Explicit Stochastic Nonlinear Predictive Control Based on Gaussian Process Models

Alexandra Grancharova, Juš Kocijan, and Tor A. Johansen

Abstract— **Nonlinear Model Predictive Control (NMPC) algorithms are based on various nonlinear models. Recently, an on-line optimization approach for stochastic NMPC based on a Gaussian process model was proposed. A significant advantage of the Gaussian process models is that they provide information about prediction uncertainties, which would be of help in NMPC design. On the other hand, an explicit solution to the stochastic NMPC problem based on Gaussian process model would allow efficient on-line computations as well as verifiability of the implementation. This paper suggests an approximate multi-parametric Nonlinear Programming approach to explicit solution of stochastic NMPC problems for constrained nonlinear systems based on *Gaussian process model*. In particular, *the reference tracking problem* is considered. The approach builds an orthogonal search tree structure of the state space partition and consists in constructing a feasible PWL approximation to the optimal control sequence.**

I. INTRODUCTION

NONLINEAR Model Predictive Control (NMPC) involves the solution at each sampling instant of a finite horizon optimal control problem subject to nonlinear system dynamics and state and input constraints [1], [2], [3], [4]. A recent survey of the main on-line optimization strategies of NMPC is given in [5]. Mathematical models of engineering systems usually contain some amount of uncertainty (unknown additive disturbances and/or uncertain model parameters). Therefore, the robust MPC problem formulation requires the model uncertainty to be taken into account. In many applications, the system to be controlled is described by a stochastic model where the probabilistic distribution of the uncertainty is assumed to be known. Several approaches for constrained MPC based on stochastic models (stochastic MPC) are proposed in [6]–[12]. The approaches [6], [7], [8] are based on linear state space

A. Grancharova is with the Institute of Control and System Research, Bulgarian Academy of Sciences, Acad. G. Bonchev str., Bl.2, P.O.Box 79, Sofia 1113, Bulgaria (phone: 3592-870-03-37; fax: 3592-870-33-61; e-mail: alexandra.grancharova@abv.bg).

J. Kocijan is with the Department of Systems and Control, Jozef Stefan Institute, Jamova 39, 1000 Ljubljana, Slovenia and with University of Nova Gorica, School of Engineering and Management, Vipavska 13, 5000 Nova Gorica, Slovenia (e-mail: jus.kocijan@ijs.si).

T. A. Johansen is with the Department of Engineering Cybernetics, Norwegian University of Science and Technology, 7491 Trondheim, Norway (e-mail: Tor.Arne.Johansen@itk.ntnu.no).

models with stochastic parameters and/or additive noise and they optimize the expected value of the cost function subject to hard input constraints [6] or probabilistic constraints [7], [8]. In [9], [10], [11], [12], stochastic MPC approaches incorporating a probabilistic cost and probabilistic constraints are developed. The method suggested in [9] is based on a moving average (MA) model with random coefficients. It was further extended to linear time-varying MA models [10] and to state space models with stochastic uncertainty in the output or the input map [11], [12].

It should be noted that the stochastic MPC approaches [6]–[12] are based on *parametric* probabilistic models. Alternatively, the stochastic systems can be modeled with *non-parametric* models which can offer a significant advantage compared to the *parametric* models. This is related to the fact that the *non-parametric* probabilistic models provide information about prediction uncertainties which are difficult to evaluate appropriately with the *parametric* models. The Gaussian process model is an example of a *non-parametric* probabilistic black-box model and up to now it has been applied to model mainly static nonlinearities. The use of Gaussian processes in the modelling of dynamic systems is a recent development e.g. [13], [14], [15]. In [16], [17], [18], an on-line optimization approach for stochastic NMPC based on Gaussian process model is proposed.

It has recently been shown that the feedback solution to linear and quadratic constrained MPC problems has an explicit representation as a piece-wise linear (PWL) state feedback defined on a polyhedral partition of the state space [19]. The benefits of an explicit solution, in addition to the efficient on-line computations, include also verifiability of the implementation, which is an essential issue in safety-critical applications. For nonlinear and stochastic MPC the prospects of explicit solutions are even higher than for linear MPC, since the benefits of computational efficiency and verifiability are even more important. An approach for efficient on-line computation of NMPC for constrained input-affine nonlinear systems has been suggested in [20]. In [21], [22], [23], approaches for off-line computation of explicit sub-optimal PWL predictive controllers for general nonlinear systems with state and input constraints have been developed, based on the multi-parametric Nonlinear Programming (mp-NLP) ideas [24]. The mentioned methods for explicit NMPC are based on *deterministic* first principle models of the systems.

This paper suggests an approximate mp-NLP approach to explicit solution of stochastic NMPC problems for constrained nonlinear systems based on a *Gaussian process model* (referred to as GP-NMPC problems). In particular, *the reference tracking problem* is considered. The contribution of the present work is the formulation of a more general GP-NMPC problem compared to [16], [17], [18], and representing it as an mp-NLP problem. Further, the approximate mp-NLP approach [23] is applied to build an orthogonal search tree structure of the state space partition and construct a feasible PWL approximation to the optimal control sequence. Thus, the approach proposed in this paper can be considered as an application of the approximate methods [22], [23], [25] for explicit solution of MPC problems to the cases where the system dynamics is described by a probabilistic (Gaussian process) model.

The following notation will be used in the paper. $A \succ 0$ means that the square matrix A is positive definite. For $x \in \mathbb{R}^n$, the Euclidean norm is $\|x\| = \sqrt{x^T x}$ and the weighted norm is defined for some symmetric matrix $A \succ 0$ as $\|x\|_A = \sqrt{x^T A x}$. For a random variable y with Gaussian distribution, $\mathcal{N}(\mu(y), \sigma^2(y))$ denotes its probability distribution, and $\mu(y)$ and $\sigma^2(y)$ are respectively its mean and variance.

II. MODELLING OF DYNAMIC SYSTEMS WITH GAUSSIAN PROCESSES

A Gaussian process is an example of the use of a flexible, probabilistic, nonparametric model which directly provides us with uncertainty predictions. Its use and properties for modelling are reviewed in [26].

