Derivation of loss using "exact local method"

Assumptions:
1. Steady-state cost \( J(u,d) \).
2. Quadratic cost
3. Linear model

1. Steady-state cost \( J(u,d) \)
\[
\min_{u,x} J(u,x,d) \Rightarrow \text{Eliminate } x,
\min_u J(u,d)
\text{subject to } \|x\| = 0
\]

2. Quadratic cost (around \( \bar{u},d \))
\[
J(u,d) = \frac{1}{2} (u - \bar{u})^T \cdot A_{uu} \cdot (u - \bar{u}) + \frac{1}{2} \cdot (u - \bar{u})^T \cdot A_{ux} \cdot \left( \frac{\partial u}{\partial d} \right) \cdot H_u(d)
\]

Nominal optimum,
\[
H_u = \left( \frac{\partial J_u}{\partial u} \right)^T \left( \frac{\partial J_u}{\partial u} \right)
\]

3. Linear measurement model
\[
y = G_u u + G_d d \quad \text{(in deviation variables, } \bar{u}, \bar{d})
\]

Question: What is loss if we control \( C = H_y y \) (i.e., \( H_y y \)) at constant value, when there are disturbances?

![Diagram of system with control input]

Note: \( u \) varies to keep \( C = G = \text{constant} \)

Need to compare with optimal case

Optimal input: Keep \( \bar{u} = 0 \) always
\[
J_u = \frac{1}{2} \cdot \left( u - \bar{u} \right)^T \cdot A_{uu} \cdot \left( u - \bar{u} \right) + \frac{1}{2} \cdot \left( u - \bar{u} \right)^T \cdot A_{ux} \cdot \left( \frac{\partial u}{\partial d} \right) \cdot H_u(d)
\]

\[
\Rightarrow \Delta J_u = - J_u \quad \text{subject to } \Delta d = \frac{\partial J_u}{\partial d} \cdot H_u(d)
\]

Note: \( \Delta J_u = G_u \Delta u + G_d \Delta d = \left( G_u H_u + G_d \right) \Delta d \)

Note: In practice, it is easier to think E by regaining for each d,
\[
\Delta J_u = \frac{\partial J_u}{\partial d} \cdot H_u(d)
\]

Evaluation of loss

\[
\text{Loss} = J(\bar{u},\bar{d}) - J(u,y,d) = \frac{1}{2} \cdot \left( \bar{u} - y \right)^T \cdot 
\]

Want to express \( \bar{z} \) as a function of \( d \) and \( n \).

We have:
\[
C = G_u \bar{u} + G_d \bar{d}
\]
\[
C_{opt} = G_y \bar{y} + G_d \bar{d}
\]

Thus:
\[
\bar{z} = \frac{1}{2} \cdot \left( C_{opt} - C \right)
\]

1. Here \( c \) is controlled at steady state. Then
\[
C_m = C_{opt} \quad \text{(assumed perfect control at steady-state)}
\]
\[
C_m = C_{opt} = G_y \bar{y} + G_d \bar{d}
\]

Also:
\[
C_m = H_{opt} = H (y + r^2) - H_y \bar{r}^2
\]

2. \( C_{opt} = H_{opt} = H_{dd} \)
\[
\bar{z} = \frac{1}{2} \cdot \left( H_{opt}ight)^T \left( - H_{opt}^2 - H_{dd} \right) = \frac{1}{2} \cdot \left( H_{opt}ight)^T \left( - H_{opt}^2 \right)
\]

So:
\[
\bar{z} = \frac{1}{2} \cdot \left( - H_{opt}^2 \right)
\]
Normalized disturbance and noise:
\[ \| d \|^2_2 \leq 1 \]

Both \( t \) allowed so sign does not matter!!

Worst-case \( Z \) (worst-case loss)

\[
\max L = \frac{1}{2} \tilde{\sigma} (M) \frac{2}{r}
\]

Need to select \( H \) such that \( \bar{\sigma} (M) \) is minimized.

Null space method

Special case: No noise \((N=0)\) and sufficient no. of measurements

Can find \( H \) such that \( HF = 0 \) (zero loss)

1. Convex formulation

\[
\min \| HF \|_F
\]

s.t. \( HG = Z \)

2. Analytical formula (provided \( Y \) full rank)

\[
H^* = (Y^T Y)^{-1} Y^T G^{-1}
\]