Stochastic inequality constrained closed-loop model predictive control

with application to chemical process operation

Dennis van Hessem

Stochastic inequality constrained closed-loop model predictive control

with application to chemical process operation

Photo: Schaapskudde Haaksbergerveen,
The Netherlands.

Stochastic inequality constrained closed-loop model predictive control

with application to chemical process operation

Proefschrift

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus prof.dr.ir. J.T. Fokkema,
voorzitter van het College voor Promoties,
in het openbaar te verdedigen op
donderdag 3 juni 2004 om 10:30 uur
door

Dennis Harald VAN HESSEM

werktuigkundig ingenieur
geboren te Dalfsen

Dit proefschrift is goedgekeurd door de promotor:
Prof. ir. O.H. Bosgra

Samenstelling promotiecommissie:

Rector Magnificus,	voorzitter
Prof.ir. O.H. Bosgra,	Technische Universiteit Delft, promotor
Prof.dr.ir. A.C.P.M. Backx,	Technische Universiteit Eindhoven
Prof.dr.-ing. W. Marquardt,	RWTH-Aachen
Prof.ir. J. Grievink,	Technische Universiteit Delft
Prof.dr. A.A. Stoorvogel,	Technische Universiteit Delft
Prof.dr. C.W. Scherer,	Technische Universiteit Delft
Prof.dr.ir. P.M.J. Van den Hof,	Technische Universiteit Delft
Prof.dr.ir. P.A. Wieringa,	Technische Universiteit Delft, reserve lid

Published and distributed by: DUP Science

DUP Science is an imprint of
Delft University Press
P.O. Box 98
2600 MG Delft
The Netherlands
Telephone: +31 15 27 85 678
Telefax: +31 15 27 85 706
E-mail: info@library.tudelft.nl

ISBN 90-407-2489-X

Keywords: Model Predictive Control, stochastic disturbances, inequality constraints.

Printed in the Netherlands.

Voorwoord

Ik heb tijdens deze afgelopen promotieperiode een stel fantastische jaren beleefd die ik voor geen goud had willen missen. Behalve het feit dat je er hier en daar ook nog iets van opsteekt, is het vooral de therapeutische en verslavende werking van in vrijheid een stuk onderzoek doen dat ik zal missen. Maar, net als vijf jaar geleden is het nu ook weer tijd voor nieuwe dingen. Ik heb het Haagse verruild voor het Amsterdamse en ik heb inmiddels ook al weer wat andere werkzaamheden opgepakt.

Een voorwoord zou geen voorwoord zijn zonder een aantal mensen te bedanken. Ten eerste bedank ik Okko voor het mogelijk maken van mijn promotie, zijn kennis en inzicht tijdens discussies, maar vooral vanwege de enorme vrijheid en steun die ik heb gehad tijdens het opzetten en uitvoeren van dit onderzoek. Ook Carsten heeft een grote rol gespeeld; zijn inzicht, dictaten en colleges en onze samenwerking voor de cursus wiskunde zijn op allerlei manieren terug te vinden in dit werkje. Verder wil ik met name Ton, Wolfgang en Johan bedanken voor hun enthousiasme, discussie en de goede en persoonlijke sfeer tijdens de INCOOP projectjaren.

Dan niet te vergeten mijn oud-collega AIO's Rob en Jog-Chem, uit de INCOOP crew, maar natuurlijk ook Tinus, Daaf, meneer Hol, Edje, Marco, Maria, Gideon, Branco, Loen, Ome Les, Mathijs, Sjoerd & Alex - Bedankt! Geen AIO bestaan zonder een stukje interactie zeg ik, lang leve de kantoortuin! Ik zal de midweekse schietoefening ook wel missen. Ook alle andere SR'ers en DCSC'ers waaronder met name Adrie, Peter, Sjoerd, Piet, Ton, Deb, Agnes & Cor. Allen bedankt voor de fantastische tijd in de groep! De andere INCOOP professionals waaronder Jobert, Wim, Peter & Peter bij IPCOS, Mario, Andrei & Siep in Eindhoven, Pieter & Peter bij STM, de Aachen-boys Martin & Jitendra, Sjoerd, Piet-Jan, Celeste & Bart bij Shell International Chemicals, Martin, Guido & Karsten at Bayer Technology Services. Thank you all for a wonderful cooperation, it has been an exiting time and we can look back upon a fantastic experience!

Katja, mijn ouders, opa & oma, familie, Joost, Joeri, Bart, Maarten en alle andere clubgenoten, vrinden en wederhelften... tsja, woorden schieten te kort, maar dat begrijpen jullie ook wel!

Dennis van Hessem
Amsterdam, 20 April 2004.

Summary

The chemical process industries are forced by the ever increasing global market competition to improve their efficiency of operation. Because of the large investment needed to build new plants it is of great interest to change the way existing plants are operated. It is therefore desirable that a plant produces a wider variety of products or that it can be switched faster between different product specifications or operation modes. To meet the requirements of the market, research and development are a necessity to innovate products, production processes and operational strategies. In almost all cases, plant and process models of varying complexity play an important role in analysis of technological problems and synthesis of solutions. Models formalize the current status of our system knowledge, point out weaknesses during validation, direct new research, are good carriers of knowledge over large time spans and they alow multi-disciplinary teamwork between engineers and scientists far beyond individual capabilities.

Model predictive control (MPC) has proved to be a successful example of model use for improvement of process operation. Typically it provides the ability for multivariable process control in which many process constraints can be included. The large number of technology vendors and successful implementations of MPC over the last twenty years points out that MPC has matured from an engineering point of view, however, the lack of predictability of the closed-loop behavior shows that it is far from theoretical maturity. The main structural limitation of MPC is that it is an open-loop predictive control method with the paradoxical property that neither future disturbances nor future measurements are considered. The implicit feedback derived from receding horizon control is typically hard to analyze and makes systematic tuning of inequality constrained MPC impossible without extensive simulation efforts. It is unknown how to choose controller parameters to influence the process sensitivity, which is a basic characteristic of any systematic feedback design method. Hence, there are no simple handles to design a predictive controller for desired closed-loop performance, to find optimal safeguarding from constraint violation or to construct optimal constraint pushing schemes. The reality of modern advanced process control, where inequality constraints *and* stochastic disturbances play a central role, points out that there is a fundamental need to reformulate the MPC problem to include these constraints and disturbances *at the same time* while optimizing the dynamic plant economy.

In this thesis a novel framework for advanced process control is presented. The aim is to develop a strategy for advanced process control in which predictive controllers are systematically designed without extensive simulations. This means that

the use of a receding horizon approach to generate feedback control is completely abandoned and a direct feedback control method is employed. This removes all the typical complications in the analysis and synthesis of predictive controllers and enables control design for *guaranteed* performance. Furthermore, techniques in high performance sensitivity-based control design are merged with the economic drive and constraint handling capabilities of advanced process control. This synergy provides a new powerful framework that exceeds the possibilities of methods in either field.

A number of steps must be taken in the development of an efficient set-up that satisfies these requirements. Crucial is the inclusion of future disturbances and future process measurements in addition to the traditional manipulated and controlled variables. The availability of future process measurements and manipulated variables allows us to define an explicit feedback control law yielding a parameterization of all realizable process sensitivity functions. Because of the unboundedness of Gaussian stochastic disturbances, the inequality constraints are enforced with a pre-defined level of certainty. The necessary amount of back-off to the constraints is then determined by the choice of the feedback controller and the second-order disturbance statistics. On top of this feedback structure, a feedforward is used for all deterministic tasks including transitions to optimal steady-states, grade and load changes and basically all economic tasks. Subtraction of the back-off from the inequality constraints reduces the search for a feedforward to a deterministic dynamic optimization problem that is simultaneously implemented with the feedback controller such that on-line feasibility is guaranteed.

Two possible predictive controllers are introduced that use this strategy. A full solution is given by the so-called closed-loop MPC problem in which the feedback controller, the back-off and the feedforward trajectory are simultaneously optimized for the global optimum. A predictive formulation of the Youla-Kučera parameterization of the closed-loop renders this problem convex such that it can be solved by modern optimization algorithms. A simplification to this procedure is obtained by the inequality constrained finite horizon LQG problem. In this case, the problem is split in two. In the first step a fixed suitably chosen feedback controller is computed that fixes the back-off, followed by a second step in which the feedforward is optimized for these fixed back-offs. The advantage of this latter approach is that its computational complexity equals that of standard open-loop MPC such that this approach can be applied to similar problems. Both problems are put in a receding horizon *implementation* (without any feedback functionality) for application to continuous processes. The optimal implementation is obtained as a predictive state-feedback in addition to the output feedback control law such that the overall optimization problem is of fixed complexity at all times. Both techniques are based on linear time-varying systems and can therefore be applied to both linear and nonlinear dynamical systems. Application of the proposed control strategy to a simulated non-linear industrial polymerization reactor shows very promising results motivating future applied and theoretic research to closed-loop predictive control methods.

Contents

1 Introduction

In the chemical process industries there is a need to operate plants within tight specifications on product quality and environmental indicators while maintaining a desired production level at a least possible realizable operational cost. Advanced process control can play an important role in optimizing chemical process operation if fundamental innovations are made in both control theoretic solutions as well as the technology. This thesis presents new developments that open the way to robust advanced model-based control guaranteeing economic profitable operation. This chapter defines the requirements on these solutions.

1.1 A knowledge-based chemical industry

The chemical process industry is a rapidly changing business that has always reacted to the changing political and market conditions. After the change from coal to oil as major source of chemicals and energy carrier to support the immense increase in transport energy needs, many chemical intermediates such as ethylene and propylene became available in large quantities and at low cost (Heaton, 1996). Combined with the advances in catalyst technology, this gave a boost in polymer production with endless demand at low production cost. After the drastic increase of energy prices in the 1970's, there has been a search for efficient production techniques reducing energy use and material loss. The chemical market place has become increasingly a global one where developing countries with large quantities of natural resources are nowadays producing large quantities of basic petrochemicals. This forces the chemical industry in western Europe and the US to improve the efficiency of the production processes for chemical products that require higher standards of research and technology. Increasing the product complexity leads to a larger diversity in products and smaller product volumes supplied to the market at demand and it then becomes increasingly important to react efficiently to market changes. Due to the large investment costs needed to build new industrial plants, it is of great interest to change the way existing (or new) plants are operated. The same plant should produce a wider variety of products or be switched faster between different product specifications or operation modes. Tighter product quality specifications attract new customers and if these tighter specs can be met using better operational tools, a chemical business can acquire the advantage over its competitors. Furthermore, there is an increasing scrutiny of the general public due to increasing global communication possibilities and media coverage that have resulted in increased legislation concerning pollution and other environmental issues. Summarizing, a chemical business must be increasingly innovative, future minded and cost effective in order not loose its market position to those who do.

1.2 Model-based research and operation

To meet the requirements of the market, research and development is a necessity to innovate products, production processes and operational strategies. To facilitate the necessary innovation in process operation, a chemical business must rise above the current way of plant operation using human operators to steer and control a plant. Although these operators will never be obsolete due to their importance in exceptional case handling for which automation is not cost effective, they should be relieved from their task of continuously making control decisions. Mental models of the plant physics are often incomplete and based on scalar steady state transfer models. By letting automata take over the multivariable dynamical control tasks from operators, better and more consistent performance in transition control, constraint handling and flexible response to the market can and should be achieved. In doing so, these computers must provide information on their control actions to operators supervising the plant in the understandable physical domain.