A Gaussian process is a collection of random variables which have a joint multivariate Gaussian distribution. Assuming a relationship of the form $y = f(z)$ between an input $z \in \mathbb{R}^D$ and output $y \in \mathbb{R}$, we have $y(1), y(2), \dots, y(M) \sim \mathcal{N}(0, \Sigma)$, where $\Sigma_{pq} = \text{Cov}(y(p), y(q)) = C(z(p), z(q))$ gives the covariance between the output points $y(p)$ and $y(q)$ corresponding to the input points $z(p)$ and $z(q)$. Thus, the mean $\mu(z)$ (usually assumed to be zero) and the covariance function $C(z(p), z(q))$ fully specify the Gaussian process. Note that the covariance function $C(z(p), z(q))$ can be any function with the property that it generates a positive definite covariance matrix. A common choice is:

$$C(z(p), z(q)) = v_1 \exp\left[-\frac{1}{2} \sum_{i=1}^{D} w_i (z_i(p) - z_i(q))^2 \right] + v_0 \alpha_{pq} \quad (1)$$

where $\Theta = [w_1, \dots, w_D, v_0, v_1]$ are the 'hyperparameters' of the covariance function, z_i denotes the i-th component of

the D-dimensional input vector z, and α_{pq} is the Kronecker operator. Other forms of covariance functions suitable for different applications can be found in [27]. For a given problem, the hyperparameters are learned (identified) using the data at hand. After the learning, one can use the w parameters as indicators of 'how important' the corresponding input components (dimensions) are: if w_i is zero or near zero it means that the inputs in dimension i contain little information and could possibly be removed.

Consider a set of M D-dimensional input vectors $\mathbf{Z} = [z(1), z(2), \dots, z(M)]^T$ and a vector of output data $Y = [y(1), y(2), \dots, y(M)]^T$. Based on the data $(\mathbf{Z}, Y)$, and given a new input vector z^*, we wish to estimate the probability distribution of the corresponding output y^*. Unlike other models, there is no model parameter determination as such, within a fixed model structure. With this model, most of the effort consists in *tuning* the parameters of the covariance function. This is done by maximizing the log-likelihood of the parameters, which is computationally relatively demanding since the inverse of the data covariance matrix ($M \times M$) has to be calculated at every iteration.

The described approach can be easily utilized for regression calculation. Based on a training set $\mathbf{Z}$, a covariance matrix $\mathbf{K}$ of size $M \times M$ is determined. As already mentioned before, the aim is to estimate the probability distribution of the corresponding output y^* at some new input vector z^*. For a new test input z^*, the predictive distribution of the corresponding output is $y^* \mid z^*, (\mathbf{Z}, Y)$ and is Gaussian, with mean and variance:

$$\mu(z^*) = k(z^*)^T \mathbf{K}^{-1} Y$$
$$\sigma^2(z^*) = k_0(z^*) - k(z^*)^T \mathbf{K}^{-1} k(z^*) + v_0 \quad (2)$$

where $k(z^*) = [C(z(1), z^*), \dots, C(z(M), z^*)]^T$ is the $M \times 1$ vector of covariances between the test and training cases and $k_0(z^*) = C(z^*, z^*)$ is the covariance between the test input and itself.

Gaussian processes can be used to model static nonlinearities and can therefore be used for modelling of dynamic systems if delayed input and output signals are used as regressors [13]. In such cases an autoregressive model is considered, such that the current predicted output depends on previous estimated outputs, as well as on previous control inputs:

$$z(t) = [\hat{y}(t-1), \hat{y}(t-2), \dots, \hat{y}(t-L), u(t-1),$$
$$u(t-2), \dots, u(t-L)]^T \quad (3)$$
$$\hat{y}(t) = f(z(t)) + \eta(t)$$

where t denotes consecutive number of data sample, L is a given lag, and $\eta(t)$ is the prediction error. The quality of the predictions with a Gaussian process model is assessed by

computing the average squared error (ASE):

$$ASE = \frac{1}{M} \sum_{i=1}^{M} [\mu(\hat{y}(i)) - y(i)]^2 \qquad (4)$$

and by the log density error (LD) [13]:

$$LD = \frac{1}{2M} \sum_{i=1}^{M} \log(2\pi) + \log[\sigma^2(\hat{y}(i))] + \frac{[\mu(\hat{y}(i)) - y(i)]^2}{\sigma^2(\hat{y}(i))} \qquad (5)$$

In (4), (5), $\mu(\hat{y}(i))$ and $\sigma^2(\hat{y}(i))$ are the prediction mean and variance, $y(i)$ is the system's output and M is the number of the training points.

The iterative multi-step ahead prediction can be done in the following ways, as described in [28]:
1) by feeding back at each time step the predictive mean only;
2) by feeding back at each time step both the predictive mean and the predictive variance;
3) by Monte Carlo simulations.

Thus, the uncertainty attached to each intermediate prediction is taken into account. The Gaussian process model now not only describes the dynamic characteristics of the non-linear system, but at the same time provides information about the confidence in the predictions. The Gaussian process can highlight areas of the input space where prediction quality is poor, due to the lack of data, by indicating the higher variance around the predicted mean. It is worthwhile noting that the derivatives of means and variances with respect to input data can be calculated in straightforward manner. For more details see [28].

