provided by the approximate solution may increase after reconfiguration, as the class of admissible solutions to the allocation problem reduces.

The minimization of an assigned cost function is often included in the control allocation scheme. In the case of a quadratic cost function with weight matrix $\Omega = \Omega^T \in \mathbb{R}^{m \times m}$, the nominal exact solution to the optimization problem

$$
\begin{align*}
\min_{u \in \mathbb{R}^m} & \quad u^T \Omega u \\
\text{subject to} & \quad \tau_c(t) = G u(t)
\end{align*}
$$

is given by (see for instance [4])

$$
u(t) = G_{\Omega}^{-R} \tau_c(t),$$

where $G_{\Omega}^{-R}$ is the weighted right-pseudo inverse

$$G_{\Omega}^{-R} = \Omega^{-1} G^T (G \Omega^{-1} G^T)^{-1}. $$

In the event of faults affecting the actuators $i_1, \ldots, i_q$, if $q < m - k$, the control reconfiguration can be performed such that the new input is the solution to the reduced optimization problem

$$
\begin{align*}
u_{i_1} = \cdots = \nu_{i_q} & \equiv 0 \\
\min_{\bar{\omega} \in \mathbb{R}^{m-q}} & \quad \bar{\omega}^T \hat{\Omega} \tilde{u} \\
\text{subject to} & \quad \tau_c(t) = \hat{G} \tilde{u}(t)
\end{align*}
$$

where $\hat{G} \in \mathbb{R}^{k \times (m-q)}$ is obtained from $G$ by neglecting the columns $i_1, \ldots, i_q$ and $\hat{\Omega} \in \mathbb{R}^{(m-q) \times (m-q)}$ contains the elements $\omega_{ij}$ from $\Omega$ such that $i, j \neq i_1, \ldots, i_q$; in particular the vector of non-faulty components $\tilde{u}$ is given by

$$
\tilde{u}(t) = \hat{G}^{-R} \tau_c(t).
$$

In the presence of constraints $u \in U$, the above reduced optimization problem has to be modified in order to take into account possible bounds and rate limitations on the control input.

As already mentioned, control allocation can be used actively also to make faults observable; in particular, by considering additional input constraints (see Section 3.3) which force control devices and surfaces to achieve common modes, one can isolate faults affecting selected groups of effectors. Whenever such constraints are not allowed to be imposed simultaneously in practice due to lack of control design freedom, an iterative control allocation scheme can be defined in order to switch periodically from one common mode to another after a prescribed time interval, until the fault isolation task is accomplished successfully.

Smooth control reconfiguration. In many practical scenarios (in the presence of actuator dynamics for instance), an abrupt change of control inputs is not allowed; as a consequence, turning off the faulty actuators $u_{i_1}, \ldots, u_{i_q}$ is not an instantaneous operation in general and it might require some time to be performed. One possible direct approach to address this problem is to consider a sufficiently smooth increasing scalar function $\chi(t) : [0, 1] \rightarrow [0, 1]$ with $\chi(0) = 0$, $\chi(1) = 1$, to be used to create a junction between the original (faulty) control input signal and the reconfigured one. Let us denote by $u_{or}(t), u_{rec}(t)$ the original control and the reconfigured control respectively and let us suppose that the time needed to move the actuators from the original to the new pose does not exceed a given constant $t_{\text{max}} > 0$; let us denote in addition by $t_0$ the time instant in which the faults are isolated and the control reconfiguration has to be initialized. We set

$$u_{\text{jun}}(t) = \left(1 - \chi \left(\frac{t}{t_0 + t_{\text{max}}}\right)\right) u_{or}(t + t_0) + \chi \left(\frac{t}{t_0 + t_{\text{max}}}\right) u_{rec}(t + t_0)$$

and

$$u(t) = \begin{cases} u_{or}(t) & t \leq t_0 \\ u_{\text{jun}}(t) & t_0 < t < t_0 + t_{\text{max}} \\ u_{rec}(t) & t \geq t_0 + t_{\text{max}} \end{cases}$$
Since by assumption both the original and the new input satisfy the control allocation equation (2) in the absence of faults, the following identity holds for the junction control

\[ Gu_{\text{jun}}(t) = (1 - \chi(t))Gu_{\text{or}}(t) + \chi(t)Gu_{\text{rec}}(t) = (1 - \chi(t))\tau_c(t) + \chi(t)\tau_c(t) = \tau_c(t), \]

this showing that the control action \( u(t) \) satisfies the allocation condition for each time step and moreover it guarantees a smooth passage from the original to the new input configuration. It is worth to note that, also in the transition interval \( (t_0, t_0 + t_{\text{max}}) \), the performances of the system are improved with respect to those provided by the original faulty control; this can be demonstrated observing that the errors on the control effect for the original control and for the junction control are assigned respectively by

\[ ||\tau_c(t) - G\Delta(t)u_{\text{or}}(t)|| = ||G(I - \Delta(t))u_{\text{or}}(t)|| \]

and

\[ ||\tau_c(t) - G\Delta(t)u_{\text{jun}}(t)|| = (1 - \chi(t))||G(I - \Delta(t))u_{\text{or}}(t)||, \]

where \( (1 - \chi(t)) < 1 \) for \( t \in (t_0, t_0 + t_{\text{max}}) \).

In the case of constrained control \( u \in U \), where \( U \subset \mathbb{R}^m \) is some prescribed range set, several methods for control allocation are available in the literature. One of the simplest approaches is to compute the exact solution in the unconstrained case and then to saturate it according to the bounds on the control inputs; on the other hand, the control effect provided by such saturated solution may be significantly different from the desired one (see [11] for instance). Other sophisticated and possibly more precise methods are based on redistributed pseudo-inverses [22] [24], Daisy-chaining [5], dynamic allocation [16] [26] and error minimization by linear or quadratic programming [3] [14] [19]. Since control reconfiguration can be regarded as a reduced-order control allocation problem in which some of the inputs are neglected, the use of such techniques for handling input constraints can be straightforwardly extended. However, by turning off the input signals corresponding to faulty actuators, the redundancy of control inputs is reduced and the error between the desired control effect and the control effect provided by the approximate solution may increase after reconfiguration, as the class of admissible solutions to the allocation problem reduces.

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### 3 Fault detection and isolation

The estimation error is defined as the difference between the true state \( x(t) \) and the estimated state \( \hat{x}(t) \):

\[ e(t) = x(t) - \hat{x}(t). \]

Our aim is to design a family of unknown input observers \( \{\mathcal{O}_h\}_{h=1} \), such that the information provided by the estimation errors allow us to detect and isolate faults. To address such target one can proceed as follows.

The input \( v(t) \) in the observer is set equal to the reference control effect \( \hat{\tau}_c(t) \), that is

\[ v(t) = Gu(t) = \hat{\tau}_c(t), \]

where \( u(t) \) is the nominal (fault free) control (3) (or (2) in the unconstrained case). Exploiting the observer structure, the dynamics of the error is ruled by the following equation

\[ \dot{e}(t) = \hat{x}(t) - \dot{\hat{x}}(t) = [(I_{n \times n} - HC)A - KC + FHC]x(t) - F\hat{x}(t) + (I_{n \times n} - HC)BG\Delta(t)u(t) - RBGu(t). \]
Setting \( K = K_1 + K_2 \), if the following conditions are satisfied
\[
R = I_{n \times n} - HC \\
F = RA - K_1 C, \quad \sigma(F) \in \mathbb{C}^- \\
K_2 = FH
\] (6)
(7)
(8)
then the latter equation reduces to
\[
\dot{e}(t) = Fe(t) + RBG(\Delta(t) - I_{m \times m})u(t),
\]
where \( \sigma(\cdot) \) stands for the spectrum of a matrix and the set \( \mathbb{C}^- \) in the left open complex semiplane. It is worth to note that a sufficient condition for \( F \) to be Hurwitz is the freedom to assign the eigenvalues of the matrix \( RA - K_1 C \) through the feedback gain \( K_1 \); to this purpose we give the following statement.

**Theorem 3.1.** ([8]) Let \( Q \in \mathbb{R}^{n \times \ell} \) be a matrix satisfying the following conditions
\[
(C1) \quad rk(Q) = rk(CQ); \\
(C2) \quad \text{the pair } (C, A_Q) \text{ is detectable, where } A_Q := A - Q((CQ)^T CQ)^{-1} (CQ)^T CA.
\]

Then the matrices \( H, R, F, K \) can be found such that equalities (6)-(8) hold true together with
\[
RQ = 0.
\] (9)

A particular solution for \( H \) is given by
\[
H_Q = Q((CQ)^T CQ)^{-1} (CQ)^T.
\]
Conversely, if equalities (6)-(9) are satisfied, then \( Q \) verifies (C1) and (C2).