Research has in the last decennia provided us with much detailed technology that has led to high standards in performance requirements. With the sharp increase of computer power, computations far beyond the reach of any hand written calculations have become possible and consequently the possible field of applications in engineering have become increasingly wide. The result of this increased luxury in computing power has however also lead to increasingly detailed knowledge on complex systems using modern software environments such as computational fluid dynamics, finite element methods, generic chemical modelling languages and so on. In almost all cases, these models play an important role in analysis of technological problems and synthesis of solutions. An important aspect that makes the development of models worthwhile is that they are good carriers of knowledge over long time periods. They formalize the current status of our understanding of physical systems and during validation they reveal their weaknesses and point to directions in which more research is needed. They allow distribution of knowledge and enhance the communication between engineers with different backgrounds enabling multi-disciplinary teamwork now and increasingly in the future beyond the expertise of individuals.

Summarizing, building models of physical and chemical processes is generally worth the effort, however, the use of models is by no means a trivial task from a technological point of view. Applications require a versatility of knowledge ranging over many areas such as mathematics, physics, chemistry, chemical engineering, control and so on and depending on the application, models must be reliable, stable, robust, fast, small, detailed or efficient. To make balanced decisions regarding these issues, research covering these many aspects is needed.

1.3 Robust and optimal process operation

The main topic in this thesis is the development and solution of *closed-loop* model predictive control. This problem has to a large extent remained unsolved ever since standard *open-loop* model predictive control (MPC) emerged in the late 1970's and this has limited the progress in predictive control tools and theory considerably. The main distinction between open-loop and closed-loop MPC is that closed-loop MPC makes use of closed-loop model predictions in which the future control action of the MPC itself is taken into account, as opposed to open-loop MPC that does not. Of course, once the open-loop calculations are set in a receding horizon control scheme we do have feedback but the resulting control law is difficult to analyze in the case of inequality constraints due to its implicit nature.

In a sense, MPC theory has not matured theoretically, although from a practical perspective it has given the current status of the theory. MPC has the inherent problem that it is an open-loop control method and consequently even after 20 years of its existence, it is still impossible to estimate the performance of the closed-loop system without extensive simulations. It is impossible to measure its robustness despite the fact that it can indeed be made robust after tuning and there is no way of extrapolating this robustness to widen its operating range beyond the use around local set-points. These limitations are key problems that must be solved for application in the chemical process industries. Contrary to the petrochemical process industries, where the investment costs of usually sufficient linear MPC are low compared to the annual pay-back, control design in chemical industries are more specific, making it risky to invest in these advanced technologies. What is needed is a method to estimate the possible benefit of control solutions before a controller is actually implemented and before extensive research hours have been spent on tuning, which can be particularly hard for nonlinear processes as encountered in this area. In the next three sections three important requirements on control solutions are discussed. Namely, control solutions should provide the ability:

- to compute the guaranteed closed-loop economic performance of a control system, thereby letting the control system justify itself (or not),
- to control nonlinear process systems on the basis of first principle models,
- to handle disturbances and plant uncertainty in the presence of inequality constraints *unconservatively*.

These three aspects will now be discussed in more detail.

1.3.1 Guaranteed closed-loop economic performance

A key issue is that a process control engineer must be able to estimate the effectiveness of a control design with respect to the desired goals that go beyond the traditional robust stability requirements of base layer control systems. Before implementation of these solutions, the benefits should be clear and pre-computed at a reasonable level of certainty. The constraints set by environmental regulation are satisfied (or its violations economically assessed) and/or the desired increase of profit or flexibility is realized. These estimates should not only be extrapolated from past experience in comparable process applications or extensive simulation studies, but should be guaranteed by theoretical foundations. In a sense, these foundations should be as strong as the conservation laws and their consequences equally predictable. The current state-of-the-art in theory as in practise is in this respect deemed inadequate.

First of all, predictive methods applied to process operation problems, including dynamic optimization and model predictive control, are open-loop methods. In the disturbance free case, there is no difference between open- and closed-loop performance as nominal systems are considered. In fact, no feedback control is needed since the future is predictable with infinite accuracy. In the disturbance case, the open-loop and closed-loop methods differ substantially as was already analyzed for some dynamic optimization problems without constraints by Dreyfus (1962) when explicit formulas for the resulting feedback laws exist. In the constrained case, no such simple analysis is generally possible and due to the implicit nature of the receding horizon control mechanism it is difficult to compute the closed-loop performance using current open-loop predictive control techniques. As a consequence, deterministic optimization scenarios are usually considered in chemical process industries. Trajectories optimized on a deterministic basis often lie on the boundary of the feasible set defined by constraints on the process variables and any disturbance to the plant pushes the trajectories out of the feasible region. In the case of input saturation, performance may be lost beyond a minimal acceptable level from which it cannot be recovered (imagine a batch process that half way down the run cannot be recovered to meet the product specification at the final time). The goal of robust dynamic optimization is to prevent such scenarios from happening by optimizing transitions while keeping the process feasible with respect to the constraints for a variety of disturbance cases.

Second, chemical processes are usually fairly complex and can exhibit difficult nonlinear dynamics for which the use of linear time invariant system theory alone is not sufficient. The question in modelling a complex chemical plant is how much detail in the model or detail of the input-output behavior is really needed for control and/or optimization purposes. A control engineer is often faced with a large model, in most cases of unnecessary complexity and impractical for use in control system design. For nonlinear systems, the concepts of a minimal realization, a Kalman decomposition or balanced and reduced models that capture the relevant input-output behavior of a dynamical system exist, (Sussmann, 1972), but these techniques are not as accessible and numerically tractable as for linear systems, see

(Gilbert, 1963; Kalman, 1963; Moore, 1981) and there seems little hope for application to large scale process models in the near future.

Third, not many publications on control theory are concerned with the direct economic impact of control; much theory is concerned with derived objectives such as disturbance rejection and regulation in some optimal sense. Control theory can in some cases be used in dynamic optimization as was discussed in Bryson (1996), Breakwell *et al.* (1962) or Athans (1971). In those contributions, unconstrained nonlinear dynamic optimization problems are approximated using linear time varying dynamics and a quadratic performance index with some properly chosen weights. The strong point in such closed-loop approaches is that disturbances around nominal trajectories are considered. However, linear terms in objectives in combination with inequality constraints on process variables are not considered in these classical methods. In fact, linear optimization problems without these inequality constraints usually produce unbounded solutions and are for that reason not very attractive. The quadratic control approach is therefore a bit arbitrary and direct methods are needed that include inequality constraints on the process variables an linear terms in the objective function.

1.3.2 Nonlinear model-based process operation

In many applications in the process industries, linear dynamical models are quite sufficient for control purposes due to the large volumes of a single product that needs to be produced in a single economically optimal operating point. Such an operating point is often determined by means of steady state optimization using a static nonlinear process model or flowsheet. A linear model obtained via step response identification and a linear MPC is then often sufficient to keep the process in place. In other areas of processing such as in small volume fine chemicals and batch processes, the process is always in a transient and its behavior is dominated by large-scale nonlinear dynamics. Profit margins may be very tight and in that case mastering nonlinear dynamics can be of prime importance to guarantee flexible operation, for instance to respond to changing market conditions. In that case, many engineers must cope with these nonlinearities in their design of the process operational strategy.

Process operation is the whole of activities employed to operate a process plant through the entire economically interesting part of the operating window in an optimal fashion, while disturbances are continuously rejected. Hence, it is inherently a dynamic problem, whether or not static methods are used to solve it. As argued in the previous section, tools to facilitate robust optimization are scarce and deterministic dynamic optimization is therefore considered as the prime tool for optimizing operation. Typically load and/or grade changes are optimized for minimal off-spec production, energy consumption and so on. In a basic sequential optimization set-up, one needs a process modelling tool/simulator and a nonlinear optimization routine, such as a Sequential Quadratic Programming routine. A classical idea to cope with disturbances (Athans, 1971) is to consider a *variational*, or *delta-mode* control scheme

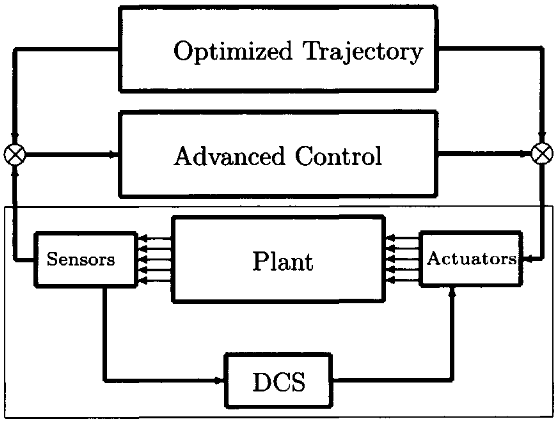


Figure 1.1: Constructive view on optimization and control.

that is very suitable for advanced process control and optimization. In this concept, dynamic optimization is used to generate optimized trajectories for the nonlinear system while feedback is used to reject all disturbances, see Figure 1.1. Then, the squared distance between the actual system trajectories and the optimized reference signals are minimized by the feedback controller. Perhaps suboptimal in its set-up, it does have the particular feature that it is easy to understand and to implement. It is the time-varying equivalent to linear control around an operating condition with that difference that the operating condition, and the linear controller may be varied in time. In advanced process control (APC), the time-varying LQG controller is generally replaced by an extended Kalman filter and a model predictive controller, as for instance discussed by Lee and Ricker (1994). Although this is a very straightforward way of controlling a plant, sub-optimality of this approach can be detected at several levels.

The first limitation in the control set-up of an extended Kalman filter and a MPC is that one falsely relies on the certainty equivalence property known from unconstrained linear control. The certainty equivalence property (Kwakernaak and Sivan, 1972) roughly states that disturbances need not be considered in determination of the optimal control law. This may appear harmless, but the result is that quite some performance may be lost either due to overly conservative back-off to constraints, (assuming that this was already subtracted from the feasible set of control solutions), or to insufficient back-off in which case many alarms and plant trips can be expected. Either way, valuable resources are lost limiting the achievable profit of the plant.

A second limitation is that in (nonlinear) dynamic optimization it is difficult to incorporate control objectives such as disturbance suppression and consequently no consideration is given to the realizability of the optimal trajectory on a real plant.

Without a doubt, one has to be conservative with respect to constraints to avoid infeasibility at a control level and it is preferable to restrict excessive movements already on a dynamic optimization level. One must somehow account for the inherent uncertainty in the model equations *before* the solution of the optimization problem is injected in the control scheme, Figure 1.1.

Besides considering control objectives on an optimization level, one can also consider economics on a controller level. Basically, one may separate disturbances in classes with high and low frequency spectra, but in advanced process control it is preferable to make a distinction between disturbances with and without economic impact on plant operation. The sample frequencies of the controller, (minute scale), or of dynamic re-optimization, (hour scale), then determines which part of the economics is transferred to which level of optimization. A consistent decomposition is far from trivial as was shown by Tousain (2002). Note that adding economics in a model predictive controller is not necessarily hard since linear terms can be added to the objective functions to account for the cost of for instance flows of steam, catalyst, feeds etc., since inequality constraints bounding the control moves are present in the problem formulation.

It is important to integrate dynamic optimization and control for nonlinear processes to maximize the benefit of implementation. In this thesis, this integration will be pursued to a high level by considering both problems to be one and solving them simultaneously. Admittedly, some process plants may be too large to use such an integrated scheme, on the other hand it will be illustrated that quite realistic applications are possible.