III. FORMULATION OF THE GP-NMPC PROBLEM AS AN MP-NLP PROBLEM

Consider a stochastic system described by an uncertain nonlinear discrete-time model:

$$x(t+1) = f(x(t), u(t)) + \xi(t) \qquad (6)$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and input variables, $\xi(t) \in \mathbb{R}^n$ are Gaussian disturbances, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear continuous function. The uncertainty consists in that the analytical expression of $f(x, u)$ is not known and neither are the mean values and the covariances of the disturbances $\xi(t)$. The relationship (6) is represented in the form:

$$y(t) = f_G(z(t)) + \xi(t) \qquad (7)$$

where $y(t) = x(t+1) \in \mathbb{R}^n$ and $z(t) = [x(t), u(t)] \in \mathbb{R}^{n+m}$. Suppose that we have an output data set $Y_i = [y_i(0), y_i(1), ..., y_i(M-1)]$, $i = 1, 2, ..., n$ corresponding to an input data set $\mathbf{Z} = [z(0), z(1), ..., z(M-1)]$. Assume that the relationship (7) is approximated with Gaussian processes with distributions:

$$Y_1 \sim \mathcal{N}(0, \mathbf{\Sigma}_1), \ Y_2 \sim \mathcal{N}(0, \mathbf{\Sigma}_2), \ ... \ , \ Y_n \sim \mathcal{N}(0, \mathbf{\Sigma}_n) \qquad (8)$$

where the covariance functions

$$\Sigma_{1,pq} = \text{Cov}_1(y_1(p), y_1(q)) = C_1(z(p), z(q)), \qquad ... \qquad ,$$
$$\Sigma_{n,pq} = \text{Cov}_n(y_n(p), y_n(q)) = C_n(z(p), z(q)) \qquad \text{with}$$

$p = 0, 1, ..., M-1$, $q = 0, 1, ..., M-1$, depend on the given input and output data sets. Having obtained the Gaussian process model (8), the probability distribution of the output $y(M)$ corresponding to a new input $z(M)$ can be determined as described in the previous section:

$$y_1(M)|z(M), (\mathbf{Z}, Y_1) \sim \mathcal{N}(\mu(y_1(M)), \sigma^2(y_1(M)))$$
$$\vdots \qquad (9)$$
$$y_n(M)|z(M), (\mathbf{Z}, Y_n) \sim \mathcal{N}(\mu(y_n(M)), \sigma^2(y_n(M)))$$

In (9), $\mu(y_i(M))$ and $\sigma^2(y_i(M))$ denote respectively the mean and the variance of the output variable $y_i(M)$, $i = 1, 2, ..., n$. We introduce the vectors $\mu_y(M) = [\mu(y_1(M)), ..., \mu(y_n(M))]$ and $\sigma_y^2(M) = [\sigma^2(y_1(M)), ..., \sigma^2(y_n(M))]$ and the matrix $\mathbf{Y} = [Y_1, Y_2, ..., Y_n]$. Then, the relation (9) is represented:

$$y(M)|z(M), (\mathbf{Z}, \mathbf{Y}) \sim \mathcal{N}(\mu_y(M), \sigma_y^2(M)) \qquad (10)$$

As shown in [28], it is possible to obtain a multi-step ahead prediction:

$$y(M+k)|z(M+k), (\mathbf{Z}, \mathbf{Y}) \sim \mathcal{N}(\mu_y(M+k), \sigma_y^2(M+k)) \qquad (11)$$
$$k = 0, 1, ... , N-1$$

Suppose the initial state $x(t) = x_{t|t}$ and the control inputs $u(t+k) = u_{t+k}$, $k = 0, 1, ... , N-1$ are given. Then, by taking into account that $y(t) = x(t+1)$ and $z(t) = [x(t), u(t)]$, from (11) we obtain the probability distribution of the predicted states $x_{t+k+1|t}$, $k = 0, 1, ... , N-1$ which correspond to the given initial state $x_{t|t}$ and control inputs u_{t+k}, $k = 0, 1, ... , N-1$:

$$x_{t+k+1|t} | x_{t+k|t}, u_{t+k} \sim \mathcal{N}(\mu(x_{t+k+1|t}), \sigma^2(x_{t+k+1|t})) \qquad (12)$$
$$k = 0, 1, ... , N-1$$

The 95% confidence interval of the random variable $x_{t+k+1|t}$ is $[\mu(x_{t+k+1|t}) - 2\sigma(x_{t+k+1|t}); \mu(x_{t+k+1|t}) + 2\sigma(x_{t+k+1|t})]$, where $\sigma(x_{t+k+1|t})$ is the standard deviation.

Here, we consider a reference tracking problem where the goal is to have the state vector $x(t)$ track the reference signal $r(t) \in \mathbb{R}^n$. In the problem formulation, the type of the cost function is like the one used in [19]. Suppose that a full measurement of the state $x(t)$ is available at the current time t. For the current $x(t)$, the reference tracking GP-NMPC solves the following optimization problem:

Problem P1:

$$V^*(x(t), r(t), u(t-1)) = \min_{U} J(U, x(t), r(t), u(t-1)) \qquad (13)$$

subject to $x_{t|t} = x(t)$ and:

$$\mu(x_{t+k|t}) - 2\sigma(x_{t+k|t}) \geq x_{\min}, \ k = 1, \dots, N \tag{14}$$

$$\mu(x_{t+k|t}) + 2\sigma(x_{t+k|t}) \leq x_{\max}, \ k = 1, \dots, N \tag{15}$$

$$u_{\min} \leq u_{t+k} \leq u_{\max}, \ k = 0, 1, \dots, N-1 \tag{16}$$

$$\Delta u_{\min} \leq \Delta u_{t+k} \leq \Delta u_{\max}, \ k = 0, 1, \dots, N-1 \tag{17}$$

$$\max\{\left\| \mu(x_{t+N|t}) - 2\sigma(x_{t+N|t}) - r(t) \right\|,$$
$$\left\| \mu(x_{t+N|t}) + 2\sigma(x_{t+N|t}) - r(t) \right\|\} \leq \delta \tag{18}$$

$$\Delta u_{t+k} = u_{t+k} - u_{t+k-1}, \ k = 0, 1, \dots, N-1 \tag{19}$$

$$x_{t+k+1|t} \mid x_{t+k|t}, u_{t+k} \sim \mathcal{N}(\mu(x_{t+k+1|t}), \sigma^2(x_{t+k+1|t}))$$
$$k = 0, 1, \dots, N-1 \tag{20}$$

with $U = [u_t, u_{t+1}, \dots, u_{t+N-1}]$ and the cost function given by:

$$J(U, x(t), r(t), u(t-1)) = \left\| \mu(x_{t+N|t}) - r(t) \right\|_P^2 +$$
$$\sum_{k=0}^{N-1} \left[\left\| \mu(x_{t+k|t}) - r(t) \right\|_Q^2 + \left\| \Delta u_{t+k} \right\|_R^2 \right] \tag{21}$$

Here, N is a finite horizon. From a stability point of view it is desirable to choose δ in (18) as small as possible [2]. However, due to the uncertainty of the $x_{t+N|t}$ prediction, characterized by the variance $\sigma^2(x_{t+N|t})$, the feasibility of problem P1 will rely on δ being sufficiently large. A part of the GP-NMPC design will be to address this tradeoff. If the system is asymptotically stable (or pre-stabilized), N is large, and the Gaussian process model has a small prediction uncertainty, then it is more likely that the choice of a small δ will be possible.