The above theorem states necessary and sufficient conditions for the existence of an unknown input observer such that (9) is satisfied. This latter condition is fundamental to make the estimation error insensitive to some of the effector/actuator faults or independent of additive uncertain inputs.

Let us denote by \( W \in \mathbb{R}^{n \times m} \) the matrix \( BG \), whose columns will be indicated with \( W_1, ..., W_m \), i.e.
\[
BG = W = [W_1 \cdots W_m].
\] (10)

It is worth to note that the matrix \( BG(\Delta(t) - I) = W(\Delta(t) - I_{m \times m}) \) appearing in the expression of \( \dot{e}(t) \) has the following structure:
\[
W(\Delta(t) - I_{m \times m}) = [(\delta_1(t) - 1)W_1 \cdots (\delta_m(t) - 1)W_m],
\]
and hence a fault in the \( j^{th} \) effector may only affect the \( j^{th} \) column of \( W \).

**Definition 3.1.** We call a multi-index any vector \( J \) of increasing natural numbers, i.e. \( J = (j_1, ..., j_\ell) \) with \( j_q \in \mathbb{N} \ \forall q = 1, ..., \ell \) and \( 1 \leq j_1 < j_2 < \cdots < j_\ell \leq r, \ r \geq \ell \). The positive integers \( \ell \) and \( r \) are defined as, respectively, the length \( L(J) \) and the domain \( D(J) \) of multi-index \( J \).

It is worth to note that the number \( s \) of distinct multi-indices having length \( \ell \) and domain \( r \) is given by the binomial coefficient
\[
s = \binom{r}{\ell} = \frac{r!}{(r-\ell)!\ell!}.
\] (11)

**Definition 3.2.** Given the system matrices \( B, G \) and defining \( W = BG \) as in (10), we call uniform sub-rank of \( W \) the positive integer \( k_0 \leq k \) computed as
\[
k_0 := \max\{\ell \leq k : rk[W_{j_1} \cdots W_{j_\ell}] = \ell, \ \forall \ \text{multi-index } J = (j_1, ..., j_\ell) \text{ with } L(J) = \ell, \ D(J) = m\}.
\] (12)

**Notation** Given a multi-index \( J = (j_1, ..., j_\ell) \) with \( D(J) = m \), we denote by \( W_J \in \mathbb{R}^{n \times \ell} \) the matrix composed by the columns of \( W \) corresponding to the indices included in \( J \), i.e.
\[
W_J = [W_{j_1} \cdots W_{j_\ell}].
\]
3.1 Zeroing faults input (ZFI)

The proposed approach consists in the design of a family of observers such that the corresponding residual signals are sensitive to some faults only; in this way, combining them together, the faults can be isolated. The next theorem provides conditions for the existence of a family of independent unknown input observers with the desired properties.

**Theorem 3.2.** Let \( k_0 \) be the uniform sub-rank of the matrix \( W \) according to Definition 3.2 and let us assume that, setting \( \ell \leq k_0 - 1 \), the following conditions hold true for any multi-index \( J \) with \( L(J) = \ell \) and \( D(J) = m \):

1. \( \text{rk}(CW_J) = \text{rk}(W_J) = \ell \);

2. the pair \((C, A_J)\) is detectable, where

\[
\]

Then a set of \( s = \binom{m}{\ell} \) unknown input observers \( \{O_h\}_{h=1}^s = \{F^{(h)}, R^{(h)}, K^{(h)}, H^{(h)}\}_{h=1}^s \), associated to the multi-indices \( J_h \) with \( h = 1, \ldots, s \), can be designed such that conditions (6)-(8) are satisfied and moreover

\[
R^{(h)}W_{J_h} = 0, \quad R^{(h)}W \neq 0. \tag{13}
\]

**Proof.** The conditions are equivalent to the hypothesis of Theorem 3.1 for \( Q = W_J \); the requirement \( \ell < k_0 \) is necessary to prevent lack of fault observability due to the rank-deficiency of \( BG \). In particular, since \( L(J_h) < k_0 \) \( \forall h = 1, \ldots, s \), condition (12) guarantees that for any \( h \) there exists at least one index \( j^* \) not included in \( J_h \) such that

\[
R^{(h)}W_{J_h, j^*} \neq 0.
\]

On the other hand if such property is not achieved, the observer \( O_h \) may result to be insensitive to all system faults. Referring to the multi-index \( J \) we define in addition the complementary input matrix \( W_J \in \mathbb{R}^{n\times m} \), which is obtained substituting with null entries the columns of \( W = BG \) which are included in \( W_J \).

Thanks to the construction provided by Theorem 3.2, the error dynamics associated to the multi-index \( J_h \) is given by

\[
e^{(h)}(t) = F^{(h)}e^{(h)}(t) + R^{(h)}W^{c}_{J_h}(\Delta(t) - I_{m\times m})u(t)
\]

with associate residual signal

\[
r^{(h)}(t) = Ce^{(h)}(t).
\]

According to Theorem 3.2, the residual signal \( r^{(h)}(t) \) is completely decoupled from faults affecting the effectors/actuators corresponding to the multi-index \( J_h \). The residual \( r^{(h)}(t) \) is said to be active, this meaning that some faults have occurred, if it overpass a known threshold \( \gamma^{(h)} \), i.e.

\[
\begin{cases} 
||r^{(h)}(t)|| \leq \gamma^{(h)} & \Rightarrow \text{no faults} \\
||r^{(h)}(t)|| > \gamma^{(h)} & \Rightarrow \text{presence of faults}.
\end{cases}
\]

The threshold \( \gamma^{(h)} \) can be computed or tuned based on known bounds for observer initialization error as well as for disturbance or noise terms affecting the plant.

**Remark 3.1.** We point out that, after the imposition of condition (13), there are still degrees of freedom in the observer parameters design. In particular a subset of the eigenvalues of the matrix \( F^{(h)} \) can be arbitrarily assigned, depending on the rank of the observability matrix for the pair \((C, R^{(h)}A)\), and moreover the range of the matrix \( R^{(h)} \) is only prescribed for (at most) \( k_0 \) directions by (13): these additional degrees of freedom can be used for tuning the observer parameters in order to cope with noise, disturbances or modeling errors and compensate for their negative effects. As a matter of fact, if the system dynamics includes an additive uncertain input \( E(t) \in \mathbb{R}^d \) with \( d \leq p - k \) and \( \text{Im}(E(t)) \cap \text{Im}(W_{J_h}^c) = 0 \forall t \geq 0 \) and for some \( h = 1, \ldots, s \), then the matrix \( R^{(h)} \) can be designed such that \( R^{(h)}[W_{J_h}, E(t)] = 0 \forall t \geq 0 \) and hence the corresponding residual signal \( r^{(h)}(t) \) is disturbance-decoupled [8].
To compute a suitable threshold one can proceed as follows. Assuming to have a bound on the initial error $e^{(h)}(0)$, i.e.

$$||e^{(h)}(0)|| = ||x(0) - \hat{x}^{(h)}(0)|| \leq \xi.$$  

Then, since in the absence of faults the error dynamics is not influenced by system inputs, one can define

$$\gamma^{(h)} = \Gamma^{(h)} ||C|| \xi,$$

where the constant $\Gamma^{(h)}$ is such that

$$||e^{F^{(h)}t}|| \leq \Gamma^{(h)} \forall t \geq 0.$$  

Moreover, it is possible to define also time-variant thresholds in order to achieve better fault sensitivity performances in the filters; this can be done in a straightforward way setting

$$\gamma^{(h)}(t) = \Gamma^{(h)} e^{-t\lambda^{(h)}_{min}} ||C|| \xi,$$

where

$$\lambda^{(h)}_{min} = \min\{||\lambda|| : \lambda \in \sigma(F^{(h)})\}.$$  

The information provided by the activation of the residual signal $r^{(h)}(t)$ can be stored in a binary $m-$dimensional vector $\mu^{(h)}$ by setting

$$\mu^{(h)}_i = \begin{cases} 1 & \text{if } i \in J_h \\ 0 & \text{otherwise} \end{cases}$$

In this way, with a simple additive procedure, one can detect and isolate up to $\ell$ faults. We define the (total) residual signature $\mu^\sharp$ as the sum of the vectors $\mu^{(h)}$ related to active residuals only; the analysis of such residual signature allow us to isolate the faults and hence to identify the effectors in which they are occurred. In particular the residual signature $\mu^\sharp$ is initialized as the null vector $0 \in \mathbb{R}^m$, then it is updated by the following algorithm:

```
for $h = 1 : s$
    if $r^{(h)}(t) > \gamma^{(h)}$ and FAULT(h)=false then
        $\mu^\sharp = \mu^\sharp + \mu^{(h)}$
        FAULT(h)=true
    end if
for $i = 1 : m$
    if $\mu^\sharp_i = s/2$ then “A fault is present in the $i^{th}$ effector”
```

We point out that the value $s/2$ can be obtained by computing the number of possible signatures $\mu^{(j)}$ with one of the entries set equal to 0 either 1. To clarify the proposed detection/isolation scheme, we give a simple abstract example.