1.3.3 An historical perspective on high performance control

During the second World War, control theory and technology has experienced important steps forward. New feedback control design techniques were introduced by Ziegler and Nichols (1942) and Bode (1945). Although their results are still used in PID control design, these methods are limited in use for complex multivariable systems and do not give high performance control. After the war, there has been a continuous effort in finding systematic techniques to compute filters and controllers. Historically, quadratic objectives in filter and control design have received much attention , because the solution to least squares problems is easily obtained and in the case of Gaussian disturbances on systems and signals the least squares solution gives the smallest variance. In the early 1940's the continuous time optimal filtering problem was solved by Wiener by showing that the solution satisfies the Wiener-Hopf integral equation of which the solution was found a decade earlier. The Wiener-Hopf equation is of the convolution type and therefore the unknown linear dynamical system is directly recovered from the spectral densities of the signals involved. The extensions by Bode and Shannon in 1950 provided a solution to construct the optimal causal filter by pre-whitening the data using spectral factorization, (Lewis, 1986).

By 1960, the frequency domain solutions were complemented by Kalman (1960) who solved the discrete time filtering problem recursively in state-space (continuous time

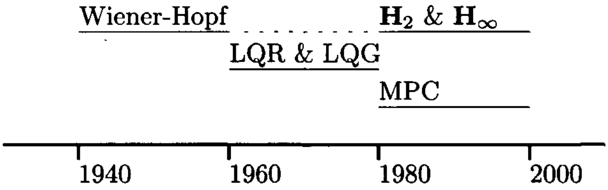


Figure 1.2: Time line in estimation and control design methods for performance

solution was given later (Kalman and Bucy, 1961)) leading to the celebrated Kalman filter. Although the solution is limited to systems that have a state-space representation, the result is comprehensive and numerically efficient. In the same papers, the solution to the noiseless regulator problem is given as a dual result strongly popularizing the concept of state-feedback. From that moment onwards, Linear Quadratic control dominated literature for nearly 15 years. Stochastic disturbance models are widely adopted and stochastic control solutions combine the Kalman filter and LQ state-feedback. LQG control is by then the most important systematic tool for multivariable control. Kailath (1968) reintroduced the pre-whitening form of the filter known as the innovations sequence approach as extension of Bode and Shannon's work and at the same time Willems (1971) related extremal solutions of the Riccati equations to storage functions and explicitly introduced Linear Matrix Inequalities. Despite the successes in LQ theoretical developments, LQG control was not widely adopted in engineering areas. The state is generally not fully accessible and the observer needed to reconstruct the state complicates control design procedures. At this stage, there is a rupture in control theory for constrained and unconstrained systems. On the one hand are the design methods for linear high performance control systems for mechanical and electrical systems and on the other advanced control design methods for petrochemical processes most significantly marked by the presence of inequality constraints.

For unconstrained systems, there is a revival of multi-variable frequency domain design techniques around 1975 with the papers by Youla *et al.* (1976b) on modern Wiener-Hopf design with guaranteed stability of the closed-loop. Still, a quadratic optimization problem is considered, but it is in a sense non-stochastic by considering frequency domain disturbance spectra marking the transition to $\mathbf{H}_2$ control design. Similar results were obtained in parallel by Kučera (1974). The basic idea is to parameterize all controllers that stabilize a given plant once an initial controller is found. However, while the LQ regulator has very good robustness properties with respect to gain and phase margin, the LQG design can have zero robustness margins (Doyle, 1978; Doyle and Stein, 1979; Safonov and Athans, 1977). Another problem is that the optimal observer is designed for one fixed disturbance spectrum only which is usually a strong limitation. This was recognized by Zames (1981) who introduced the alternative $\mathbf{H}_\infty$ control design problem considering sensitivity of the closed-loop to plant uncertainty embedded in the frequency domain. Then, instead of finding all controllers that stabilize a single plant, it is more important to find a controller that stabilizes a whole family of plants. In that same paper and a next (Zames and Francis, 1983) the Q-parameterization is introduced that is related to the works of Youla and Kučera. The affine parameterization of the closed-loop in the state-space is at that time the way to find a solution via a model-matching procedure (Maciejowski, 1994), using results of Desoer *et al.* (1980) and Nett *et al.* (1984). The direct algebraic state-space solution (not using the Q-parameterization) is given by Doyle *et al.* (1989), by solving two coupled Riccati equations. Although less important to our storyline, LMI solutions became available as well which are in turn related to the frequency domain by the Kalman-Yakubovitch-Popov lemma.

For constrained systems, or process systems in general, a different approach was taken. The lack of inherent robustness in LQG control, the presence of constraints, process nonlinearities, the fact that the plant economy is not given by quadratic performance criteria and cultural reasons are the main reasons for being inadequate for process control (Garcia *et al.*, 1989). Essentially, similar objections hold for high performance control systems, although in this case, the choice of $\mathbf{H}_\infty$ control is arbitrary. Instead, model predictive control arose in industry as a generic tool to control multivariable, time delayed and strongly coupled process systems with many inputs and outputs. It was pioneered by Richalet *et al.* (1978) as Model Predictive Heuristic Control (MPHC) and Cutler and Ramaker (1980) as Dynamic Matrix Control (DMC) during the 1970's though not published until the end of the decade. An attractive property of the MPC approach is that it allows physical interpretation of the controller predictions in the time domain. An interesting point is that around the same time as Zames (1981), Garcia and Morari (1982) published their work on the Q-parameterization as Internal Model Control (IMC) in application to MPC. The important observation is that the controller consist of a plant model put in parallel to the process. Another strong point of MPC is that inequality constraints on process variables can be included in the problem formulation leading to extensions such as QDMC (Garcia *et al.*, 1986). Furthermore, the same concept can be extended easily to nonlinear process systems (Gattu and Zafiriou, 1992; Lee and Ricker, 1994). Nevertheless, once inequality constraints are considered directly in design, stability of the receding horizon control strategy is difficult to analyze because of the finite horizon and the unpredictability of the active set of constraints. Some results on infinite horizons are available in literature by assuming the constraints to be inactive after the prediction horizon (Chmielewski and Manousiouthakis, 1996; Scokaert and Rawlings, 1998). This assumption is not unreasonable in regulation problems, but it makes little sense in the case of constraint pushing strategies.

1.3.4 From open-loop to closed-loop MPC

In a recent survey paper by Mayne *et al.* (2000) and a tutorial paper by Rawlings (2000) it was concluded that robustness remains a problem due to the indirect nature of receding horizon feedback. Further, MPC does *not* provide a systematic way of dealing with (stochastic) disturbances. Lee and Ricker (1994) and Robertson *et* *al.* (1996) propose to decompose the problem into an optimal Gaussian estimation problem and a deterministic prediction problem. This view has become a main line of MPC research which considers stochastics in the past but not in the future. Consequently, this approach does not provide a solution for the situation that the process variables are close to the constraints. The MPC will force the system to violate its constraints due to its ignorance towards disturbances. A receding horizon implementation cannot prevent this as its corrective action is always one step late. Secondly, standard model predictive control suffers from the limitation of any open-loop strategy, namely that the possibility of shaping the (process) sensitivity, a basic characteristic of feedback design methods, is completely absent. As a consequence, robustness is and always has been a problem with MPC, (Rawlings, 2000; Campo and Morari, 1987; Bemporad and Morari, 1999). Hence, standard MPC is no solution in the long run if one aims at systematic methods in process control where disturbances are large and plant model mismatch not so easily identified as for linear systems. A consistent way of dealing with disturbances is to consider feedback in the prediction. Shaping the sensitivity of the system trajectories to future disturbances is particularly important when the states and inputs are near their constraints. A few contributions to formulate closed-loop MPC's that exist in literature relate to this problem. Scokaert and Mayne (1998) and Lee and Yu (1997) propose a worst-case state feedback control law (min-max MPC) proceeding via dynamic programming. Although this would, in absence of stochastic disturbances, lead to the desired properties of closed-loop MPC, the computational burden originating from the combinatorial explosion of the optimization problem after gridding the state and disturbance spaces prevents application to real systems. Bemporad (1998;1999) proposes an alternative by using a scheduled fixed state-feedback, however, the structural choices in this nonlinear programming formulation to achieve feedback are considerable concessions on the achievable performance but still at a high computational burden. Batina *et al.* (2002) takes a truly stochastic approach to closed-loop MPC in which constraint violations are exponentially penalized. However, because no explicit parameterization of the feedback control law is chosen, their solution is based on a direct optimization over (an approximation of) the nested set of conditional expectations. This leads to a computationally demanding randomized programming approach in which no simple relation between the closed-loop performance and the controller parameters can be given.

The research presented in this thesis aims at unifying high performance control theory for constrained and unconstrained systems. To achieve this goal, the closed-loop MPC problem is solved in which a number of crucial aspects are identified (boldfaced below) and integrated to give a concise result. The future effect of disturbances are directly controlled using future feedback via **closed-loop prediction** which provides necessary **robustness** and directly links design to closed-loop performance by avoiding receding horizon control. **Feedforward** aims at optimizing the **plant economy** in analogy to a real time optimizer commanding a regulatory advanced controller for constraint pushing. This allows us to **separate** and differently tune the two important tasks namely **transition control** (for grade and load changes) and **regulatory control** resulting in a **two degrees of freedom control design** strategy. Another important issue is that **stochastic disturbances** are treated in the presence of **inequality constraints** in a systematic way. **Back-off** to the constraints is one of the main concepts that gives the closed-loop system its linearity such that a meaningful closed-loop **process sensitivity** is defined. The sensitivity then enables to choose a feedback controller to minimize that same back-off to the constraints as in a bootstrap method, such that feedforward can be used for constraint pushing. It will be shown that this **simultaneous feedback/feedforward optimization** problem is **convex** such that it can **efficiently** be solved for the global optimum. Furthermore, the solution can be constructed **recursively** leading automatically to a **receding horizon implementation** of fixed complexity for continuous processes. Finally, the whole framework is developed for a linear time-varying **generalized plant** that can be applied to **nonlinear processes** using a model in the form of differential algebraic equations.

2 Problem Formulation

In this chapter, the discussion in the introduction of this thesis is formalized in a research objective. This objective is then divided in seven smaller research questions each related to steps in the control design trajectory.

2.1 The research objective

The research is aimed at applications in the field of chemical process operation. In this field, there is a need for advanced control solutions that allow to operate process plants in a very flexible way, facilitating fast 'tracking' response to market conditions. As argued in Chapter 1, fundamental requirements on such technological solutions are that a process control engineer is able to compute the benefits of an advanced control system, is able to deal in a model-based fashion with nonlinear dynamical systems and has the availability of robust control tools that can handle model uncertainties and disturbances while keeping the process well within its operational constraints at all times. The objective of this research is as follows:

Develop a generic control theory that supports model-based integration of dynamic economic optimization and feedback control for chemical process applications.

Let us highlight several crucial elements in this objective. First of all, *"a generic control theory"* reflects the *need* for control; the plant is not behaving according to some nominal model but is instead driven by exogenous disturbances while plant-model mismatch is complicating the computation of optimal control moves. Secondly, control design is model-based and not achieved by extensive simulation efforts. Integration of *"dynamic economic optimization"* and *"feedback control"* underlines the fact that dynamic economic optimization needs some form of feedback control and reflects the desire to create control solutions that justify their implementation economically. Last but not least, *"chemical process application"* reflects the necessity to deal with nonlinear systems dominated by inequality constraints for which there exist nonlinear possibly first-principles models[1].