A more general stochastic MPC problem is formulated in [9], [10], [11], [12], where a probabilistic formulation of the cost is introduced that includes the probabilistic bounds of the predicted variable. Also in these references, a probabilistic formulation of the constraints is used, i.e. the random variable should not exceed a certain bound with a given probability. The stochastic MPC problem considered in this paper (problem P1) is of a more special form compared to the general problem formulated in [9]–[12]. Here, the cost function (21) includes the mean value of the random variable and the constraints (14), (15) and (18) are equivalent to the following probabilistic constraints:

$$\Pr(x_{t+k|t} \geq x_{\min}) = p, \ k = 1, \dots, N \tag{22}$$

$$\Pr(x_{t+k|t} \leq x_{\max}) = p, \ k = 1, \dots, N \tag{23}$$

$$\Pr(\left\| x_{t+N|t} - r(t) \right\| \leq \delta) = p \tag{24}$$

where the probability p is 0.95 (the confidence interval used in (14), (15) and (18) is associated with this level of probability).

The following assumptions are made:
A1. $P, Q, R \succ 0$.

A2. $x_{\min} < 0 < x_{\max}$, $u_{\min} < 0 < u_{\max}$ and $\Delta u_{\min} < 0 < \Delta u_{\max}$.

We introduce an extended state vector:

$$\tilde{x}(t) = [x(t), r(t), u(t-1)] \in \mathbb{R}^{\tilde{n}}, \ \tilde{n} = 2n + m \tag{25}$$

Let $\tilde{x}$ be the value of the extended state at the current time t. Then, the optimization problem P1 can be formulated in a compact form as follows:

Problem P2:

$$V^*(\tilde{x}) = \min_U J(U, \tilde{x}) \tag{26}$$

subject to:

$$G(U, \tilde{x}) \leq 0 \tag{27}$$

The GP-NMPC problem defines an mp-NLP, since it is NLP in U parameterized by $\tilde{x}$. An optimal solution to this problem is denoted $U^* = [u_t^*, u_{t+1}^*, \dots, u_{t+N-1}^*]$ and the control input is chosen according to the receding horizon policy $u(t) = u_t^*$. Define the set of N-step feasible initial states as follows:

$$X_f = \{\tilde{x} \in \mathbb{R}^{\tilde{n}} \mid G(U, \tilde{x}) \leq 0 \text{ for some } U \in \mathbb{R}^{Nm}\} \tag{28}$$

If δ in (18) is chosen such that the problem P1 is feasible, then X_f is a non-empty set and due to assumption A2, the origin is an interior point in X_f.

In parametric programming problems one seeks the solution $U^*(\tilde{x})$ as an explicit function of the parameters $\tilde{x}$ in some set $X \subseteq X_f \subseteq \mathbb{R}^{\tilde{n}}$ [24]. The explicit solution allows us to replace the computationally expensive real-time optimization with a simple function evaluation. In this paper we suggest a computational method for constructing an explicit PWL approximate solution of the reference tracking GP-NMPC problem.

IV. APPROXIMATE MP-NLP APPROACH TO EXPLICIT GP-NMPC

Here, the computational issues related to the non-convexity of the optimization problem are treated in a way similar to that in [23].

A. Close-to-global solution of mp-NLP

In general, problem P2 can be non-convex with multiple local minima. Therefore, it would be necessary to apply an efficient initialization of problem P2 so to find a close-to-global solution. One possible way to obtain this is to find a close-to-global solution at a point $v_0 \in X_0$ by comparing the local minima corresponding to several initial guesses and then to use this solution as an initial guess at the neighbouring points $v_i \in X_0$, $i = 1, 2, \dots, N_1$, i.e. to propagate the solution. The following procedure is used to generate a set of points $V_0 = \{v_0, v_1, v_2, \dots, v_{N_1}\}$, where $v_i \in X_0$, $i = 0, 1, 2, \dots, N_1$.

Procedure 1 (generation of set of points):
Consider any hyper-rectangle $X_0 \subseteq X_f$ with vertices $\Lambda^0 = \{\lambda_1^0, \lambda_2^0, \dots, \lambda_{N_\lambda}^0\}$ and center point v_0. Consider also

the hyper-rectangles $X_0^j \subset X_0$, $j = 1, 2, ..., N_j$ with vertices respectively $\Lambda^j = \left\{ \lambda_1^j, \lambda_2^j, ... , \lambda_{N_\lambda}^j \right\}$, $j = 1, 2, ..., N_j$.

Suppose $X_0^1 \subset X_0^2 \subset ... \subset X_0^{N_j}$. For each of the hyper-rectangles X_0 and $X_0^j \subset X_0$, $j = 1, 2, ..., N_j$, determine a set of points that belongs to its facets and denote this set $\Phi^j = \left\{ \phi_1^j, \phi_2^j, ... , \phi_{N_\phi}^j \right\}$, $j = 0, 1, 2, ..., N_j$. Define the set of all points $V_0 = \left\{ v_0, v_1, v_2, ... , v_{N_1} \right\}$, where

$$ v_i \in \left\{ \bigcup_{j=0}^{N_j} \Lambda^j \right\} \cup \left\{ \bigcup_{j=0}^{N_j} \Phi^j \right\} , \ i = 1, 2, ... , N_1 . $$

The following procedure is applied to find a close-to-global solution at the points $v_i \in V_0$, $i = 0, 1, 2, ... , N_1$:

Procedure 2 (close-to-global solution of problem P2):
Consider any hyper-rectangle $X_0 \subseteq X_f$ with a set of points $V_0 = \left\{ v_0, v_1, v_2, ... , v_{N_1} \right\}$ determined by applying Procedure 1. Then:

   a). Determine a close-to-global solution of problem P2 at the center point v_0 through the following minimization:

$$ U^*(v_0) = \arg \min_{U_i^{local} \in \left\{ U_1^{local}, ..., U_{N_U}^{local} \right\}} J(U_i^{local}, v_0) , \quad (29) $$

where U_i^{local} , $i = 1, 2, ... , N_U$ correspond to local minima of the cost function $J(U, v_0)$ obtained for a number of initial guesses U_i^0 , $i = 1, 2, ... , N_U$.

   b). Determine a close-to-global solution of problem P2 at the points $v_i \in V_0$, $i = 1, 2, ... , N_1$ in the following way:

   1. Determine a close-to-global solution of P2 at the center point v_0 by solving problem (29). Let $i = 1$.

   2. Let $V^s = \left\{ v_0, v_1, v_2, ... , v_{N_2} \right\} \subset V_0$ be the subset of points at which a feasible solution of P2 has been already determined.

   3. Find the point $\bar{v} \in V^s$ that is most close to the point v_i, i.e. $\bar{v} = \arg \min_{v \in V^s} \| v - v_i \|$. Let the solution at $\bar{v}$ be $U^*(\bar{v})$.

   4. Solve P2 at the point v_i with initial guess for the optimization variables set to $U^*(\bar{v})$.

   5. If a solution of P2 at the point v_i has been found, mark v_i as feasible and add it to the set V^s . Otherwise, mark v_i as infeasible.

   6. Let $i = i + 1$. If $i \leq N_1$, go to step 2. Otherwise, terminate.

   B. Computation of feasible PWL solution

Definition 1 (Feasibility on a discrete set):
Let $X = \left\{ v_1, v_2, ... , v_Q \right\} \subset \mathbb{R}^{\tilde{n}}$ be a discrete set. A function $U(\tilde{x})$ is feasible on X if $G(U(v_i), v_i) \leq 0$, $i \in \left\{ 1, 2, ... , Q \right\}$.

We restrict our attention to a hyper-rectangle $X \subset \mathbb{R}^{\tilde{n}}$ where we seek to approximate the optimal solution $U^*(\tilde{x})$ to problem P2. We require that the state space partition is orthogonal and can be represented as a $k - d$ tree. The main idea of the approximate mp-NLP approach is to construct a feasible piecewise linear (PWL) approximation $\hat{U}(\tilde{x})$ to $U^*(\tilde{x})$ on X , where the constituent affine functions are defined on hyper-rectangles covering X . In case of convexity, it suffices to compute the solution of problem P2 at the $2^{\tilde{n}}$ vertices of a considered hyper-rectangle X_0 by solving up to $2^{\tilde{n}}$ NLPs. In case of non-convexity, it would not be sufficient to impose the constraints only at the vertices of the hyper-rectangle X_0 . One approach to resolve this problem is to include some interior points in addition to the set of vertices of X_0 [23]. These additional points can represent the vertices and the facets centers of one or more hyper-rectangles contained in the interior of X_0 . Based on the solutions at all points, a feasible local linear approximation $\hat{U}_0(\tilde{x}) = K_0 \tilde{x} + g_0$ to the optimal solution $U^*(\tilde{x})$, valid in the whole hyper-rectangle X_0 , is determined by applying the following procedure:

Procedure 3 (computation of explicit approximate solution):
Suppose A1 and A2 hold. Consider any hyper-rectangle $X_0 \subseteq X_f$ with a set of points $V_0 = \left\{ v_0, v_1, v_2, ... , v_{N_1} \right\}$ determined by applying Procedure 1. Compute K_0 and g_0 by solving the following NLP:
Problem P3:

$$ \min_{K_0, g_0} \sum_{i=0}^{N_1} (J(K_0 v_i + g_0, v_i) - V^*(v_i) $$
$$ + \beta \| K_0 v_i + g_0 - U^*(v_i) \|_2^2) \quad (30) $$

subject to:

$$ G(K_0 v_i + g_0, v_i) \leq 0 , \ \forall v_i \in V_0 \quad (31) $$

In (30), the parameter $\beta > 0$ is a weighting coefficient.

   C. Estimation of error bounds

Suppose that a state feedback $\hat{U}_0(\tilde{x})$ that is feasible on $V_0 \subseteq X_0$ has been determined by applying Procedure 3. Then, for the cost function approximation error in X_0 we have:

$$ \varepsilon(\tilde{x}) = \hat{V}(\tilde{x}) - V^*(\tilde{x}) \leq \varepsilon_0 , \ \tilde{x} \in X_0 \quad (32) $$

where $\hat{V}(\tilde{x}) = J(\hat{U}_0(\tilde{x}), \tilde{x})$ is the sub-optimal cost and $V^*(\tilde{x})$ denotes the cost corresponding to the close-to-global solution $U^*(\tilde{x})$, i.e. $V^*(\tilde{x}) = J(U^*(\tilde{x}), \tilde{x})$. The following

procedure can be used to obtain an estimate $\hat{\varepsilon}_0$ of the maximal approximation error ε_0 in X_0.

Procedure 4 (computation of the error bound):

Consider any hyper-rectangle $X_0 \subseteq X_f$ with a set of points $V_0 = \{v_0, v_1, v_2, \dots, v_{N_1}\}$ determined by applying Procedure 1. Compute an estimate $\hat{\varepsilon}_0$ of the error bound ε_0 through the following maximization:

$$\hat{\varepsilon}_0 = \max_{i \in \{0,1,2,\dots,N_1\}} (\hat{V}(v_i) - V^*(v_i)) \qquad (33)$$

D. Approximate mp-NLP algorithm for explicit GP-NMPC

Assume the tolerance $\bar{\varepsilon} > 0$ of the cost function approximation error is given. The following algorithm is proposed to design explicit reference tracking GP-NMPC:

Algorithm 1 (explicit reference tracking GP-NMPC)

   1. Initialize the partition to the whole hyper-rectangle, i.e. $\Pi = \{X\}$. Mark the hyper-rectangle X as unexplored.

   2. Select any unexplored hyper-rectangle $X_0 \in \Pi$. If no such hyper-rectangle exists, terminate.

   3. Compute a solution to problem P2 at the center point v_0 of X_0 by applying Procedure 2a. If P2 has a feasible solution, go to step 4. Otherwise, split X_0 into two hyper-rectangles X_1 and X_2 by applying the heuristic rule 2 from [23]. Mark X_1 and X_2 unexplored, remove X_0 from Π, add X_1 and X_2 to Π, and go to step 2.