**Example**

Let us consider a system having the following matrix dimensions $n = 5, m = p = 4, k = 3$ with $k_0 = 3$. In this case we can detect and isolate up to $\ell = 2 = k_0 - 1$ faults affecting the system; let us assume that a suitable family of $s = \binom{m}{\ell} = 6$ ZFI residual signals $r^{(h)}(t)$ has been designed, whose corresponding binary vectors $\mu^{(h)}$ are listed below

$$\mu^{(1)} = (1,1,0,0), \quad \mu^{(2)} = (1,0,1,0), \quad \mu^{(3)} = (1,0,0,1),$$  

$$\mu^{(4)} = (0,1,1,0), \quad \mu^{(5)} = (0,1,0,1), \quad \mu^{(6)} = (0,0,1,1).$$  

- Suppose that effector $u_2$ undergoes a fault; only the sensitive residual signals will be activated and as a consequence we have a total residual signature

$$\mu^\sharp = \mu^{(1)} + \mu^{(4)} + \mu^{(5)} = (1,3,1,1)$$

The maximum entry of the residual signature is the second one, $\mu^\sharp_2 = 3 = s/2$: this implies that a fault has occurred in the corresponding effector.
• Suppose that failures are present in both \( u_1 \) and \( u_4 \). All residual will be activated except \( r^{(5)}(t) \) which is insensitive to both faults; the residual signature is therefore

\[
\mu^* = \mu^{(1)} + \mu^{(2)} + \mu^{(3)} + \mu^{(4)} + \mu^{(6)} = (3, 2, 3, 2).
\]

As before, the faulty effectors are those corresponding to higher entries in the residual signature.

• Suppose now that effectors \( u_1, u_2 \) and \( u_4 \) fail at the same time. In this case all residual signal will be activated and we get a saturated residual signature

\[
\mu^* = \sum_{i=1}^{6} \mu^{(i)} = (3, 3, 3, 3)
\]

this indicating that more than two faults have occurred. On the other hand, due to rank deficiency of the matrix \( BG \), it is not possible to isolate such faults or to know exactly how many they are; in particular, referring to the present example, one will get the same residual signature if all the four effectors fail, as well as if the fault involves any other combination of three of them.

We summarize in the following statement, which can be regarded as a corollary to Theorem 3.2, the main properties of the presented method.

**Theorem 3.3.** For \( \ell < k_0 \), let us consider the set \( \{J_h\}_{h=1}^{\sharp} \) of all distinct multi-indices of length \( \ell \) and domain \( m \). If the conditions of Theorem 3.2 are fulfilled, then a family of observers \( \{O_h\}_{h=1}^{\sharp} = \{F^{(h)}, R^{(h)}, K^{(h)}, H^{(h)}\}_{h=1}^{\sharp} \) with corresponding residual signals \( r^{(h)}(t) \), can be designed such that

- The residual \( r^{(h)}(t) \) is insensitive to faults entering the system with input matrix \( W_{J_h} \).
- The residual \( r^{(h)}(t) \) is sensitive to faults entering the system with input matrix \( W_{J_h}^c \).
- Combining the information provided by all activated residuals, it is possible to isolate up to \( \ell \) faults; this can be done considering the residual signature \( \mu^* \) and checking whose entries of such vector are equal to \( s/2 \): they correspond to the positions of the faulty actuators.
- The critical case in which the system is affected by more than \( \ell \) faults leads to a completely saturated residual signature having all entries equal to the maximum admissible value \( s/2 \).

A sufficient condition for the existence of a family of observers with the desired properties can be deduced from the next technical result.

**Proposition 3.1.** Given \( E = [E_1, \ldots, E_r] \) with \( rk(E) = r \) and \( rk(CE) = r \), let \( E_{(q)} \in \mathbb{R}^{n \times (r-1)} \) be the matrix obtained from \( E \) neglecting the \( q^{th} \) column, i.e. \( E_{(q)} = [E_1, \ldots, E_{q-1}, E_{q+1}, \ldots, E_r] \), and define

\[
R = I - HC, \quad H = E[(CE)^TCE]^{-1}(CE)^T, \quad R^* = I - H^*C, \quad H^* = E_{(q)}[(CE_{(q)})^TCE_{(q)}]^{-1}(CE_{(q)})^T.
\]

Then, for any \( A \in \mathbb{R}^{n \times n} \), the detectability of \((C, RA)\) implies the detectability of \((C, R^*A)\).

**Proof.** Let \( \text{ker}(C) = \text{span}\{\omega_1, \ldots, \omega_{n-r}\} \) and let \( v_1, \ldots, v_{p-r} \) be linearly independent vectors such that

\[
Cv_j \neq 0, \quad HCv_j = 0 \quad \forall \ j = 1, \ldots, p-r.
\]

By definition we have

\[
RE = 0, \quad R^*E_{(q)} = 0,
\]

\[
R\omega_j = R^*\omega_j = \omega_j \quad \forall \ j = 1, \ldots, n-p,
\]

\[
Rv_j = R^*v_j = v_j \quad \forall \ j = 1, \ldots, p-r.
\]
Moreover the vector $E_q$, representing the missing column in the matrix $E_{(q)}$, can be decomposed as

$$E_q = \bar{E} + w,$$

with

$$R^* w = 0, \quad R^* \bar{E} = \bar{E}, \quad C \bar{E} \neq 0$$

and

$$rk([v_1 \cdots v_{p-h} \bar{E}]) = p - r + 1.$$  

Hence, by the above constructions, the following basis of $\mathbb{R}^{n \times n}$ is well defined:

$$B := \{\omega_1, \ldots, \omega_{n-p}, v_1, \ldots, v_{p-r}, \bar{E}, E_1, \ldots, E_{q-1}, E_{q+1}, \ldots, E_r\};$$

in particular any element in $\mathbb{R}^n$ can be written as a linear combination of such vectors. Let $x \in \mathbb{R}^n$ an observable state for $(C, RA)$ such that

$$C(\text{RA})x \neq 0, \quad (\text{RA})^j x = 0 \quad \forall i = 0, \ldots, h - 1; \quad \text{(14)}$$

we claim that

$$C(\text{RA})^i x \neq 0 \quad \text{for some } 1 \leq i \leq h.$$  

Since for $j = 0$ condition (14) reduces to $Cx = 0$, one has necessarily $x = \sum_{j=1}^{n-p} \alpha_j \omega_j$ for some coefficients $\alpha_j \in \mathbb{R}$; on the other hand if $Cx \neq 0$ there is nothing to prove. Let us consider the case $h = 1$ first. By definition we have

$$\ker(R^* A) \subset \ker(\text{RA})$$

and hence

$$\text{RA}x \neq 0 \Rightarrow R^* Ax \neq 0.$$  

Moreover if CRAx $\neq 0$ one has necessarily

$$Ax = \sum_{j=1}^{n-p} \beta_j \omega_j + \sum_{j=1}^{p-r} \gamma_j v_j + \zeta_1 \bar{E} + \sum_{i \neq j, i=1}^{r} \tau_i E_i$$  

with $\gamma_i \neq 0$ for at least one index $i$: in fact the following identities hold

$$\text{RA}x = \sum_{j=1}^{n-p} \beta_j \omega_j + \sum_{j=1}^{p-r} \gamma_j v_j,$$

$$\text{CRA}x = \sum_{j=1}^{p-r} \gamma_j Cv_j,$$

On the other hand one has

$$R^* Ax = \sum_{j=1}^{n-p} \beta_j \omega_j + \sum_{j=1}^{p-r} \gamma_j v_j + \zeta_1 \bar{E}$$  

and hence, as by construction $C \bar{E} \neq 0$, one can deduce

$$C R^* Ax = \sum_{j=1}^{p-r} \gamma_j Cv_j + \zeta_1 C \bar{E} \neq 0.$$  

Let us treat now the general case $h \geq 2$. Condition $C(\text{RA})^j x = 0 \quad \forall j = 1, \ldots, h - 1$ implies that

$$(\text{RA})^j x \in \text{span}\{\omega_1, \ldots, \omega_{n-p}\} \quad \forall j = 1, \ldots, h - 1.$$  