2.2 Available tools of online process operation

Model predictive control (MPC) is by all means a well established way of advanced process control as can be seen from a recent survey of industrial MPC technology (Qin and Badgewell, 2003) and a tutorial overview (Rawlings, 2000). Its roots lie in industry, where people developed their own control algorithms now known by the collective name model predictive control (Richalet *et al.*, 1978; Cutler and Ramaker, 1980). The strong point of MPC is that physical interpretation of the variables is possible, it previews future process behavior, it handles multi-input multi-output systems as easily as single-input single output systems, it automatically generates feedforward, it handles time delays and includes inequality constraints in its quadratic programming structure. No other control technology in the process industries had such versatile qualities and its popularity was immediate. The downside was also obvious. There was no theory for the closed-loop behavior of the controller in the case of inequality constraints and contrary to normal design techniques no systematic approaches were available for tuning. As a consequence, within 10 years a large number of different versions of MPC arose such as GPC, DMC, MAC, PCA and so on, all with their little differences and based on different rules of thumb and sets of tuning parameters (Soeterbeek, 1990; Qin and Badgewell, 2003). Systems and control theory found its way into MPC research by comparisons with state-space LQG theory as discussed by Robertson (1995,1996) and Muske (1995). Closely related were the studies on infinite horizon strategies (Chmielewski and Manousiouthakis, 1996; Scokaert and Rawlings, 1998) which allow to investigate stability properties of the receding horizon control laws, see also the review by Mayne *et al.* (2000). Related to these infinite horizon controller schemes are the dual mode controllers, however these approaches still consider open-loop prediction despite a feedback law on the terminal set (Michalska and Mayne, 1995). Furthermore, most studies above mainly consider nominal cases and leave the true robust studies largely open. Robust MPC formulations have quite a history already, however, these techniques are based also on open-loop prediction (Campo and Morari, 1987). More related to our interest are the worst-case solutions solved via dynamic programming techniques (Lee and Yu, 1997) and (Scokaert and Mayne, 1998). However, these approaches suffer from the *curse of dimensionality* after the necessary gridding of the state and disturbance spaces and are therefore not applicable to our process operation objective. The alternative is the the concept of closed-loop prediction which was studied by Bemporad (1998;1999). However, the resulting synthesis technique is a complex set of nested non-convex optimization problem leading to a feedback control structure that seems hard to analyze using system theoretic ideas. Sum-

[1]The term first-principles model is ambiguous but here we mean models based on (approximations of) generally accepted conservation laws, which may very well contain empirical relations of any kind.

marizing, let us wrap up the discussion up with the recent conclusion of Rawlings (2000) that *"the difficulty that MPC introduces into the robustness question is the open-loop nature of the optimal control problem and the implicit feedback produced by the receding horizon implementation"* and of Mayne *et al.* (2000) *"while the problem has been studied, the outcome of the research are conceptual controllers that work in principle but are hard to employ"*. These observations lead to the conclusion that this area has not matured in the sense that robust tools are available for real application and therefore this issue is considered to be the main technical difficulty to be removed.

2.3 Scope of the research

Process operation is a wide and multi-disciplinary area of engineering and it is impossible to cover the whole range of knowledge and expertise needed in real applications. We balance the need for realistic ready-to-apply tools for model-based process operation and the need for fully generic theory by assuming the following.

- *The model and its structure.* A process model is available that contains smooth nonlinearities only and is regular with respect to its index in the case of DAE systems, which allows to use perturbation analysis at any process state. We will make use of linear time-varying dynamics along trajectories of the nonlinear model and therefore we assume the approximation to be sufficiently accurate for control purposes. Then, the updates on the feedforward trajectory and the feedback controller are based on the same dynamics. Hence, if there are inequality constraints present in the model, these are either removed and enforced on a dynamic optimization level or are otherwise approximated using smooth interpolation. It is often not directly clear whether this is restrictive or not from open-loop simulations with the model, the adequacy must be established in closed-loop.

- *The optimization problem formulation.* The objective function and the constraints are convex. It is assumed that all non-convex functions needed to describe the objective and constraints are taken up in the model as additional algebraic variables. This might conflict with the smoothness issue above depending on the application at hand.

- *The process/model complexity.* The process complexity is restricted by the currently available computing power. This applies to the number of states and algebraic variables as well as the number of constraints. The model "size" we aim at is characterized by approximately 5000 variables of which 100 to 200 are states variables. Ill-conditioning of the model equations will be left undiscussed and is assumed to be manageable. From an operational point of view one is therefore committed to model a process with a bounded level of detail in which only important input-output dynamics are accounted for. This modelling problem is highly non-trivial and usually requires model reduction and physical insight.

Once we have arrived at this point, we have a smooth nonlinear DAE model and optimization problem of moderate size. Our aim is to give a framework that allows to formulate process operational problems in terms of *feedforward* and *feedback* in a dynamic optimization setting. Analysis and synthesis of these two basic forms of control is central in this thesis. There are many other feedback mechanisms including adaptive strategies, feedback linearization, neural-net MPC and other typical aspects in process control such as inter-sample behavior, multi-rate and delayed estimation, supervisory interaction, safeguarding against loop-failure and so on, but these specific topics lie outside the scope of this research. The set-up discussed in this thesis provides a broad enough framework for new research in those directions.

2.4 A decomposition in research questions

In this section the overall research objective is decomposed in several smaller goals by formulating a number of research questions fundamental to the underlying problem.

Question 1: Formulating an advanced process control problem

Q1 *What is a suitable framework for model predictive control problems that considers the plant economy, constraints on state and input variables, exogenous disturbances and model uncertainty?*

What we are aiming at is to present generalized plant framework for inequality constrained systems. This framework is highly appreciated in many application areas of linear unconstrained control but in literature on inequality constrained MPC, the generalized plant framework is used less extensively. This is mainly due the complex implicit feedback mechanism of receding horizon control and the fact that future disturbances and measured outputs are not explicitly considered. We will argue that the alternative view on model predictive control in this thesis removes these limitations and does fit in the generalized plant framework.

Question 2: Integration of optimization and control

Q2 *How can economic optimization and control be integrated such that the design and implementation of advanced control technology can be justified on economic grounds without extensive simulation investments?*

Feedback control and dynamic optimization are generally considered as separate tasks originating from a decomposed supervisory integrated optimization problem. Clearly, better control leads to the possibility of implementing more complex dynamic optimization results, but it is hard to quantify the exact relation between controller tunings and the feasible set in the dynamic optimization problem. The closed-loop properties of MPC are not that well understood and they cannot immediately be translated into constraint relaxations, especially in the case of (stochastic) disturbances.

Question 3: Feedback, sensitivity and constraint handling

Q3 *How can inequality constraints be enforced, having a meaningful process sensitivity function that is optimally shaped in a numerically efficient way?*

In MPC literature either 1) the process sensitivity function is considered for tuning, but in that case inequality constraints are *not* (added in a final stage after the tuning parameters are chosen), or 2) one considers inequality constraints directly in the problem formulation and no process sensitivity function is available for design. This is why MPC is fundamentally a poor control strategy from a theoretic point of view and there has hardly been any unifying improvement in this area. Our viewpoint is that a control problem either has inequality constraints or not; if there are inequality constraints they should be considered explicitly in the design procedure for which a meaningful sensitivity function is then needed. If there are no inequality constraints then other design methods than MPC should be employed not based on receding horizon control. This point lies at the core of this thesis and we will show that indeed a very meaningful integration is possible.

Question 4: Feedforward trajectory design

Q4 *What is the role of feedforward in enhancing performance of the control system and how can feedforward trajectories be designed in line with the feedback control structure?*

The key idea is to use feedforward trajectories to optimize the economic behavior of the plant, while feedback is used to suppress disturbances. Better feedback control action reduces back-off to the constraints and enlarges the feasible set of trajectories in the feedforward optimization problem. The mathematical difficulty is the simultaneous optimization of feedforward and feedback control to find the global optimum. This is a largely unexplored territory of classical and predictive control because classical schemes do not explicitly consider inequality constraints while model predictive control is an open-loop control method. As a major contribution of this thesis it is shown how to construct these optimal controllers and feedforward trajectories by numerically efficient methods.

Question 5: Convergence and stationary behavior

Q5 *What is the stationary behavior of the control law and how can it be computed?*

In modern model predictive control theory, the prediction horizon is extended to infinity by assuming that the constraints are no longer active from a certain point onwards (Chmielewski and Manousiouthakis, 1996; Scokaert and Rawlings, 1998) which allows to compute a state end-point weighting to account for the quadratic cost beyond the prediction horizon. In the closed-loop predictive control problem considered in this thesis, inequality constraints are an integral part of the control problem formulation and they can never be assumed inactive because of constraint pushing. Therefore, inequality constraints must also be considered in the stationary problem. In this stationary problem, we seek the linear time invariant controller that allows us to shift the optimal set-point as close as possible to the constraints. The finite time solution cannot be used to solve this problem and the stationary solution must therefore be solved directly in terms of a time-invariant controller and optimal steady state.

Question 6: Receding horizon implementation

Q6 *How can the control solution of the closed-loop predictive problem be put in a receding horizon implementation?*

To apply the results to continuous processes, the control law is implemented in a receding horizon fashion because any finite horizon depletes as time proceeds. To do so, a separation theory is needed which decomposes the stochastic optimization problem into a prediction part and an estimation part. Then, measurement data is processed recursively and the size of the predictive control problem in terms of the number of free optimization variables remains bounded by a fixed number for all time. Although the receding horizon implementation does not generate feedback itself, as in open-loop MPC, there are some non-trivial aspects to generate a sequence of feedback controllers and feedforward trajectories. By showing equivalence to some LQG controller, we can indeed construct a sequence of solutions that satisfy the optimality conditions of the finite horizon problem at each time instant without having to use dynamic programming. As it turns out, a predictive state-feedback is necessary and sufficient to compactly represent the contribution of the process history to the optimal control corrections.

Question 7: Application to nonlinear process systems

Q7 *How can the control strategy be applied to chemical processes for which there exist smooth nonlinear dynamical differential algebraic models based on the laws of conservation?*

Chemical process systems are generally described by nonlinear dynamics based on a description of the conservation laws because nonlinear system identification is still a very difficult and a largely unsolved problem (Ljung, 2003). The dominant part of the dynamics implied by the conservation laws are often smooth for which linear time varying perturbation models can be derived. Then, nonlinear control is setup by exploiting ideas from sequential dynamic optimization to give iterative improvements as the horizon is shifting through time. A moderately sized model of a gas-phase polyethylene polymerization reactor is used for demonstration of these techniques.

2.5 Structure of this thesis

In open-loop MPC, the future predictions are in essence deterministic such that back-off is a non-existing issue in the problem formulation. Once future disturbances are taken into account, the problem shifts from deterministic to stochastic dynamic optimization under inequality constraints. Then, the necessary amount of back-off to the constraints that is needed to avoid violation becomes a key issue. In Chapter 3, it is shown how to use the first- and second-order statistic properties of the disturbances and the open-loop plant dynamics to systematically compute the back-off that is needed in the uncontrolled situation. In open-loop this is of course a very conservative approach but it immediately reveals an important property namely that due to the back-off, the constraints are inactive with respect to the actual trajectories or sample paths of the system. The crucial issue here is that the process dynamics then remain linear and the analysis is not based on switching dynamics. In Chapter 4, the open-loop system is put in a feedback loop with a controller and the linearity introduced by the back-off is exploited to define a meaningful sensitivity function. Then, by using a (sort of) bootstrap technique, the back-off can be minimized by the proper choice of controller parameters to reduce the conservatism in the feedforward optimization. The key observation in Chapter 4 is that the Youla-Kučera parameterization of the closed-loop renders this optimization convex such that the global optimum can be found by numerically efficient techniques. In Chapter 5, insight in the structural properties of the control law is obtained by relating the optimality conditions of the resulting closed-loop MPC problem to the familiar finite horizon LQG problem. This reveals the intuitive notion that reduction of variance in the direction of important constraints increases the profit rate of the plant. In Chapter 6, the optimal steady-state and linear time-invariant controller are computed by using asymptotic methods that lead to semi-definite optimization problems. In Chapter 7, a recursive solution is given which makes it possible to apply the results to continuous processes. The fundamental solution that transforms the snapshot solution of Chapter 4 into a recursive solution, is the introduction of a state feedback in addition to the output feedback controller. The proof proceeds via the optimality conditions derived in Chapter 5 and the known properties of LQG control. The importance of this result is that the optimization problem is of fixed complexity independent of the total number amount of measurement data. In Chapter 8, the theory is applied to a simulation of an industrial polymerization reactor. The performance in compared to straightforward application of an extended Kalman filter and a linear time-varying MPC. Chapter 9 gives the conclusions and recommendations.