   4. Define a set of points $V_0 = \{v_0, v_1, v_2, \dots, v_{N_1}\}$ by applying Procedure 1. Compute a solution to problem P2 for $\tilde{x}$ fixed to each of the points v_i, $i = 1, 2, \dots, N_1$ by applying Procedure 2b. If P2 have a feasible solution at all these points, go to step 6. Otherwise, go to step 5.

   5. Compute the size of X_0 using some metric. If it is smaller than some given tolerance, mark X_0 infeasible and explored and go to step 2. Otherwise, split X_0 into hyper-rectangles X_1, X_2, …, X_{N_s} by applying the heuristic rule 1 from [23]. Mark X_1, X_2, …, X_{N_s} unexplored, remove X_0 from Π, add X_1, X_2, …, X_{N_s} to Π, and go to step 2.

   6. Compute an affine state feedback $\hat{U}_0(\tilde{x})$ using Procedure 3, as an approximation to be used in X_0. If no feasible solution was found, split X_0 into two hyper-rectangles X_1 and X_2 by applying the heuristic rule 3 from [23]. Mark X_1 and X_2 unexplored, remove X_0 from Π, add X_1 and X_2 to Π, and go to step 2.

   7. Compute an estimate $\hat{\varepsilon}_0$ of the error bound ε_0 in X_0 by applying Procedure 4. If $\hat{\varepsilon}_0 \leq \bar{\varepsilon}$, mark X_0 as explored and feasible and go to step 2. Otherwise, split X_0 into two

hyper-rectangles X_1 and X_2 by applying Procedure 4 from [23]. Mark X_1 and X_2 unexplored, remove X_0 from Π, add X_1 and X_2 to Π, and go to step 2.

V. SIMULATION EXAMPLE

A. The nonlinear system

Consider the stochastic system described by the following nonlinear state space model:

$$\dot{x} = -\tanh(x + u^3) + \xi_1 \qquad (34)$$

where ξ_1 is white noise with variance 0.0025 and zero mean. The sampling time, determined according to system dynamics, was selected to be $T_s = 0.5$. The Euler approximation of system (34) is:

$$x(t+1) = x(t) - T_s \tanh(x(t) + u(t)^3) + \xi_2(t) \qquad (35)$$

where $\xi_2(t) = T_s \xi_1(t)$.

B. Gaussian process model identification

The control signal u was generated by a random number generator with normal distribution. The control signal blocking was $T_u = 6T_s$, i.e. it is kept constant for 6 time instants. The number M of the input signal samples used for the identification determines the dimension of the covariance matrix. In our case, $M = 200$. Let x_m be the mean value of the state of system (35) obtained for the generated control signals, i.e. $x_m = \dfrac{1}{M} \sum_1^M x(t)$. By introducing the variable $y(t) = x(t) - x_m$, we would like to obtain a Gaussian process model for the following discrete-time stochastic system:

$$y(t+1) = y(t) - T_s \tanh(y(t) + u(t)^3) + \xi_2(t) \qquad (36)$$

Based on the generated data set, the discrete-time system (36) is approximated with Gaussian process with zero mean and covariance function of the form (1). The maximum likelihood framework was used to determine the hyperparameters. The optimization method applied for identification of the Gaussian process model was the conjugate gradient method with line searches [28]. The following set of hyperparameters was found:

$$\Theta = [w_1, w_2, v_0, v_1] = [0.3952, 0.9754, 1.0333, 0.0354] \quad (37)$$

A validation control input signal was generated by random number generator with normal distribution and rate of change that is different from the one used for the identification signal. The response of the Gaussian process model to the validation signal is shown in Fig. 1. The associated average squared error and log density error are respectively $ASE = 0.0017$ and $LD = -2.1476$.
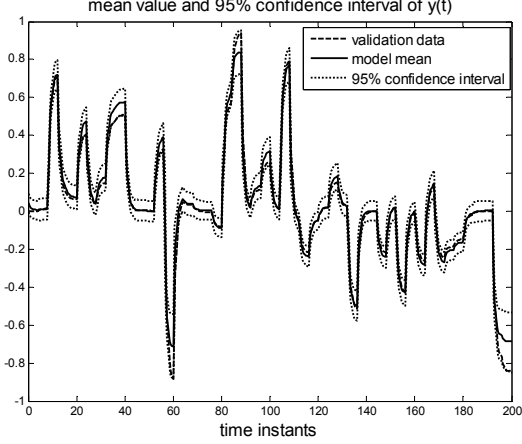
mean value and 95% confidence interval of y(t)

Fig. 1. Response of the Gaussian process model to the excitation signal used for validation.

C. Design of explicit reference tracking GP-NMPC controller

The mp-NLP approach described in section IV is applied to design an explicit reference tracking GP-NMPC controller for the system (35) based on the obtained Gaussian process model.

In the GP-NMPC problem formulation (problem P1), the predicted state $x_{t+k+1|t}$ of system (35) is used. This prediction is obtained in the following way. First, we obtain the prediction of $y_{t+k+1|t}$ from the Gaussian process model of system (36):

$$y_{t+k+1|t} \mid y_{t+k|t}, u_{t+k} \sim \mathcal{N}(\mu(y_{t+k+1|t}), \sigma^2(y_{t+k+1|t})) \quad (38)$$
$$k = 0, 1, \ldots, N-1$$

Then, the predicted $x_{t+k+1|t}$ is:

$$x_{t+k+1|t} = y_{t+k+1|t} + x_m \quad (39)$$

The iterative multi-step ahead prediction was done by feeding back at each time step the predictive mean only.