On the other hand one has

$$R^* Ax = RAX + \zeta_1 \bar{E};$$

if $\zeta_1 \neq 0$ one can conclude observing that

$$C R^* Ax = \zeta_1 C \bar{E} \neq 0,$$
otherwise
\[ R^*Ax = RAx \text{ with } AR^*Ax = ARAx \neq 0. \]

In the latter case, as for the previous step, one obtains an equality of the form
\[ R^*AR^*Ax = RARAx + \zeta_2\bar{E}. \]

This procedure can be iterated according to the scheme:
\[ (R^*A)^jx = (RA)^jx + \zeta_j\bar{E} \]

where
\[
\begin{cases}
\zeta_j \neq 0 \Rightarrow C(R^*A)^jx \neq 0 \\
\zeta_j = 0 \Rightarrow A(R^*A)^jx = A(RA)^jx
\end{cases}
\]

As a consequence, if \( \zeta_j \neq 0 \) for some \( j < h \), the state \( x \) turns out to be observable for \( (C, R^*A) \); on the other hand if \( \zeta_j = 0 \forall j = 1, \ldots, h-1 \), then one has
\[ C(R^*A)^hx = C(RA)^hx + \zeta_kC\bar{E}; \]

recalling that \( ker(C) \cap \text{span}\{v_1, \ldots, v_{p-r}, \bar{E}\} = 0 \) and that \( C(RA)^hx \neq 0 \) by assumption, one can conclude that
\[ C(R^*A)^hx \neq 0 \]

in both cases \( \zeta_h = 0 \) and \( \zeta_h \neq 0 \). The claim is proved. \( \square \)

**Corollary 3.1.** Let us assume that \( rk(CW) = rk(W) = k \) and that \( (C, A_W) \) is a detectable pair; then, as a consequence of the above result, the necessary conditions of Theorem 3.2 are satisfied for any multi-index \( J_h, h = 1, \ldots, s \).

The next corollary presents a straightforward extension of Proposition 3.1, that will be used in the design of cluster residuals.

**Corollary 3.2.** Given \( E = [E_1, \ldots, E_s] \) with \( rk(E) = r \) and \( rk(CE) = r \), let \( \hat{E} \in \mathbb{R}^{n \times (r-1)}, \bar{E} = [\bar{E}_1 \cdots \bar{E}_{r-1}] \), with \( rk(\bar{E}) = r-1 \) and \( \text{Im}(\bar{E}) \subset \text{Im}(E) \); let us define
\[ R = I - HC, \quad H = E[(CE)^TCE]^{-1}(CE)^T, \]
\[ \hat{R} = I - \hat{H}C, \quad \hat{H} = \hat{E}[(CE)^TCE]^{-1}(CE)^T. \]

Then, for any \( A \in \mathbb{R}^{n \times n} \), the detectability of \( (C, RA) \) implies the detectability of \( (C, \hat{R}A) \).

### 3.2 Second method: constrained output fault directions (COFD)

This subsection is dedicated to the presentation of an alternative approach to UIO-based fault detection and isolation in overactuated systems. This second method consists in constraining the residuals in prescribed subspaces of the output space (see for instance [?] [?]). We still refer to the observer structure given by (1)-(3); this approach is valid also for different classes of estimators indeed, such as classical Luenberger observers. On the other hand, the properties of the UIO may be very useful since in this case the construction of the detection filters can be readily adapted to cope with the presence of disturbance terms (see Remark 3.3 at the end of this section).

Throughout the presentation of this method it will be assumed \( m \leq p \). Let us consider the canonical basis of \( \mathbb{R}^p \), namely \( e_1, \ldots, e_p \); since by assumption the output matrix \( C \) is full-rank, there exists \( S \in \mathbb{R}^{n \times p} \) such that
\[ CS = I_{p \times p} = [e_1 \cdots e_p]. \]

The general solution of such equation is given by
\[ S = C^T(CC^T)^{-1} + [I_{n \times n} - C^T(CC^T)^{-1}C]S_*, \] (17)
where $S_* \in \mathbb{R}^{n \times p}$ is an arbitrary matrix. Denoting by $S_1, ..., S_p$ the columns of the matrix $S$, the basic idea of the method is to design the observer parameters in order to guarantee that, if a fault occurs in the $i^{th}$ actuator, then the estimation error maintains the direction $S_i$ during the system evolution, this corresponding to a fixed direction $e_i$ for the residual. It is worth to note that a first strong design constraint for the achievability of this condition is that directions $S_1, ..., S_p$ need to correspond to eigenvectors of the observer matrix $F$. Moreover, due to the rank deficiency of $BG$, it is not possible in general to address a decoupled distribution of the faults effects over the columns of the matrix $S$ and we are required to deal with linear combinations of such characteristic directions. We recall that the dynamics of the estimation error is given by the equation

$$\dot{e}(t) = F e(t) + RW (\Delta(t) - I_{m \times m}) u(t).$$

Since $\text{rank}(W) = k < m \leq p$, we can arbitrarily assign only $k$ columns of the matrix $RW$ through the design parameter $R$, as the remaining $m - k$ are consequently constrained; we need therefore to consider several independent observers to achieve a correct fault isolation. One can proceed as follows. Let $k_0 \leq k$ the uniform sub-rank of the matrix $W$. Then, if by tuning the matrix $R$ we prescribe the first $k_0$ columns of $RW$, for example imposing that they have to be equal to $S_1, S_2, ..., S_{k_0}$, we get

$$RW = [S_1 \cdots S_{k_0} V_1 \cdots V_{m-k_0}],$$

where $V_j = \sum_{i=1}^{k_0} \alpha_{ij} S_i$ for some coefficients $\alpha_{ij}$. As a consequence we have

$$CRW = [e_1 \cdots e_{k_0} \omega_1 \cdots \omega_{m-k_0}],$$

with $\omega_j = \sum_{i=1}^{k_0} \alpha_{ij} e_i$. Setting $R^{(1)} = R$, we can iterate this construction by designing $R^{(h)}$ such that

$$R^{(h)} [W_{j_1h} \cdots W_{j_{k_0}h}] = [S_{i_1h} \cdots S_{i_{k_0}h}]$$  \hspace{1cm} (18)

as multi-indices $J_h = (j_1^h, ..., j_{k_0}^h)$ and $I_h = (i_1^h, ..., i_{k_0}^h)$ vary; at the end of the construction we obtain a family of matrices $\{R^{(h)}\}_{h=1}^s$, with

$$s = \binom{m}{k_0} \binom{p}{k_0} = \frac{m!p!}{(m - k_0)!(p - k_0)!(k_0)!^2}.$$

We point out that the information provided by the residuals associated to such family of matrices is redundant. In order to reduce the computational burden and avoid overlapping of information, we need to investigate which is the minimum number of matrices $R^{(h)}$ required for a proper fault isolation. We see from the explicit construction of $R^{(1)}$ that, if $k_0 > 1$, in this case faults affecting the first $k_0$ actuators lead to residual signals directed as $e_1, ..., e_{k_0}$ respectively, while faults affecting the other actuators lead to residual signal obtained as linear combination of two or more vectors $e_j$, $j = 1, ..., k_0$. On the other hand, if more than one fault occurs we get a residual signal defined by a linear combination of vectors $e_j$ as well, this meaning that with the information provided by this unique observer we are not able to distinguish multiple faults from individual faults affecting one of the last $m - k_0$ actuators. For this reason, the number of observers to be considered has to be sufficient to decouple effects of single and multiple faults. To this purpose we note that, since the observers are designed independently, the particular choice of vectors $S_j$ among $\{S_1 \cdots S_p\}$ in (18) does not influence the fault isolation procedure: nevertheless, this freedom of choice may result to be helpful in the stabilization process of the matrix $F$. We claim that the maximum number of isolable faults is $k_0 - 1$ with a required number of observers equal to