The chapters are roughly related to the research questions as follows: **Q1,Q2** are discussed in Chapter 3, **Q3,Q4** are discussed in Chapters 4 and 5, **Q5** is discussed in Chapter 6, **Q6** is discussed in Chapter 7 and finally **Q8** is discussed in Chapter 4 and 8.

2.6 A brief guideline on notation

Throughout the thesis, the following basic notation will be used. Any vector valued variable such as inputs, states and outputs will be assigned a letter

$$u_k, x_k, y_k, z_k$$

with a subscript k related to a specific time instant t_k. These variables are related to linear time-variant or time-invariant systems usually derived locally or along trajectories of an underlying nonlinear dynamic system. The variables in the original nonlinear system are denoted using an overbar

$$\bar{u}_k, \bar{x}_k, \bar{y}_k, \bar{z}_k.$$

Stochastic variables are always denoted with their argument ξ

$$u_k(\xi), x_k(\xi), y_k(\xi), z_k(\xi)$$

to distinguish them from deterministic variables. This should not be confused with the realization of these stochastic processes which are given by u_k, x_k, y_k without ξ. Estimates of a random variable is denoted with a hat

$$\hat{y}_k.$$

The feedforward or reference signals are denoted by a superscript ^{r}

$$u_k^r, x_k^r, y_k^r, z_k^r.$$

A superscript ^{c} is used to denote tracking errors which are defined as the difference between the actual process and the reference trajectory

$$u_k^c(\xi) = u_k(\xi) - u_k^r.$$

A signal is a variable considered over a time horizon of n samples and they are stacked in long vectors and are printed in boldface. For technical reasons, we will often consider shrinking horizons which have a fixed final time t_n. Subscripts on signals will then mean the remaining part of the samples that are left in the horizon.

$$\mathbf{u}_k = \begin{pmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_n \end{pmatrix}, \mathbf{x}_k = \begin{pmatrix} x_k \\ x_{k+1} \\ \vdots \\ x_n \end{pmatrix}, \mathbf{y}_k = \begin{pmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_n \end{pmatrix}, \mathbf{z}_k = \begin{pmatrix} z_k \\ z_{k+1} \\ \vdots \\ z_n \end{pmatrix},$$

Mappings between signals are systems and are denoted by G and have subscripts related to the specific input and output signals of the system. In some cases it is necessary to give G a superscript k to reference a specific time instant t_k. For instance

$$G_{yu}^k$$

is the system mapping $\mathbf{u}_k$ to $\mathbf{y}_k$. Mappings from parameters or initial conditions to signals are called impulse response systems and are denoted in similar fashion.

3 The Generalized Plant for Constrained Predictive Control

The generalized plant framework enables a control engineer to formulate advanced control problems directly in terms of control inputs, disturbance inputs, measured outputs and performance outputs. This generic set-up is discussed in this chapter in relation to stochastic systems, inequality constraints and open-loop dynamic optimization. As shown in later chapters, there are efficient methods for predictive controller synthesis in the generalized plant framework that build on the foundations given here.

3.1 Introduction

Modern control technology for chemical processes uses mathematical representations or dynamic models to describe the input-output behavior between the manipulated and controlled variables. In current state-of-the-art industrial applications, linear dynamical models are used for which very reliable black-box identification methods exist using experimental step response data. Whether the final model is given as a pulse/step response model or in state space is completely irrelevant to this input-output behavior. The key issue is that the behavior is assumed to be linear time-invariant and therefore inherently limited to set-point regulation tasks in the case of nonlinear dynamics. The up-side is that, as long as the controller is active, the process remains in the vicinity of its origin and thereby the controller ensures the validity of its own internal model representation of the plant dynamics. However, to make substantial progress in transition and batch control, new robust and efficient nonlinear model-based predictive control techniques must be developed. As mentioned before, we assume that such a nonlinear first principles model is available, possibly after extensive engineering effort in modelling, validation and size reduction.

A typical requirement on these control methods is that the process is operated highly autonomously, while dealing with disturbances, plant-model mismatch, inequality constraints and a wide range of operating conditions. The nonlinear model is used to cover this large operating window in the background, however, full nonlinear optimization based MPC is generally infeasible and often unnecessary. In the end, control is always based on perturbation or sensitivity analysis and not on nonlinear simulation and the computed solution trajectory itself does not give information on the sensitivity of the solution. Therefore, in pursuit of efficient methods, iterative application of linear time-varying (LTV) models is used as in sequential dynamic optimization. To overcome the mismatch between the nonlinear model and its local LTV approximation, robust rather than exact control is used. Inevitably, even nonlinear models are approximations of reality, due to limited process knowledge, purposely limited modelling of the physics/chemistry and partly because there are exogenous forces continuously acting on the system. Let us now set-up this LTV control framework in a fairly general way.

In robust linear control, the generalized plant framework is popular because it provides a unifying framework for control problems but also because there exist many solutions for numerous control problems formulated in this framework giving *guaranteed closed-loop performance*, and there exist powerful numerical tools for solving Linear Matrix Inequalities or algebraic equations involved in the synthesis. This is an important issue because in advanced process control the number of systematic solutions for many interesting control problems are either absent or not matured. For instance, typical questions such as

- what is the minimal model accuracy or complexity needed to guarantee some level of closed-loop performance?
- what is the actual robustness of the controlled plant or how much uncertainty can be allowed before loosing performance or even stability?
- how much will the closed-loop performance increase by adding a new sensor/actuator and does the increase in economic benefit justify the extra cost?

cannot be answered by simple design calculations with the existing control solutions and even standard model predictive control is in this respect lacking. However, the controller synthesis tools in linear theory are not immediately suited for process control because the control problems themselves differ substantially. For instance in linear systems theory, inequality constraints are generally not explicitly included in the problem formulation. In contributions where these inequality constraints are considered such as the works by Boyd and Vandenberghe (2002) and Gökcek *et al.* (2001), regulator control problems are considered in which desirable economic aspects such as *constraint pushing* are not discussed.

3.2 The generalized plant

In this section we will discuss the generalized plant framework as *the* starting point for the central control problem discussed in this thesis. In general, the starting point for the control problems in the chemical process industries is a nonlinear dynamic process so these will be discussed first. Because we will use linear time-varying dynamics linearized along the feedforward trajectories to compute search directions in the nonlinear optimization problem, LTV models are discussed next. In the case the process linear time-varying itself, the control problem is convex as will be shown in Chapter 4 such that we can find the global optimum.

3.2.1 The nonlinear case

In robust linear control, the generalized plant framework is the set-up to formulate control problems in a precise manner. Despite this success, it has not achieved a broad support in the advanced process control community, because future measured outputs and future disturbances are not used directly in the standard MPC control design methods. To overcome these limitations an formulation is given that explicitly considers constraints. We start from the nonlinear dynamic system described by a set of differential-algebraic equations generally referred to as a DAE model

$$\overline{G} := \begin{cases} 0 = f(\dot{\bar{x}}, \bar{x}, \bar{v}, \bar{u}, \bar{w}, \bar{d}) \\ \bar{y} = C_y^x \bar{x} + C_y^v \bar{v} + D_y^u \bar{u} + D_y^w \bar{w} + D_y^d \bar{d} \\ \bar{z} = C_z^x \bar{x} + C_z^v \bar{v} + D_z^u \bar{u} + D_z^w \bar{w} + D_z^d \bar{d} \end{cases} \tag{3.1}$$

$\bar{d}(t) \in \mathbf{R}^{n_d}$	model parameters
$\bar{u}(t) \in \mathbf{R}^{n_u}$	input variables
$\bar{v}(t) \in \mathbf{R}^{n_v}$	algebraic variables
$\bar{w}(t) \in \mathbf{R}^{n_w}$	disturbance variables
$\bar{x}(t) \in \mathbf{R}^{n_x}$	state variables
$\dot{\bar{x}}(t) \in \mathbf{R}^{n_x}$	time derivatives of states
$\bar{y}(t) \in \mathbf{R}^{n_y}$	measured outputs
$\bar{z}(t) \in \mathbf{R}^{n_z}$	performance outputs

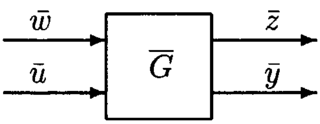
It is customary to use the term *assigned* variables for the set of control inputs $\bar{u}$, disturbance inputs $\bar{w}$ and the model parameters $\bar{d}$ and the term *algebraic* variables for the variables $\bar{v}$ that appear in the model equations f, are not assigned variables and do not appear as time derivatives as opposed to state-variables. The state variables together with the algebraic variables form the set of latent variables. The measured outputs $\bar{y}$ and $\bar{z}$ are linear transformations of the variables appearing in the model equations f.

Then, we want the performance output $\bar{z}$ to satisfy a set of linear inequality constraints

$$\mathcal{P}_t := \{h_j^T \bar{z}(t) \leq g_j, \; j = 1, \ldots, m\} \tag{3.2}$$

for each time instant t in the interval $t \in [0, T]$ for some matrix $h_j \in \mathbf{R}^{n_z}$ and vector $g_j \in \mathbf{R}$. Both H_j and g_j are allowed to be time-varying. It may appear if nonlinear constraints are not allowed here, but in fact one can use the freedom in the DAE model f to move nonlinearities from the constraints to the model via introduction of additional algebraic variables $\bar{v}$. How we will enforce these constraints and for which types of disturbances is discussed later on in this chapter.

The basic scheme of a generalized plant is given in figure 3.1, where the plant $\overline{G}$ is shown with two multivariable inputs $\bar{u}, \bar{w}$ and two multivariable outputs $\bar{y}, \bar{z}$. The



$\bar{w}$: *disturbances inputs*

$\bar{z}$: *performance outputs/controlled variables*

$\bar{u}$: *control inputs/manipulated variables*

$\bar{y}$: *measured outputs/process variables*

Figure 3.1: The input-output structure of a generalized plant.

generalized plant is the dynamic system mapping all inputs to all outputs mentioned above. The measured outputs or process variables (PV's) are denoted by $\bar{y}(t) \in \mathbf{R}^{n_y}$ and the performance outputs or controlled variables (CV's) are denoted by $\bar{z}(t) \in \mathbf{R}^{n_z}$. The control inputs or manipulated variables (MV's) is denoted by $\bar{u}(t) \in \mathbf{R}^{n_u}$ and are generated by the controller. The disturbance input $\bar{w}(t) \in \mathbf{R}^{n_w}$ is generally used to define transfer functions to the controlled variables $\bar{z}$, but $\bar{w}$ can be given a physical disturbance interpretation. Then $\bar{w}$ contains any exogenous signal determined outside of the system boundary not under the influence of the controller. For ease of presentation $\bar{d}$ is included in the signal $\bar{w}$. It is customary not to show the latent variables (LV's) of the system explicitly.