The following control input and rate constraints are imposed on the system:

$$-1 \le u \le 1 \; ; \; -0.5 \le \Delta u \le 0.5 \quad (40)$$

The prediction horizon is $N = 8$ and the terminal constraint is:

$$\max\{\|\mu(x_{t+N|t}) - 2\sigma(x_{t+N|t}) - r(t)\|, \\ \|\mu(x_{t+N|t}) + 2\sigma(x_{t+N|t}) - r(t)\|\} \le \delta \quad (41)$$

where $\delta = 0.015$. The weighting matrices in the cost function (21) are $Q = 10$, $R = 1$, $P = 10$. The GP-NMPC minimizes the cost function (21) subject to the Gaussian process model (38)–(39) and the constraints (40), (41).

The formulated GP-NMPC problem results in optimization problem P2 with 8 optimization variables and 33 constraints. One internal region $X_0^1 \subset X_0$ is used in Procedures 1, 2, 3 and 4. This results in problem P3 which has 32 optimization variables and 285 constraints. In (30), it

is chosen $\beta = 10$. The approximation tolerance is chosen in the following way:

$$\overline{\varepsilon}(X_0) = \max(\overline{\varepsilon}_a, \overline{\varepsilon}_r \min_{\tilde{x} \in X_0} V^*(\tilde{x})), \quad (42)$$

where $\overline{\varepsilon}_a = 0.005$ and $\overline{\varepsilon}_r = 0.05$ are the absolute and the relative tolerances, respectively. The extended state vector is $\tilde{x}(t) = [x(t), r(t), u(t-1)] \in \mathbb{R}^3$, which leads to a 3-dimensional state space to be partitioned. The latter is defined by $X = [-1.2, 1.2] \times [-0.7, 0.7] \times [-1, 1]$.

The partition has 1419 regions and 18 levels of search. Totally, 24 arithmetic operations are needed in real-time to compute the control input (18 comparisons, 3 multiplications and 3 additions).

The performance of the closed-loop system was simulated for the following set point change:

$$r(t) = -0.5, \; t \in [0;50] \; ; \; r(t) = -0.2, \; t \in [51;100] \\ r(t) = 0.2, \; t \in [101;150] \; ; \; r(t) = 0.5, \; t \in [151;200] \quad (43)$$

and initial conditions for the state and control variable $x(0) = 0$ and $u(0) = 0$, respectively. The resulting closed-loop response is depicted in Fig. 2 to Fig. 4.
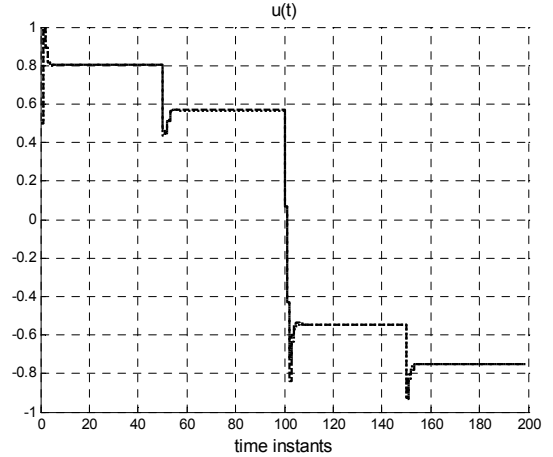

u(t)

Fig. 2. The control input. The dashed curve is with the approximate explicit GP-NMPC and the dotted curve is with the exact GP-NMPC.
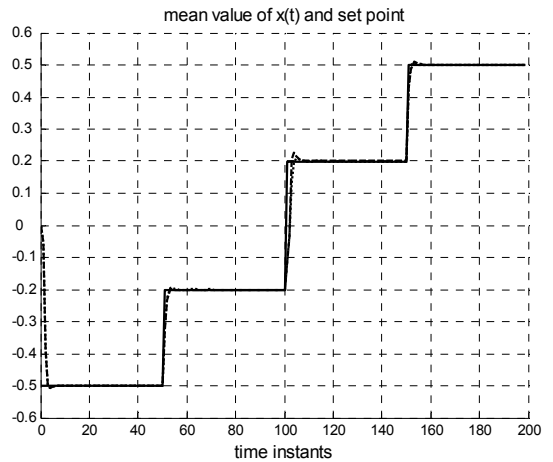

mean value of x(t) and set point

Fig. 3. The mean value of the state variable predicted with the Gaussian process model. The dashed curve is with the approximate explicit GP-NMPC, the dotted curve is with the exact GP-NMPC, and the solid curve is the set point.
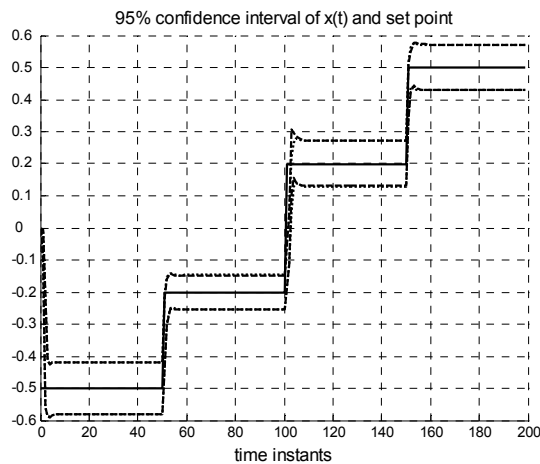
Fig. 4. The 95% confidence interval of the state variable predicted with the Gaussian process model. The dashed curve is with the approximate explicit GP-NMPC, the dotted curve is with the exact GP-NMPC, and the solid curve is the set point.

The results show that the exact and the approximate solutions are almost indistinguishable.

VI. CONCLUSIONS

In this paper, an approximate mp-NLP approach to explicit solution of reference tracking NMPC problems based on Gaussian process models is developed. The approach builds an orthogonal search tree structure of the state space partition and consists in constructing a feasible PWL approximation to the optimal control sequence.

REFERENCES

[1] S. S. Keerthi and E. G. Gilbert, "Optimal infinite horizon feedback laws for a general class of constrained discrete-time systems: Stability and moving horizon approximations", *Journal of Optimization Theory and Applications*, vol. 57, pp. 265–293, 1988.