$$\bar{s} = \binom{m}{k_0}.$$

This can be verified observing that, in order to isolate the faults, at least one residual $r^{(h)}(t)$ needs to have null projection along one of the basis vectors $e_j$: as a consequence, the number of isolable faults has to be less than the maximum admissible number of independent components of each residual $r^{(h)}(t)$, that is $k_0$. The integer $\bar{s}$ can be obtained simply computing the number of all distinct vectors of $m$ elements with $k_0$ assigned entries. For sake of simplicity we fix the vectors $S_j$ in (18), assuming that the right-hand side is equal to $\hat{S} := [S_1 \cdots S_{k_0}]$ for any $h$. Once the properties of the matrices $R^{(h)}$ are defined, one have to deal with the stabilization of the matrix $F^{(h)}$ together with the fulfillment of the rank condition

$$\text{rank}[S_j F^{(h)} S_j \cdots (F^{(h)})^{n-1} S_j] = 1 \ \forall \ j = 1, ..., k_0,$$  \hspace{1cm} (19)
which corresponds to the requirement for $S_j$ to be an eigenvector of the matrix $F^{(h)}$. We point out that for any $h = 1, \ldots, s$, the solution of (18) is given by

$$\begin{align*}
R^{(h)} &= (I_{n \times n} - H^{(h)}C) \\
H^{(h)} &= (W_{J_h} - \hat{S})(CW_{J_h})^{-L} + H^*_r(C - (CW_{J_h})(CW_{J_h})^{-L}C)
\end{align*}$$

(20)

where $(\cdot)^{-L}$ stands for the left pseudo-inverse and $H^*_r \in \mathbb{R}^{n \times p}$ is an arbitrary matrix. Let $M^{(h)} \in \mathbb{R}^{k_0 \times k_0}$ be a diagonal and negative definite matrix. Recalling that $F^{(h)} = R^{(h)}A - K_1^{(h)}C$, condition (19) can be achieved solving

$$K_1^{(h)}(C\hat{S}) = R^{(h)}A\hat{S} - \hat{S}M^{(h)},$$

(21)

which gives

$$K_1^{(h)} = (R^{(h)}A\hat{S} - \hat{S}M^{(h)})(C\hat{S})^{-L} + K^*_r(I_{p \times p} - (C\hat{S})(C\hat{S})^{-L}),$$

(22)

where $K^*_r \in \mathbb{R}^{n \times p}$ is an arbitrary matrix. Denoting by $e^{(h)}(t)$ the estimation error associated to the observer $O_h$, we define the (vectorial) residual signals:

$$r^{(h)}(t) = C e^{(h)}(t).$$

We are now ready to state the main result of this section.

\textbf{Theorem 3.4.} Let $S, R^{(h)}, H^{(h)}, K_1^{(h)}$ be assigned by (17), (20) and (22). Let us assume that, for any multi-index $J_h = (j_1^h, \ldots, j_k^h)$, $h = 1, \ldots, \bar{s}$, the following conditions hold true

1. $\text{rank}(W_{J_h}) = \text{rank}(CW_{J_h}) = k_0$;

2. the matrices $S_r, H^*_r$ and $K^*_r$ can be found such that the matrix $F^{(h)} = R^{(h)}A - K_1^{(h)}C$ is Hurwitz.

Then $(F^{(h)}, H^{(h)}, R^{(h)}, K^{(h)})$ is an unknown input observer and it satisfies condition (18) for any $h = 1, \ldots, \bar{s}$, hence the residual signals are able to detect and isolate up to $k_0 - 1$ faults affecting simultaneously the system actuators.

We can represent residuals as ordered sums of the basis vectors and their combinations; we will employ the $\oplus$ to indicate a logic sum of the vectors $v, w$ depending on the order, i.e. $v \oplus w \neq w \oplus v$. For example we obtain the following logic representation of the first residual:

$$r^{(1)} = e_1 \oplus e_2 \oplus \cdots \oplus e_{k_0} \oplus \omega^1_1 \oplus \cdots \oplus \omega^1_{m-k_0}$$

where $\omega^1_j$ is an arbitrary combination of the vectors $e_1, \ldots, e_{k_0}$; in a similar way, the residual associated to the multi-index $J_h = (j_1^h, \ldots, j_{k_0}^h)$ can be represented by

$$r^{(h)} = \omega^h_1 \oplus \cdots \oplus \omega^h_{j_1-1} \oplus e_1 \oplus \omega^h_{j_1} \oplus \cdots \oplus e_{k_0} \oplus \cdots \oplus \omega^h_{m-k_0}$$

We use again the abstract example of the previous section to illustrate the method efficiency.

\textbf{Example}

Let us consider a system having the following matrix dimensions $n = 5, m = p = 4, k = 3$. In this case we can detect and isolate up to 2 faults affecting the system. The residual signals to be considered are 4:

$$\begin{align*}
r^{(1)} &= e_1 \oplus e_2 \oplus e_3 \oplus \omega_1 \\
r^{(2)} &= e_1 \oplus e_2 \oplus \omega_2 \oplus e_3 \\
r^{(3)} &= e_1 \oplus \omega_1 \oplus e_2 \oplus e_3 \\
r^{(4)} &= \omega_4 \oplus e_1 \oplus e_2 \oplus e_3
\end{align*}$$
Suppose that actuator $u_2$ undergoes a fault; the first residual will be directed as $e_2$, this indicating the presence of a single fault in the second actuator.

Suppose that actuator $u_1$ and $u_3$ fail. The first residual is directed as a combination of $e_1$ and $e_3$, while the third one as a combination of $e_1$ and $e_2$: according to the positions of these vectors in the logic representation of residuals, the information provided by $r^{(1)}$ and $r^{(3)}$ is sufficient to isolate the faults.

Suppose that three actuators fail at the same time. In this case all residuals will be directed as combination of all $e_1$, $e_2$ and $e_3$, this meaning that no isolation can be accomplished.

Remark 3.2. It is worth to note that condition (19) is sufficient for prescribing a fixed direction to the residual signals only if the observer initialization error is zero or if the estimation error $e(t)$ evolves in a steady-state regime at the moment of fault occurrence.

Remark 3.3. Suppose the system to be affected by an additive disturbance term $b(t) \in \text{Im}(B_0)$ $\forall t \geq 0$, for some known matrix $B_0 \in \mathbb{R}^{n \times d}$. If $\text{Im}(W) \cap \text{Im}(B_0) = 0$, $p \geq k_0 + d$ and rank$(CB_0) = d$, the observer matrix $R^{(k)}$ fulfilling (18) can be designed such that
\[
R^{(k)}B_0 = 0,
\]
and hence residual signals are disturbance-decoupled.

3.3 Cluster residuals

We consider here an extended framework, in which the actuators are grouped into clusters:
\[
A_1 = \{u_1, \ldots, u_{i_1}\}, \quad A_2 = \{u_{i_1+1}, \ldots, u_{i_2}\}, \quad \ldots, \quad A_q = \{u_{i_q-1+1}, \ldots, u_m\},
\]
where $1 \leq i_1 < i_2 < \cdots i_{q-1} < m$. The introduction of this new model is motivated by the need of isolate common mode faults which may affect simultaneously actuators or effectors sharing the same auxiliaries. For sake of clarity only the case of non-overlapping clusters is presented; on the other hand, the proposed methods can be readily modified in order to be used also in the case of possibly overlapping clusters.