For this system we shall aim at finding some optimal reference trajectory $\bar{u}^r$ for the control inputs for a given reference trajectory $\bar{w}^r$ of the disturbances (disturbance feedforward). Suppose we are given an initial guess $\bar{u}_0^r, \bar{w}_0^r$ for the input trajectories. These reference values induce reference values $\bar{y}_0^r, \bar{z}_0^r$ for the measured and performance outputs if we let these trajectories satisfy the model equations

$$\begin{aligned} 0 &= f(\dot{\bar{x}}_0^r, \bar{x}_0^r, \bar{v}_0^r, \bar{u}_0^r, \bar{w}_0^r), \qquad \bar{x}_0^r(0) = \bar{x}_0^r \\ \bar{y}_0^r &= C_y^x \bar{x}_0^r + C_y^v \bar{v}_0^r + D_y^u \bar{u}_0^r + D_y^w \bar{w}_0^r \\ \bar{z}_0^r &= C_z^x \bar{x}_0^r + C_z^v \bar{v}_0^r + D_z^u \bar{u}_0^r + D_z^w \bar{w}_0^r \end{aligned} \tag{3.3}$$

for some trajectories $\bar{x}_0^r, \bar{v}_0^r$. These reference signals are of course very natural candidates for the basic trajectories along which we derive the linear time-varying model of the system (which we will do in the next section). The actual trajectories $\bar{u}, \bar{w}, \bar{y}, \bar{z}$ are given by their reference trajectories plus additional variational terms $\bar{u}^c, \bar{w}^c, \bar{y}^c, \bar{z}^c$ as shown in Figure 3.2

$$\begin{aligned} \bar{u} &= \bar{u}^r + \bar{u}^c, & \bar{w} &= \bar{w}^r + \bar{w}^c, \\ \bar{v} &= \bar{v}^r + \bar{v}^c, & \bar{x} &= \bar{x}^r + \bar{x}^c, \\ \bar{y} &= \bar{y}^r + \bar{y}^c, & \bar{z} &= \bar{z}^r + \bar{z}^c. \end{aligned}$$

The differences observed in the actual and reference values of the measured and performance outputs $\bar{y}^c$ and $\bar{z}^c$ respectively are caused by the *unpredictable* error $\bar{w}^c$

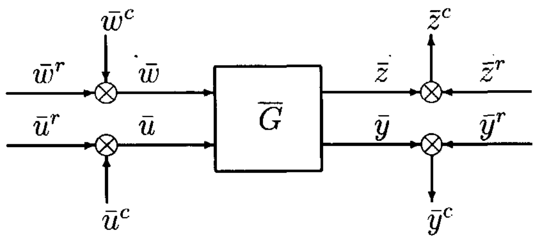


Figure 3.2: Adding reference signals to the generalized plant set-up.

and are called tracking errors in the remainder of this thesis. An obvious choice to reduce the tracking error is to use a feedback controller to compensate these errors by means of an additive control input $\bar{u}^c$. Therefore, to define a closed-loop system, we introduce the *linear* time-varying controller K

$$\bar{u}^c = K\bar{y}^c$$

where K is assumed to have zero initial conditions. This produces a closed-loop interconnection between the nonlinear system $\overline{G}$ and the linear controller K as shown in Figure 3.3.

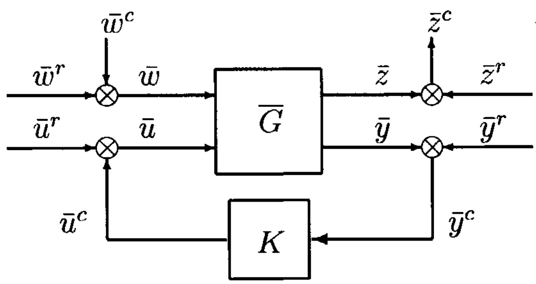


Figure 3.3: Interconnecting the generalized plant with the controller.

The goal will eventually be to efficiently compute the optimal feedback controller K as well as the optimal feedforward $\bar{u}^r$ maximizing some objective function $J(z^r)$.

3.2.2 The linear time-varying case

To find the optimal reference input and feedback controller, we need two types of sensitivities, namely:

1) the sensitivity of the system response $(\bar{y}, \bar{z})$ with respect to the disturbance signal $\bar{w}^c$ and the controller response $\bar{u}^c$ in order to quantify the controller performance,

2) the sensitivity of the system trajectories $(\bar{y}^r, \bar{z}^r)$ with respect to perturbations in the reference trajectories $\bar{u}^r$ and (optionally) $\bar{w}^r$.

Suppose the system would have been linear from the start, then we would have found in the same way as for the nonlinear system $\overline{G}$ the signals for the linear generalized plant as

$$\begin{array}{ll} u = u^r + u^c, & w = w^r + w^c, \\ x = x^r + x^c & v = v^r + v^c \\ y = y^r + y^c, & z = z^r + z^c \end{array} . \tag{3.4}$$

The relation of these signals to their nonlinear counterparts are given as follows. The perturbation of the disturbance signal $\bar{w}^c$ is determined outside of the system boundary and is therefore equal to the one appearing in the linear case

$$\bar{w}^c = w^c.$$

Our aim is here to find an update on the reference signal $\bar{u}^r$ for the control input that improves the response of the performance output $\bar{z}^r$ in some (to be defined) sense. Assume that we are given an initial guess $\bar{u}_0^r$ and that we seek an update u^r on the control inputs and for a given update w^r on the reference of the disturbance. This leads to the update laws

$$\bar{u}^r = \bar{u}_0^r + u^r, \quad \bar{w}^r = \bar{w}_0^r + w^r.$$

Let the trajectories x^r, v^r, y^r, z^r satisfy the linear time-varying dynamics along the reference trajectories of the nonlinear system

$$\begin{aligned} 0 &= \partial_{\dot{\bar{x}}} f|_0 \, \dot{x}^r + \partial_{\bar{x}} f|_0 \, x^r + \partial_{\bar{v}} f|_0 \, v^r + \partial_{\bar{u}} f|_0 \, u^r + \partial_{\bar{w}} f|_0 \, w^r, \quad x^r(0) = x_0^r \\ y^r &= C_y^x x^r + C_y^v v^r + D_y^u u^r + D_y^w w^r \\ z^r &= C_z^x x^r + C_z^v v^r + D_z^u u^r + D_z^w w^r \end{aligned}$$

The notation $\cdot|_0$ denotes the derivation along the given reference trajectory, that is

$$\partial_\star f|_0 = \partial_\star f(\bar{x}_0^r, \bar{v}_0^r, \bar{u}_0^r, \bar{w}_0^r).$$

Then, the effect of the perturbations u^r, w^r on the outputs is given, in a first order approximation, by

$$\bar{y}^r \simeq \bar{y}_0^r + y^r, \quad \bar{z}^r \simeq \bar{z}_0^r + z^r$$

leading to a change in the objective function of approximately

$$J(\bar{z}^r) \simeq J(\bar{z}_0^r) + \partial_z J(\bar{z}_0^r) z^r + \frac{1}{2} z^{rT} \partial_z^2 J(\bar{z}_0^r) z^r$$

where for ease of presentation we assume J to be twice differentiable.

In precisely the same way we can analyze the effect of perturbations u^c, w^c by letting x^c, v^c, y^c, z^c satisfy the same linearized dynamics

$$\begin{aligned} 0 &= \partial_{\dot{\bar{x}}} f|_0 \, \dot{x}^c + \partial_{\bar{x}} f|_0 \, x^c + \partial_{\bar{v}} f|_0 \, v^c + \partial_{\bar{u}} f|_0 \, u^c + \partial_{\bar{w}} f|_0 \, w^c, \quad x^c(0) = 0 \\ y^c &= C_y^x x^c + C_y^v v^c + D_y^u u^c + D_y^w w^c \\ z^c &= C_z^x x^c + C_z^v v^c + D_z^u u^c + D_z^w w^c \end{aligned}$$

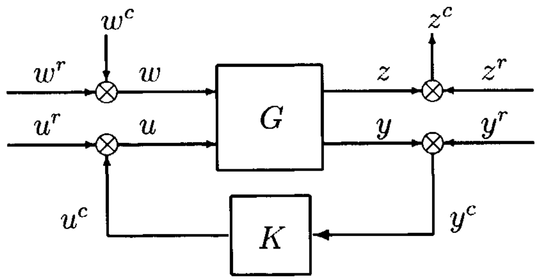


Figure 3.4: Interconnecting the generalized plant with the controller.

and as before we find the first-order effects on the variables in the nonlinear system as

$$\bar{y}^c \simeq y^c, \quad \bar{z}^c \simeq y^c$$

because the initial guesses $\bar{y}_0^c, \bar{z}_0^c$ for these signals are zero. For the linear system we also define the control law

$$u^c = Ky^c$$

such that the effect of the disturbance input $\bar{w}^c$ on the input $\bar{u}^c$ is approximated as

$$\bar{u}^c = K\bar{y}^c \simeq Ky^c = u^c$$

Finally, from the definition (3.4) and by linearity it follows that the trajectories u, v, w, x, y, z also satisfy the linearized dynamics and as such we also find approximations of the actual process variables

$$\bar{y} = \bar{y}^r + \bar{y}^c \simeq \bar{y}_0^r + y^r + y^c, \quad \bar{z} = \bar{z}^r + \bar{z}^c \simeq \bar{z}_0^r + z^r + z^c$$

such that we end up with the generalized plant for the linearized dynamics as shown in Figure 3.4. Throughout the thesis we will therefore largely be focussed on the problem of finding an update u^r on an existing initial guess $\bar{u}_0^r$ and a controller K. We will be very explicit on how to do this in this Chapter 3 which is mainly concerned with the open-loop case and Chapter 4 which is focussed on the closed-loop case. Finally, the inequality constraints are given by

$$h_j^T \bar{z}(t) \leq g_j, \ j = 1, \ldots, m, \ \forall t \in [0, T]$$

which are (iteratively) approximated by

$$h_j^T z(t) \leq g_j - h_j^T \bar{z}_0^r(t), \ j = 1, \ldots, m, \ t \in [0, T]$$

In the remaining sections of this chapter we will be concerned on how to explicitly deal with such constraints that depend on the disturbances.

3.2.3 The discrete time case

In this thesis we are only looking at discrete time controllers. To do so, we will consider the process only on discrete time instances

$$t_k = t_0 + kT_s, \quad k \in \mathbf{N}_0^+$$

where T_s is the sampling time. This allows us not only to give very simple representations of the LTV dynamics G, mapping the perturbation of the inputs into perturbations on the outputs, and the controller K, mapping perturbations in outputs back into perturbations on the inputs. To do so, we will parameterize the control inputs u using a zero-order hold mechanism such that

$$u_k = u(t_k), \quad \forall\, t \in [t_k, t_{k+1})$$

and we discretise the disturbances in a similar fashion. This is achieved by sampling the stochastic process and adapting the stochastic properties (Lewis, 1986). Then, the latent variables and outputs will only be considered on the sample times t_k

$$x_k = x(t_k), \quad v_k = v(t_k), \quad y_k = y(t_k), \quad z_k = z(t_k).$$

The last step in getting a generalized plant for predictive control is to introduce the lifted signals

$$\mathbf{u}_k = \begin{pmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+n} \end{pmatrix}, \mathbf{w}_k = \begin{pmatrix} w_k \\ w_{k+1} \\ \vdots \\ w_{k+n} \end{pmatrix} .\mathbf{y}_k = \begin{pmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+n} \end{pmatrix}, \mathbf{z}_k = \begin{pmatrix} z_k \\ z_{k+1} \\ \vdots \\ z_{k+n} \end{pmatrix}$$

in which the variables at each time instant are stacked in long vectors denoted by bold faced letters. The inequality constraints will only be enforced during on these sample time instances that is, we will consider the following inequality constraints

$$\mathcal{P}_k = \{z_k : H_k^T z_k \leq g_k\}$$

The model or system G is then a mapping between these stacked vectors and since linear discrete time-varying systems are considered, the representation of the dynamic system or impulse response function G is a large structured matrix

$$\begin{aligned} \mathbf{y}_0 &= G_{yx}x_0 + G_{yu}\mathbf{u}_0 + G_{yw}\mathbf{w}_0 \\ \mathbf{z}_0 &= G_{zx}x_0 + G_{zu}\mathbf{u}_0 + G_{zw}\mathbf{w}_0. \end{aligned}$$

where we use the subscripts of G to identify which signal or initial condition is mapped to which. There are two main ways to arrive at these system matrices, 1) in a control oriented approach via local discretisation of the system dynamics or 2) in a dynamic optimization oriented approach via sensitivity integration. These two techniques are briefly discussed below.