[2] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, "Constrained model predictive control: Stability and optimality", *Automatica*, vol. 36, pp. 789–814, 2000.

[3] F. Allgöwer and A. Zheng (eds.), *Nonlinear Model Predictive Control, Progress in system and control theory*, Vol. 26, Birkhäuser Verlag, Basel, 2000.

[4] B. Kouvaritakis and M. Cannon (eds.), *Nonlinear predictive control, Theory and practice*, IEE Control Engineering Series 61, IEE, 2001.

[5] L. Magni and R. Scattolini, "Stabilizing model predictive control of nonlinear continuous time systems", *Annual Reviews in Control*, vol. 28, pp. 1–11, 2004.

[6] J. H. Lee and B. L. Cooley, "Optimal feedback control strategies for state-space systems with stochastic parameters", *IEEE Transactions on Automatic Control*, vol. 43, No. 10, pp. 1469–1475, 1998.

[7] D. van Hessem, C. W. Scherer, and O. H. Bosgra, "LMI-based closed-loop economic optimization of stochastic process operation under state and input constraints. *Proceedings of the 40-th IEEE Conference on Decision and Control*, 2001, pp. 4228–4233.

[8] J. Yan and R. R. Bitmead, "Incorporating state estimation into model predictive control and its application to network traffic control", *Automatica*, vol. 41, pp. 595–604, 2005.

[9] B. Kouvaritakis, M. Cannon, and P. Couchman, "MPC as a tool for sustainable development integrated policy assessment", *IEEE Transactions on Automatic Control*, vol. 51, No. 1, pp. 145–149, 2006.

[10] P. Couchman, B. Kouvaritakis, and M. Cannon, "LTV models in MPC for sustainable development", *International Journal of Control*, vol. 79, No. 1, pp. 63–73, 2006.

[11] P. Couchman, M. Cannon, and B. Kouvaritakis, "Stochastic MPC with inequality stability constraints", Automatica, vol. 42, No. 12, pp. 2169–2174, 2006.

[12] P. Couchman, M. Cannon, and B. Kouvaritakis, "MPC for stochastic systems", *Proceedings of the International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control*, Freudenstadt-Lauterbad, Germany, 26-30 August, 2005, pp.61–72.

[13] J. Kocijan, A. Girard, B. Banko, and R. Murray-Smith, "Dynamic systems identification with Gaussian processes", *Mathematical and Computer Modelling of Dynamic Systems*, vol. 11, No. 4, pp. 411–424, 2005.

[14] E. Solak, R. Murray-Smith, W. E. Leithead, D. J. Leith, and C. E. Rasmussen, "Derivative observations in Gaussian process models of dynamic systems", *NIPS 15*, Vancouver, Canada, MIT Press, 2003.

[15] A. Girard, C. E. Rasmussen, J. Quinonero Candela, and R. Murray-Smith, "Gaussian process priors with uncertain inputs & application to multiple-step ahead time series forecasting", *NIPS 15*, Vancouver, Canada, MIT Press, 2003.

[16] J. Kocijan, and R. Murray-Smith, "Non-linear predictive control with a Gaussian process model", In *Switching and Learning in Feedback Systems*, Eds. R. Shorten and R. Murray-Smith, Lecture Notes in Computer Science 3355, Springer-Verlag, Heidelberg, Germany, 2005.

[17] B. Likar and J. Kocijan, "Predictive control of a gas-liquid separation plant based on a Gaussian process model", *Computers & Chemical Engineering*, vol. 31, pp. 142–152, 2007.

[18] R. Murray-Smith, D. Sbarbaro, C. E. Rasmussen, and A. Girard, "Adaptive, cautious, predictive control with Gaussian process priors", *Proceedings of 13-th IFAC Symposium on System Identification*, Rotterdam, 2003.

[19] A. Bemporad, M. Morari, V. Dua, and E. N. Pistikopoulos, "The explicit linear quadratic regulator for constrained systems". *Automatica*, vol. 38, pp. 3–20, 2002.

[20] M. Bacic, M. Cannon, and B. Kouvaritakis, "Constrained NMPC via state-space partitioning for input-affine non-linear systems". *Proceedings of American Control Conference*, Denver, Colorado, 2003.

[21] T. A. Johansen, "On multi-parametric nonlinear programming and explicit nonlinear model predictive control", *Proceedings of IEEE Conference on Decision and Control*, Las Vegas, NV, 2002, vol. 3, pp. 2768–2773.

[22] T. A. Johansen, "Approximate explicit receding horizon control of constrained nonlinear systems", *Automatica* vol. 40, pp. 293–300, 2004.

[23] A. Grancharova, T. A. Johansen, and P. Tøndel, "Computational aspects of approximate explicit nonlinear model predictive control", *Proceedings of the International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control*, Freudenstadt-Lauterbad, Germany, 26-30 August, 2005, pp. 85–96.

[24] A. V. Fiacco, *Introduction to sensitivity and stability analysis in nonlinear programming*. Orlando, Fl: Academic Press, 1983.

[25] T. A. Johansen and A. Grancharova, "Approximate explicit constrained linear model predictive control via orthogonal search tree", *IEEE Transactions on Automatic Control*, vol. 48, pp. 810–815, 2003.

[26] C. K. I. Williams, "Prediction with Gaussian processes: From linear regression to linear prediction and beyond", In *Learning in Graphical Models*, M.I. Jordan, (ed.), Kluwer Academic, Dordrecht, pp. 599–621, 1998.

[27] C. E. Rasmussen, *Evaluation of Gaussian Processes and other Methods for Non-Linear Regression*, Ph.D. Disertation, Graduate department of Computer Science, University of Toronto, Toronto, 1996.

[28] A. Girard and R. Murray-Smith, "Gaussian process: Prediction at a noisy input and application to iterative multiple-step ahead forecasting of time-series", In *Switching and Learning in Feedback Systems*, Eds. R. Shorten and R. Murray-Smith, Lecture Notes in Computer Science 3355, Springer-Verlag, Heidelberg, Germany, 2005.