Let us denote by $\alpha_i$ the cardinality of the cluster $A_i$, i.e. $\alpha_i = i_i - i_{i-1}$ and hence $\sum_{i=1}^{q} \alpha_i = m$. The faults are supposed to act uniformly on the whole cluster $A_i$, so that they can be modeled as the block-diagonal multiplicative matrix:
\[
\Delta(t) = \text{diag}(d_1(t)I_{\alpha_1 \times \alpha_1}, d_2(t)I_{\alpha_2 \times \alpha_2}, \cdots, d_q(t)I_{\alpha_q \times \alpha_q})
\]

Let $k_0$ be the uniform sub-rank of $W$ and let us suppose that $k_0 \geq 3$ and
\[
\max_{i=1,\ldots,q} \alpha_i \leq k_0 - 1.
\]
As a consequence if a fault is present in a single cluster it can be detected and isolated; in the same way, if there exist two (or more clusters) with $\alpha_i + \alpha_j \leq k_0 - 1$, a multiple fault can be isolated as well. This can be done, referring to the ZFI observers design scheme proposed in Section 3.1, by selecting the matrix $H$ such that $R = I - HC$ satisfies
\[
R[W_{A_i}, W_{A_i}] = 0,
\]
where with a slight abuse of notation we have denoted by $W_{A_i}$ the matrix formed by the columns of $W$ corresponding to the actuators/effectors belonging to the cluster $A_i$. On the other hand, if for some pair of indices $i, j$, one has $\alpha_i + \alpha_j \geq k_0$, a multiple fault on the associated effectors will lead to completely saturated residuals, this meaning that no fault isolation can be achieved at the present step. However, by introducing additional constraints in the control allocation scheme, one can design a new set of observers to be used to identify faulty clusters of actuators/effectors. For sake of simplicity let us consider first the following case:
\[
\alpha_1 + \alpha_2 \geq k_0;
\]
in addition, let us assume that
\[ m - k \geq \alpha_1 + \alpha_2 - 2. \]  
(24)

We select two finite sequences of real numbers \( \{\zeta_1^{(1)}, \cdots, \zeta_{\alpha_1-1}^{(1)}\} \) and \( \{\zeta_1^{(2)}, \cdots, \zeta_{\alpha_2-1}^{(2)}\} \); using the control redundancy ensured by (24), we are allowed to impose the constraints
\[
\frac{u_j(t)}{u_i(t)} = \zeta_j^{(1)}, \quad j = 1, \ldots, \alpha_1 - 1,
\]
(25)
\[
\frac{u_{i_1+j}(t)}{u_i(t)} = \zeta_j^{(2)}, \quad j = 1, \ldots, \alpha_2 - 1
\]

\[
t(t) = Gu(t).
\]

Remark 3.4. The conditions (25) are practically meaningful: they may correspond to several control surfaces moving together or in opposite directions, as well as to fixed rotations of control devices. However, constraints of this type may not be possible to impose all the time and for all relevant faults: as a consequence an additional method to switch in/out constraints (25) to actively search for faults location may need to be introduced.

Due to (25), the overall input signal associated to the first cluster \( A_1 \) turns out to be
\[
\sum_{i=j}^{i_1} W_j u_j(t) = u_{i_1}(t) \left( W_{i_1} + \sum_{j=1}^{\alpha_1-1} \zeta_j^{(1)} W_j \right),
\]
while the overall signal corresponding to \( A_2 \) is
\[
\sum_{j=i_1+1}^{i_2} W_j u_j(t) = u_{i_2}(t) \left( W_{i_2} + \sum_{j=1}^{\alpha_2-1} \zeta_j^{(2)} W_{j+i_1} \right).
\]

Let us denote by \( W^{(1)} \) and \( W^{(2)} \) the constant vectors
\[
W^{(1)} := W_{i_1} + \zeta_1^{(1)} W_j, \quad W^{(2)} := W_{i_2} + \zeta_1^{(2)} W_{j+i_1};
\]

since, without loss of generality, the coefficients \( \zeta_j^{(\ast)} \), \( \ast = i_1, i_2 \), can be chosen such that the latter vectors are independent, by definition one has
\[
\text{rk}[W^{(1)} \ W^{(2)} \ W_{i_1+1} \cdots \ W_m] \geq 3.
\]
(26)

If the cluster \( A_1 \) undergoes a fault, the dynamics of the estimation error turns out to be
\[
\dot{e}(t) = Fe(t) + RW^{(1)} (\delta_1(t) - 1) u_{i_1}(t)
\]
and a similar condition holds for faults in \( A_2 \). Now, recalling that \( \text{rk}[W^{(1)} \ W^{(2)}] = 2 \) and designing the observer matrix \( R \) such that
\[
RW^{(1)} = RW^{(2)} = 0,
\]
one obtains a residual signal decoupled from faults affecting the first two clusters of actuators: we point out that, due to (26), the residual will be sensitive to faults affecting other effectors/actuators. We will refer to such signal as a cluster residual. The above construction can be readily extended to the case of faults involving more than two effector clusters, if there is enough redundancy to use control allocation. To this purpose, let us set
\[
W^{(h)} = W_{i_h} + \sum_{j=1}^{\alpha_h-1} \zeta_j^{(h)} W_{i_{h-1}+1}, \quad h = 1, \ldots, q
\]
(27)
where the coefficients $\zeta^{(i)}_j \in \mathbb{R}$ have to be defined. Let us denote by $k_2$ the integer

$$k_2 = \max \{ \ell < q : rk[W_{(j_1)} \cdots W_{(j_{\ell+1})}] = \ell + 1, \forall \text{multi-index } J = (j_1, \ldots, j_{\ell+1}) \}. \quad (28)$$

The number of distinct multi-indices of length $k_2$ with entries chosen within a set of cardinality $q$ is given by

$$s = \binom{k_2}{q};$$

these can be organized in an ordered family $J_\ell$, $\ell = 1, \ldots, s$. We indicate by $W_{(J_\ell)}$ the matrix whose columns are the vectors $W_{(h)}$ having index $h$ included in the multi-index $J_\ell$ and we set

$$N(k_2) = \max \left\{ \sum_{\ell=1}^{k_2} \alpha_{i_\ell} : i_\ell < i_{\ell+1}, \ i_\ell \in \{1, \ldots, q\} \ \forall \ell = 1, \ldots, k_2 \right\}.$$  

The integer $N(k_2)$ represents the total number of effectors belonging to the $k_2$ largest clusters. A general result on the existence of cluster residuals can be stated now.

**Theorem 3.5.** Suppose that the following conditions are satisfied for $W = BG$:

1. $rk(CW) = rk(W) = k$

2. the pair $(C, A_W)$ is detectable, where

   $$A_W := A - W((CW)^T CW)^{-1}(CW)^T CA$$

   and $W \in \mathbb{R}^{n \times k}$ is an arbitrary matrix such that $\text{Im}(W) = \text{Im}(W)$.

If the coefficients $\zeta^{(i)}_j$ are determined such that $(m-k) \geq (N(k_2) - k_2)$, where $k_2$ is given by (28), then

a) the control $u(t)$ can be allocated according to

$$u(t) \in U, \quad \tau_c(t) = Gu(t), \quad \frac{u_{i_{h-1}+j}(t)}{u_{i_h}(t)} = \zeta^{(i)}_j, \quad j = 1, \ldots, \alpha_h - 1, \quad h = 1, \ldots, q \quad (29)$$

b) a family of $s = \binom{q}{k_2}$ unknown input observers $\{F^{(h)}, R^{(h)}, H^{(h)}, K^{(h)}\}_{h=1}^s$, can be designed with the property

$$R^{(h)}W_{(J_h)} = 0, \quad h = 1, \ldots, s \quad (30)$$

$$R^{(h)}W \neq 0 \quad (31)$$

**Proof.** The first condition ensures that matrix $A_W$ is well defined; moreover since $\text{Im}(W_{(J_h)}) \subset \text{Im}(W)$, by Corollary 3.2, the second condition guarantees that for any $h = 1, \ldots, s$, the observer parameters $F^{(h)}, R^{(h)}, H^{(h)}, K^{(h)}$ can be designed such that

- $R^{(h)} = K^{(h)}_1 + K^{(h)}_2, \quad K^{(h)}_2 = F^{(h)}H^{(h)}$;

- condition (30) holds true;

- $F^{(h)} = R^{(h)}A - K^{(h)}_1 C = (I_{n \times n} - H^{(h)}C)A - K^{(h)}_1 C$ is Hurwitz.

Since by assumption $(m-k) \geq (N(k_2) - k_2)$, the redundancy is sufficient to impose conditions (29) on the control input by the allocation procedure; lastly by definition one has $k_2 < \min\{k, q\}$ for any choice of the coefficients $\{\zeta^{(i)}_j\}$, and hence for any $h = 1, \ldots, q$ there exists at least one vector $W_{(h)}$ with $\ell \notin J_h$ and $R^{(h)}W_{(h)} \neq 0$: therefore condition (31) is satisfied, as the subspaces inclusion $\text{Im}(W_{(h)}) \subset \text{Im}(W)$ holds by construction. \qed
Remark 3.5. The only theoretical requirement to be imposed on the coefficients \{c^{(h)}_k\} is the fulfillment of the uniform condition \((m - k) \geq (Nk_1 - k_2)\); however, other non-uniform and possibly weaker conditions can be introduced “ad hoc” with the aim of designing cluster residuals associated to selected groups of effectors with different cardinalities.