3.2.4 The control approach

The algebraic representation of a dynamical system can be obtained in several ways. In the case that the dynamical system is linear time invariant this representation is obtained by standard, well defined operations. In the nonlinear case, the lifted sensitivity systems can be derived in several ways and a computationally cheap method (and strong competitor to full sensitivity integration in the next example), exploits the exact time discretization of linear models derived from the continuous time nonlinear system along a trajectory. For ease of presentation, assume that our DAE model has the following structure

$$\dot{\bar{x}} = f_x(\bar{x}, \bar{v}, \bar{u}), \quad f_v(\bar{x}, \bar{v}, \bar{u}) = 0, \quad \bar{x}(t_0) = \bar{x}_0 \tag{3.5}$$

Then, along a solution $(\bar{x}, \bar{u}, \bar{v})$ of (3.5), the linear time-varying dynamical system is derived as

$$\begin{aligned} \dot{x} &= \partial_x f_x(\bar{x}, \bar{v}, \bar{u})x + \partial_v f_x(\bar{x}, \bar{v}, \bar{u})v + \partial_u f_x(\bar{x}, \bar{v}, \bar{u})u \\ 0 &= \partial_x f_v(\bar{x}, \bar{v}, \bar{u})x + \partial_v f_v(\bar{x}, \bar{v}, \bar{u})v + \partial_u f_v(\bar{x}, \bar{v}, \bar{u})u \end{aligned}$$

leading to the description

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ v(t) &= C(t)x(t) + D(t)u(t) \end{aligned}$$

where the continuous time system matrices are related to the dynamical system (3.5) as

$$\begin{aligned} A &= \partial_x f_x(\bar{x}, \bar{u}, \bar{v}) - \partial_v f_x(\bar{x}, \bar{v}, \bar{u})\partial_v f_v(\bar{x}, \bar{v}, \bar{u})^{-1}\partial_x f_v(\bar{x}, \bar{v}, \bar{u}) \\ B &= \partial_u f_x(\bar{x}, \bar{u}, \bar{v}) - \partial_v f_x(\bar{x}, \bar{v}, \bar{u})\partial_v f_v(\bar{x}, \bar{v}, \bar{u})^{-1}\partial_u f_v(\bar{x}, \bar{v}, \bar{u}) \\ C &= -\partial_v f_v(\bar{x}, \bar{v}, \bar{u})^{-1}\partial_x f_v(\bar{x}, \bar{v}, \bar{u}) \\ D &= -\partial_v f_v(\bar{x}, \bar{v}, \bar{u})^{-1}\partial_u f_v(\bar{x}, \bar{v}, \bar{u}) \end{aligned}$$

where $\partial_v f_v(\bar{x}, \bar{v}, \bar{u})$ is assumed to be invertible such that any perturbation pair (x, u) uniquely defines a perturbation in the algebraic variables v. Suppose the rate of change in the open-loop dynamics is small over one sample instant, then

$$(A(t), B(t), C(t), D(t)) \approx (A(t_k), B(t_k), C(t_k), D(t_k)) \quad \forall t \in [t_k, t_{k+1})$$

such that the discrete time dynamics are obtained in standard fashion using a zero-order hold circuit on the control input (Aström and Wittenmark, 1990)

$$A_k = e^{A(t_k)(t_{k+1}-t_k)}, \; B_k = \int_{t_k}^{t_{k+1}} e^{A(t_k)(t_{k+1}-s)}B(t_k)ds, \; C_k = C(t_k), D_k = D(t_k).$$

We assume that the dynamics do not change very fast in the sense that we will only allow very small changes in the system dynamics at each time instant. This is usually a requirement on the closed-loop system such that high frequent closed-loop behavior is avoided to enhance robustness against inter-sample behavior and plant-model mismatch. This procedure is easily extended to include the performance outputs and disturbance inputs, then leading to the recursive dynamical system

$$\begin{pmatrix} x_{k+1} \\ z_k \\ y_k \end{pmatrix} = \begin{pmatrix} A_k & B_k^w & B_k \\ C_k^z & E_k^z & D_k^z \\ C_k & D_k^w & O \end{pmatrix} \begin{pmatrix} x_k \\ w_k \\ u_k \end{pmatrix} \tag{3.6}$$

and suppose we lift this system over n samples, see for instance Furuta (1993,1995), then we obtain the desired algebraic LTV description

$$G_{yx}^0 = \begin{pmatrix} C_0 \\ C_1\Phi_{1,0} \\ \vdots \\ C_n^y\Phi_{n,0} \end{pmatrix}, \quad G_{yu}^0 = \begin{pmatrix} O & O & \cdots & O \\ C_1B_0 & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ C_n\Phi_{n,1}B_0 & C_n\Phi_{n,2}B_1 & \cdots & O \end{pmatrix} \tag{3.7}$$

$$G_{yw}^0 = \begin{pmatrix} D_0^w & O & \cdots & O \\ C_1G_0 & D_1^w & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ C_n\Phi_{n,1}G_0 & C_n\Phi_{n,2}G_1 & \cdots & D_n^w \end{pmatrix} \tag{3.8}$$

where the transition matrices are given by, see (Rhodes, 1971)

$$\Phi_{k,j} = A_{k-1}A_{k-2}\cdots A_j, \quad \Phi_{j,j} = I. \tag{3.9}$$

The systems transfer matrices G_{zx}, G_{zu}, G_{zw} are obtained in similar fashion.

3.2.5 The dynamic optimization approach

If the approximate procedure above leads to unsatisfactory *closed-loop* performance due to strong nonlinearities, then an alternative is to compute the exact sensitivity equations. This increased accuracy comes at a cost of a higher computational complexity and this means that the length of the prediction horizon is significantly smaller, possibly leading to loss of stability or control performance.

Consider the case in which we want to find a scalar optimal input trajectory parameterized using a vector $p \in \mathbf{R}^n$ as follows

$$\bar{u} = \bar{u}^r + \sum_i p_i\phi_i, \quad \forall t \in [t_0, t_1] \tag{3.10}$$

where $\bar{u}^r$ is some reference signal and the signals ϕ_i are suitably chosen to generate an interesting control space. In the case of zero-order hold sampled inputs

$$\bar{u}_i := \bar{u}(t_i) = p_i, \forall [t_i, t_{i+1})$$

one makes the specific choice for ϕ

$$\phi_i(t) := \mathbf{1}_{[t_i,t_{i+1}]}(t) = \begin{cases} 1 & t \in [t_i, t_{i+1}) \\ 0 & t \notin [t_i, t_{i+1}) \end{cases} \tag{3.11}$$

Consider the nonlinear dynamical system again

$$0 = f(\dot{\bar{x}}, \bar{x}, \bar{v}, \bar{u}, \bar{w})$$

of which the solution depends on the parameter p via the input $\bar{u}$. Due to the simple structure of the discontinuities in the control input, the solution of the differential equations is considered only on the separate open intervals (t_i, t_{i+1}). Then let the sensitivities s_i be trajectories that satisfy the linearized dynamics for zero disturbances, input ϕ_i and some trajectory v_i, then

$$0 = \partial_{\dot{\bar{x}}} f|_0 \, \dot{s}_i + \partial_{\bar{x}} f|_0 \, s_i + \partial_{\bar{v}} f|_0 \, v_i + \partial_{\bar{u}} f|_0 \, \phi_i, \quad s_i(0) = 0$$

Then s_i is the sensitivity of the solution of the differential equation with respect to the parameter p_i in the sense that the the perturbed solution is in first-order approximated by

$$\bar{x} = \bar{x}^r + \bar{x}^c \simeq \bar{x}^r + x^c, \text{ where } x^c = \sum_i s_i p_i = Sp, \quad S := \begin{pmatrix} s_1, & \cdots, & s_n \end{pmatrix}.$$

Hence, in a lifted system representation using a impulse response matrix notation G_{xp}, the result is a sampled version of S times the perturbation p

$$\mathbf{x}^c = \begin{pmatrix} x^c(t_0) \\ x^c(t_1) \\ \vdots \\ x^c(t_N) \end{pmatrix} = G_{xp} p, \quad G_{xp} = \begin{pmatrix} S(t_1) \\ S(t_2) \\ \vdots \\ S(t_N) \end{pmatrix}.$$

which gives an alternative procedure to determine the matrices G_{xu}. The LTV model obtained from the time discretized dynamics of the previous example is in fact a low cost approximation of the sampled sensitivity matrix S. This last approach is the more common view in nonlinear dynamic optimization and yields more accurate predictions than the control approach. The control approach is on the other hand cheaper in computational cost.

3.3 The stochastic generalized plant

In the preceding sections we have not yet specified the nature of the disturbances nor the effect it would have on the system description. To include stochastic disturbances in the standard plant, a stochastic framework is needed. The theory of probability, random variables and stochastic processes has well developed in a very general and abstract way heavily building on measure and integration theory, (Doob, 1953). It is a fact that full rigor is needed to define the concepts in probability in a precise and concise way, nevertheless, only a very small portion of the machinery is needed for our set-up. In this thesis all, random variables are **linear** or **affine** functions on $\mathbf{R}^n$, with a **Gaussian** probability density function. For the class of problems in this category, the treatment of stochastic in discrete time systems leads to very compact and easy to understand results, see for instance the treatise on LQG control in (Tse, 1971). The approach to set up our stochastic problem follows that found in (Papoulis, 1965), a good reference for applied probability calculations in connection to dynamical systems.

3.3.1 Stochastic disturbance models

In the generalized plant framework, it is necessary to build a disturbance model besides the model of the physical system based on the conservation laws. This disturbance model captures the interaction of the system with the world outside the system boundary not under our control. These disturbance systems are obtained by identification on process data as in (Ljung, 1987) or by modelling in which case one chooses simple structures such as steps and ramps. Disturbance modelling is the action of specifying 1) how disturbances w enter a dynamical system and 2) the set W of possible disturbances w. The basic model for the stochastic disturbances are given by the following definition.