The above theorem gives sufficient conditions for the isolation of faults distributed over \(k_1\) actuator clusters. In particular, due to (23), the estimation error \(e^{(h)}(t)\) associated to the \(h^{th}\) observer is given by

\[
e^{(h)}(t) = F^{(h)}e(t) + R^{(h)}\left[(d_1(t) - 1)W^{(1)} \cdots (d_q(t) - 1)W^{(q)}\right]^T[u_1(t) \cdots u_q(t)],
\]

where \(R^{(h)}W^{(l)} = 0\) if \(l \in J_h\). The cluster residuals are defined as the family of output errors

\[
r^{(h)}(t) = ||Ce^{(h)}(t)|| = ||y(t) - C\hat{x}^{(h)}(t)||.
\]

Following a similar procedure to the one described in Section 3.1, one can associate to each cluster residual its cluster residual signature in order to simplify the fault isolation operations.

4 Case study: overactuated marine vessel

The presented fault detection/isolation and control reconfiguration setup is typical for redundant propulsion and dynamic positioning systems where groups of thrusters usually share electric power distribution, but also for auxiliaries plants such as cooling water circulation, diesel generator fuel distribution and pressurized air supply, where failures may be more difficult to detect. This section is focused on illustrating the application of theoretical results to the case of an overactuated marine vessel. We consider the following ship model

\[
\begin{align*}
\dot{\eta} &= P(\eta)\nu \\
M\dot{\nu} &= -V(\eta, \nu) + \tau + P^T(\eta)b(t)
\end{align*}
\]

where \(M\) is the inertia matrix, \(\eta = [x_G, y_G, \psi]^T\) is the ship position coordinates in the earth-fixed reference frame and \(\nu = [\nu_x, \nu_y, \theta]\) contains surge, sway and yaw angular velocities with respect to the body-fixed reference frame, which is identified with the craft center of mass; the vector \(V(\eta, \nu)\) includes Coriolis and damping terms, the actual thrust force in surge, sway and the yaw moment are given by \(\tau = [\tau_x, \tau_y, m_\theta]\) and \(P(\eta) = P(\psi)\) is a standard rotation matrix

\[
P(\psi) = \begin{bmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The perturbation term \(b(t)\), which is assumed to be bounded by a known constant, is used to model disturbances affecting the systems, such as slowly-varying forces and moments caused by wave loads, ocean currents or winds (see [12] for further details). Following [2], both \(\eta\) and \(\nu\) are assumed to be measured (possibly through a state estimation), the constant matrix \(M\) is known and \(V(\eta, \nu)\) is a known function in the state variables \(\eta, \nu\). We consider a ship equipped with 3 azimuth thrusters (rotatable) \(T_1, T_2, T_3\) and 2 transverse tunnel thrusters (fixed orientation) \(T_4, T_5\). A sketch of the vessel model is depicted in Figure 1.

The actual thrust force is related to the control input through the linear equation

\[
\tau = Gu(t),
\]

with

\[
G = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8
\end{bmatrix},
\]

where the moment arms \(\gamma_j\) are associated to azimuth thrusters and the moment arms \(\chi_j\) to tunnel thrusters instead. In particular such arms can be computed as

\[
\begin{align*}
\gamma_{2s-1} &= d_s \sin \phi_s & s &= 1, 2, 3 \\
\gamma_{2s} &= d_s \cos \phi_s & s &= 1, 2, 3 \\
\chi_s &= d_s \cos \phi_s & s &= 4, 5
\end{align*}
\]
where $d_s$ are the distances of thrusters and $\phi_s$ are the angles from the rotation point; for the considered thrusters location (see Figure 1) we have

$$
\gamma_1 = -\gamma_3, \quad \gamma_2 = \gamma_4, \quad \gamma_5 = 0,
\gamma_6 = d_3, \quad \chi_7 = d_4, \quad \chi_8 = d_5.
$$

Assuming that the vessel rotation is negligible with respect to translation motion, setting $X = [\eta, \nu]^T$, for sake of simplicity and without loss of generality, the above nonlinear model can be linearized as follows

$$
\dot{X}(t) = AX(t) + Br(t) + E(t)
$$

(33)

with

$$
A = \begin{bmatrix}
0_{3 \times 3} & P(\tilde{\psi}) \\
0_{3 \times 3} & -M^{-1}D
\end{bmatrix}, \quad
B = \begin{bmatrix}
0_{3 \times 1} \\
M^{-1}
\end{bmatrix}, \quad
E(t) = \begin{bmatrix}
0_{3 \times 1} \\
M^{-1}P^T(\tilde{\psi})b(t)
\end{bmatrix},
$$

where $\tilde{\psi}$ is a constant angle associated to some reference heading direction, $D = D(\bar{\nu})$ is a constant damping matrix depending on a nominal reference velocity $\bar{\nu}$ and

$$
E(t) = \begin{bmatrix}
0_{3 \times 1} \\
M^{-1}P^T(\tilde{\psi})b(t)
\end{bmatrix}.
$$

Since by assumption all state variables are measured, without loss of generality the output matrix $C$ is supposed to be equal to the identity matrix

$$
C = I_{6 \times 6}.
$$

Assuming that the marine vessel has a mass $\mu = 6 \cdot 10^6 Kg$, with length $L = 76m$ and width $w = 16m$, the following parameters are obtained [12]:

$$
M = 10^9 \begin{bmatrix}
0.0068 & 0. & 0. \\
0. & 0.0113 & -0.0340 \\
0. & -0.0340 & 4.4524
\end{bmatrix}, \quad D = 10^8 \begin{bmatrix}
0.0008 & 0. & 0. \\
0. & 0.0025 & -0.0203 \\
0. & -0.0340 & 3.8481
\end{bmatrix}.
$$

Without loss of generality $\tilde{\psi} = 0$, that is $P(\tilde{\psi}) = I_{3 \times 3}$. Moreover, taking $d_1 = d_2 = 20m$, $d_3 = 18.5m$, $d_4 = 30m$, $d_5 = 35m$ and $\phi_1 = \pi + 0.3$, $\phi_2 = \pi - 0.3$, the matrix $G$ is given by

$$
G \simeq \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
-5.91 & -19.1 & 5.91 & -19.1 & 0 & 18.5 & 30 & 35
\end{bmatrix}.
$$
We suppose the faults to occur in effectors (thrusters) rather than in single actuators: this corresponds to consider a fault matrix $\Delta(t) = \text{diag}(\delta_1(t), \delta_1(t), \delta_2(t), \delta_2(t), \delta_3(t), \delta_3(t), \delta_4(t), \delta_4(t))$ with coupled entries representing failures in the thrusters $T_j$, $j = 1, ..., 5$. The matrix $W = BG$ is

$$W = [W_1 \ W_2 \ W_3 \ W_4 \ W_5 \ W_6 \ W_7 \ W_8] \simeq 10^{-4}$$

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
14.78 & 0 & 14.78 & 0 & 14.78 & 0 & 0 & 0 \\
0.41 & 7.71 & 0.41 & 7.71 & 0 & 10.30 & 11.09 & 11.44 \\
-0.13 & -0.37 & 0.13 & -0.37 & 0 & 0.49 & 0.76 & 0.87 \\
\end{array}
\]

where the vertical rules have been added to easily individuate the actuators corresponding to each thruster.

We can design a family of $s = 5$ ZFI unknown input observers $\{O_h\}$ to isolate faults affecting singularly each thruster. In particular, following the steps of Theorem 3.3, we select the observers matrices $R(h)$ in order to have

$$R(h)[W_{2h-1}W_{2h}] = 0 \quad h = 1, 2, 3$$

$$R(h)W_{h+3} = 0 \quad h = 4, 5.$$

The observer gains $K_1^{(h)}$ can be chosen in order to assign the eigenvalues of the observer matrices $F^{(h)}$: on the other hand, since the whole state $X$ is measurable, i.e. $C = I_{6 \times 6}$, the matrices $F^{(h)}$ turn out to be diagonal. For sake of simplicity we assume $F^{(h)} = F$ for any $h = 1, ..., 5$, with $F = \text{diag}(-0.5, -0.5, -0.3, -0.2, -0.2, -0.1)$. Different single or multiple thruster fault events have been simulated, assuming the initial conditions $\eta_0 = \eta(0) = (1, 1, 0)$ and $\nu_0 = \nu(0) = (2.2, 1.9, 0)$. The disturbance term is supposed to be given by the sum of two contributions: a constant term with random but fixed input direction representing an irrotational ocean current and an oscillating term with varying input direction representing waves; the overall disturbance effect $b(t)$ is assumed to be bounded by the known constant $\epsilon = 5 \cdot 10^9$ and the fault detection threshold $\varrho = \varrho(\epsilon) = 2 \cdot 10^{-4}$. Can be consequently computed by the residual signals dynamics. The ship is supposed to be equipped with an $xy$-joystick control device together with a heading autopilot; the nominal operating conditions of the vessel are defined by a constant translational speed regime, this corresponding to the commanded control effect

$$\tau_c(t) = D \nu(t) + [0 \ 0 \ a_\psi]^T,$$

where $a_\psi$ is a PID controller for the yaw angle.

**Remark 4.1.** *In this case study we do not take into account measurement noise; however, if bounded additive uncertainties $\omega(t)$ affect the output equation, i.e. $y(t) = Cx(t) + \omega(t)$, these can be handled by increasing the fault detection threshold.*

**Single faults**

We first suppose the azimuth thruster $T_1$ to be affected by a fault that gradually fades and its effect is $\delta_1(t) = e^{-0.01t}$. The residual behavior is depicted in Figure 2-(a): all residuals signals expect $r^{(1)}(t)$ overpass the detection threshold, this corresponding to a residual signature:

$$\mu^{(1)} = \mu^{(2)} + \mu^{(3)} + \mu^{(4)} + \mu^{(5)} = (4, 4, 3, 3, 3, 3)$$

As a consequence the fault can be isolated and the control reconfiguration policy can be applied. In Figure 2-(b) a different scenario is illustrated, in which a fault $\delta_3(t) = e^{-0.02t}$ affects the tunnel thruster $T_4$; in this case the residual signature is

$$\mu^{(1)} = \mu^{(2)} + \mu^{(4)} + \mu^{(5)} = (3, 3, 3, 3, 4, 4, 3),$$

this allowing to isolate the faulty thruster $T_4$. In both considered cases, some of the residuals, i.e. $r^{(2)}$ in the first example and $r^{(3)}$ in the second one, due to disturbance effect switch over/under the detection threshold after initialization. In order to prevent possible false alarms, such transient behavior can be avoided by considering a suitable residual filtering/projection. On the other hand, a time window can always be defined such that only residuals which remain active for a sufficient large time interval will be associated to the occurrence of relevant faults.
Multiple faults

If multiple faults affecting several thrusters occur, due to observer design limitation it is not possible to isolate them using the previous set of filters; in particular, in the considered scenario, all residuals signals will be activated after a short time, so that no useful information about the location of faults is provided.

The most relevant cases of faults affecting simultaneously two thrusters due to common auxiliaries or power supply failures are reported in the following table:

<table>
<thead>
<tr>
<th>Thruster</th>
<th>T1</th>
<th>T2</th>
<th>T3</th>
<th>T4</th>
<th>T5</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1 - The circles indicate that the case of a fault occurring in the corresponding pair of thrusters is relevant

In particular, it is assumed that T1 and T4 share the same auxiliaries, as well as T2 and T5 do; the thruster T3 is supposed to be equipped with a switching device that enables it to be connected arbitrarily to one sub-group or to the other, depending on the operating conditions of the system.

Using actively the control allocation, a new family of observers can be designed to obtain cluster residuals (as showed in Section 3.3); for sake of simplicity we assume the evolution time to be re-initialized at the present step. We choose the coefficients $\zeta^{(1)} = 0.45, \zeta^{(2)} = 0.29$ and $\zeta^{(3)} = 0.72$ in Theorem 3.5 according to the initial values of the control inputs, i.e. $\zeta^{(i)} = u_{2i-1}(0)/u_{2i}(0)$; we impose the following conditions on the control input

$$\zeta^{(1)}u_1(t) = u_2(t), \quad \zeta^{(2)}u_3(t) = u_4(t), \quad \zeta^{(3)}u_5(t) = u_6(t)$$

and we set

$$W_{\{1\}} = W_1 + W_2, \quad W_{\{2\}} = \zeta^{(2)}W_3 + W_4, \quad W_{\{3\}} = \zeta^{(3)}W_5 + W_6, \quad W_{\{4\}} = W_7, \quad W_{\{5\}} = W_8.$$  

We point out that in the considered case $m - k = 5$ and therefore the degrees of freedom in control allocation are enough to allow imposing the constraints (34): in particular we have $k_4 = 2$ and $N(k_4) = 4$. By the transformation (34)-(35), each thruster $T_i$ is associated to its overall input vector $W_{\{i\}}, i = 1, 2, 3, 4, 5$, in a scheme that corresponds to fixed directions of azimuth thrusters. A family of ZFI unknown input observers $\{O_h\}_{h=1}^{10}$ is then defined such that the corresponding matrices $R^{(h)}$ satisfy:

$$R^{(h)}[W_{\{1\}} W_{\{h+1\}}] = 0 \quad h = 1, 2, 3, 4$$
$$R^{(h)}[W_{\{2\}} W_{\{h-2\}}] = 0 \quad h = 5, 6, 7$$
$$R^{(h)}[W_{\{3\}} W_{\{h-4\}}] = 0 \quad h = 8, 9$$
$$R^{(h)}[W_{\{4\}} W_{\{5\}}] = 0 \quad h = 10$$

In this way, for example, the cluster residual $r^{(1)}(t)$ (see (32)) results to be insensitive to faults in both thrusters $T_1$ and $T_2$ (even if in practice, due to motion control design, this is an unlikely event), while the cluster residual $r^{(3)}(t)$ is decoupled from faults in both $T_2$ and $T_3$. Figure 3-(a) illustrates the case of faults $\delta_1(t) = e^{-0.008t}$ and $\delta_3(t) = e^{-0.006t}$ affecting the azimuth thrusters $T_1, T_3$. The residual $r^{(2)}$ (insensitive to faults in thrusters $T_1$ and $T_3$ by design) evolves without violating the fault detection threshold which only depends on the disturbance: this allow us to individuate the faulty effectors. A similar scenario is depicted in Figure 3-(b), where faults $\delta_1(t) = e^{-0.004t}$ and $\delta_5(t) = e^{-0.002t}$ are supposed to affect simultaneously the azimuth thruster $T_3$ and the tunnel thruster $T_5$.

Control reconfiguration

The control re-allocation can be performed through the reduced order pseudo-inverse method (5). Figure 4-(a) and Figure 4-(b) show the evolution of the ship position in the case of a fault affecting the thruster $T_1$; the reconfiguration procedure is supposed to be activated after $t_0 = 180s$. The control is successfully re-allocated, and the commanded control effect is modified in order to track the original vessel velocities in surge and sway, while the rotational speed is automatically updated by the heading PID controller given by $a_{\psi}$. The evolution of control inputs is depicted in Figure 4-(c) and Figure 4-(d): the reconfigured inputs corresponding to the
faulty thruster $T_1$, namely $\hat{u}_1(t)$ and $\hat{u}_2(t)$, are null for $t \geq t_0$ and the generation of commanded control effect $\tau_c(t)$ is redistributed over the other thrusters $T_2, T_3, T_4$ and $T_5$.

Figure 2: Faults affecting individual thrusters: residual signals evolution

(a) Fault affecting the azimuth thruster $T_1$

(b) Fault affecting the tunnel thruster $T_4$

Figure 3: Faults affecting multiple thrusters: cluster residual signals evolution

(a) Faults affecting the thrusters $T_1$ and $T_3$

(b) Faults affecting the thrusters $T_3$ and $T_5$
5 Conclusions

The present report focuses on fault detection/isolation and control reconfiguration in overactuated systems using unknown input observers (UIO). Depending on structural conditions on the system matrices, a suitable number of observers can be designed in order to detect and isolate single or multiple faults affecting the actuators/effectors. The rank-deficiency of the input matrix may complicate the fault isolation task, as the virtual force commanded to the system is given by an overlap of redundant inputs and thus the individual effect of each input is not always distinguishable. The main contribution of the paper is the study of advantages of control allocation in this framework. One first and classical benefit of input redundancy is the availability of sufficient degrees of freedom to reconfigure the control in order to cope with unexpected changes in the system dynamics. A less immediate but strategic feature of overactuated systems, which have been emphasized in this work, is the possibility to use actively control allocation in the fault isolation process by imposing additional input constraints. The proposed theoretical results have been validated by the case study of an overactuated marine vessel whose thrusters are subject to a loss of efficiency; several faulty events have been considered in order to illustrate how control allocation can be successfully used for the tasks of fault isolation and control reconfiguration.

References