Definition 1 *Stochastic disturbances.* Let ξ be a zero mean unit variance **Gaussian** random vector with n_ξ components. A stochastic disturbances $\mathbf{w}$ is an affine function defined on the random vector ξ parameterized as

$$\mathbf{w}(\xi) = \mathbf{w}^r + F_W \xi$$

for some vector $\mathbf{w}^r \in \mathbf{R}^{Nn_w}$ and some matrix $F_W \in \mathbf{R}^{Nn_w \times n_\xi}$. □

Then, the mean of $\mathbf{w}(\xi)$ is given by

$$E\mathbf{w}(\xi) = \mathbf{w}^r$$

and the covariance matrix is given by

$$E(\mathbf{w}(\xi) - \mathbf{w}^r)(\mathbf{w}(\xi) - \mathbf{w}^r)^T = F_W F_W^T = W$$

Note that the class of affine random variables considered here is quite large and sufficient to model quite relevant disturbance scenarios for process control purposes. The stochastic disturbances $\mathbf{w}(\xi)$ originate from a Gaussian distribution in the sense that each component $\mathbf{w}_k(\xi)$ of $\mathbf{w}(\xi)$ lies in the space $W_k = \text{span}\left\{v_j^k\right\}, v_j^k \in \mathbf{R}^N$

$$w_k(\xi) = w_k^r + \sum_j v_j^k \xi_j^k$$

Consequently, $\mathbf{w}(\xi) - \mathbf{w}^r$ lies in the Cartesian product $W := \Pi_k W_k$. Note in particular that one can also *model* persistent disturbances or biases of which the actual value is distributed normally, one can consider ramps of which the slope is a random variable or even day-night temperature fluctuations with unknown amplitude and so on. Then, the stochastic disturbance would be the sum of a bias $\mathbf{b}(\xi)$, a ramp $\mathbf{r}(\xi)$, and a sine $\mathbf{s}(\xi)$ (with fixed frequency) where $\xi \in \mathbf{R}^3$ is the outcome of an experiment with a Gaussian density function

$$\mathbf{w}(\xi) = \mathbf{b}(\xi) + \mathbf{r}(\xi) + \mathbf{s}(\xi), \quad \xi = (\xi_1, \xi_2, \xi_3).$$

Along the same lines we let the initial condition be given by

$$x_0(\xi) = x_0^r + F_P \xi, \quad \hat{x}_0 = x_0^r, \quad E(x_0(\xi) - x_0^r)(x_0(\xi) - x_0^r)^T = F_P F_P^T = P.$$

It is emphasized here that F_P and F_W are fixed a priori or otherwise assumed to be known beforehand. Furthermore, we will generally assume that the disturbances $w_k(\xi)$ on the different time instances are independent random variables and the initial condition $x_0(\xi)$ and disturbance $\mathbf{w}(\xi)$ are uncorrelated. In the computations that lie ahead, we need the Gaussian probability density function which is defined by

$$f(x) = \frac{1}{\sqrt{(2\pi)^n \det(R)}} e^{-\frac{1}{2}(x-\mu)^T R^{-1}(x-\mu)}$$

where $\mathbf{R}^{n \times n} \ni R = R^T > 0$ and $\mu \in \mathbf{R}^n$. For a given vector $x \in \mathbf{R}^n$ the event

$$(-\infty, x] = (-\infty, x_1] \times \ldots \times (-\infty, x_n]$$

then defines the distribution function of a random vector $\mathbf{x}(\xi)$ via

$$F_{\mathbf{x}}(x) = \int_{(-\infty,x]} f_{\mathbf{x}}(x) dx.$$

This bring us to the stochastic generalized plant, where from hereon we will include the argument ξ explicitly in the system description. This leads to notation of the form

$$\begin{aligned} \mathbf{y}_0(\xi) &= G_{yx} x_0(\xi) + G_{yu} \mathbf{u}_0(\xi) + G_{yw} \mathbf{w}_0(\xi) \\ \mathbf{z}_0(\xi) &= G_{zx} x_0(\xi) + G_{zu} \mathbf{u}_0(\xi) + G_{zw} \mathbf{w}_0(\xi) \end{aligned}$$

It is emphasized that this notation including ξ is used to identify random variables among other deterministic signals. We assume that there is no feedthrough from the inputs $u(\xi)$ to the measured outputs $y(\xi)$ to avoid algebraic loops in the closed-loop system (but the results can be extended to this case). Further note that $\mathbf{u}(\xi)$ depends in a causal fashion on the future disturbances. Next to these equations, we define the deterministic reference signals

$$\mathbf{y}_0^r = G_{yx} x_0^r + G_{yu} \mathbf{u}_0^r + G_{yw} \mathbf{w}_0^r \tag{3.12}$$

$$\mathbf{z}_0^r = G_{zx} x_0^r + G_{zu} \mathbf{u}_0^r + G_{zw} \mathbf{w}_0^r \tag{3.13}$$

The reference trajectories defined for the measured and outputs $\mathbf{y}^r$ and the controlled variables $\mathbf{z}^r$ by (3.12),(3.13) are to be optimized in the sense that the objective and constraints are defined using $\mathbf{z}^r$, while $\mathbf{u}^r$ is a vector of free manipulated variables.

3.3.2 Conditional expectation and estimation

Although the set of all possible outcomes is known beforehand, it generally impossible to actually measure the outcome ξ. Instead, one may have access to some other measurement y that is known to depend on ξ via some relation $\mathbf{y}$

$$\mathbf{y}(\xi) = y.$$

In estimation problems in control one is faced with the problem that (a linear combination of) the state of a dynamical system $\xi \in \mathbf{R}^n$ is only measured through a limited number of sensors $\mathbf{y}(\xi) = C\xi$, $C \in \mathbf{R}^{k \times n}$, $\mathbf{y}(\xi) \in \mathbf{R}^k$. If ξ were the outcome of a stochastic process, the function $\mathbf{y}$ would be a random variable with realization $y = \mathbf{y}(\xi)$. An estimation problem amounts to finding the best estimate $\hat{\xi}$ for ξ given the measurement $\mathbf{y}(\xi) = y$.

Conditional expectation of a random variable $\mathbf{x}$ with respect to another random variable $\mathbf{y}$ plays a crucial role in estimation problems. Consider the following estimation problem. Let $\mathbf{x} : \Omega \to \mathbf{R}^n$ be a random variable with second-order moment matrix

$$P = E\mathbf{x}(\xi)\mathbf{x}(\xi)^T,$$

let $g : \mathbf{R}^n \to \mathbf{R}^k$ a linear function $g(x) = Lx$ and let $\mathbf{y}$ be the composite function defined by $\mathbf{y} = g \circ \mathbf{x}$. Then $\mathbf{y}$ is the random variable defined by

$$\mathbf{y}(\xi) = g(\mathbf{x}(\xi)) = L\mathbf{x}(\xi).$$

If a measurement or realization y of $\mathbf{y}$ is available, the estimation problem amounts to finding an approximation $\hat{\mathbf{x}}(\xi)$ of $\mathbf{x}(\xi)$ in some optimal sense. To be specific, of all such approximations we seek the best linear one

$$\hat{\mathbf{x}}(\xi) = N\mathbf{y}(\xi).$$

parameterized by some matrix N, (Kalman, 1960; Lewis, 1986), such that the approximation is in fact a random variable itself. Let us formulate this optimization problem to find the optimal estimate as

$$\begin{array}{ll} \min\limits_{N} & \operatorname{tr} E(\mathbf{x}(\xi) - \hat{\mathbf{x}}(\xi))(\mathbf{x}(\xi) - \hat{\mathbf{x}}(\xi))^T \\ \text{s.t.} & \mathbf{y}(\xi) = L\mathbf{x}(\xi),\ \hat{\mathbf{x}}(\xi) = N\mathbf{y}(\xi) \end{array}$$

An alternative problem formulation is that we seek the estimate maximizing the conditional probability density function subject to the constraint that any allowable estimate must reproduce the measured output, thus $\mathbf{y}(\xi) = L\hat{\mathbf{x}}(\xi)$, which implies $LN = I$, (Cox, 1964). Either way, by substitution of the model one directly finds

$$\mathbf{x}(\xi) - \hat{\mathbf{x}}(\xi) = (I - NL)\mathbf{x}(\xi)$$

and it is well known and easily shown that the optimal solution satisfies the matrix equation

$$N(LPL^T) = PL^T \quad \text{or} \quad N = PL^T(LPL^T)^{-1}$$

if the inverse $(LPL^T)^{-1}$ exists. Therefore, the best estimate is given as

$$\hat{\mathbf{x}}(\xi) = PL^T(LPL^T)^{-1}\mathbf{y}(\xi)$$

which happens to be the conditional expectation of x

$$\hat{\mathbf{x}}(\xi) = E\left\{\mathbf{x}(\xi)|\mathbf{y}(\xi)\right\}.$$

Then, as the actual measurements or realization $\mathbf{y}$ of the stochastic process $\mathbf{y}(\xi)$ becomes available, the conditional expectation is evaluated to obtain the *realization* of the estimate

$$\hat{\mathbf{x}} = E(\mathbf{x}(\xi)|\mathbf{y}(\xi) = \mathbf{y}) = N\mathbf{y}.$$

The power of this result is its simplicity and generality. For instance one can use this result for recursive estimation, a popular choice for numerical efficiency, but the solution may just as well be used for horizon estimation.

3.4 Standard MPC in the generalized plant framework

It is interesting to put the standard MPC formulation in the generalized plant framework to see the development needed to give a closed-loop MPC formulation. In the vast amount of MPC literature one considers the following standard open-loop MPC control problem (Garcia *et al.*, 1989; Rawlings, 2000; Morari and Lee, 1997). Given a discrete time LTI system

$$x_{k+1} = A_k x_k + B_k u_k, \quad k \geq 0,$$

and an initial condition x_0, find the control sequence $\{u_k\}_{k=0}^{N}$ minimizing some quadratic cost function for some horizon length $N \in \mathbb{N}$, $Q, P \geq 0$, while satisfying the linear inequality constraints on the states and inputs for all $k \geq 0$

$$J(u) = x_N^T P x_N + \sum_{k=0}^{N-1} \begin{pmatrix} x_k \\ u_k \end{pmatrix}^T \begin{pmatrix} Q_{xx} & Q_{xu} \\ Q_{ux} & Q_{uu} \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \quad H_k^T \begin{pmatrix} x_k \\ u_k \end{pmatrix} \leq g_k.$$

In practice, many alternative MPC formulations are given mostly in the following form. Given a linear step response model H (impulse response models are treated similar)

$$\mathbf{z} = \hat{\mathbf{z}} + H\Delta\mathbf{u} + \mathbf{w}, \quad \hat{\mathbf{z}} = T\hat{x}_0, \quad \Delta u_k = u_k - u_{k-1}$$

where the outputs, inputs and disturbances $\mathbf{z}, \Delta\mathbf{u}, \mathbf{w}$ are stacked vectors over multiple samples. The disturbance $\mathbf{w}$ is chosen as a constant bias on the future prediction equal to the last mismatch between the predicted and measured outputs. Then the open-loop control problem amounts to finding the control sequence $\{u_k\}_{k=0}^{N}$ minimizing the quadratic cost functional for some horizon length $N \in \mathbb{N}$, $Q \geq 0$, while satisfying the linear inequality constraints on the outputs and inputs for all $k \geq 0$

$$J(u) = \sum_{k=0}^{N-1} \begin{pmatrix} \hat{z}_k - z_k^r \\ \Delta u_k \end{pmatrix}^T \begin{pmatrix} Q_{zz} & Q_{zu} \\ Q_{uz} & Q_{uu} \end{pmatrix} \begin{pmatrix} \hat{z}_k - z_k^r \\ \Delta u_k \end{pmatrix}, \quad H^T \begin{pmatrix} \hat{z}_k \\ \Delta u_k \end{pmatrix} \leq g$$

The prediction $\hat{\mathbf{z}} = T\hat{x}_0$ is used to account for the past or the memory of the system and whether this future prediction is obtained via an optimal state-estimate or suboptimal as in dynamic matrix control (DMC) is irrelevant (Lundström *et al.*, 1995).

If we choose $z_k = (x_k, u_k)$, we can readily see that the whole problem can be formulated with the open-loop framework of figure 3.5: we have a control input u and a performance output z and no future disturbances nor future measurements are considered explicitly in the optimization problem. Because the disturbances and measured outputs are only considered in the past, their continuation into the future is deterministic. Many variants on the above two MPC formulations exist, but they are fundamentally the same: the unknown future disturbances and measurements are not considered contrary to the control inputs u and the controlled variables z. The weakness in model predictive control and the paradox in its use become clear.

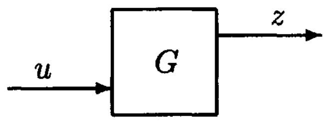


Figure 3.5: Standard open-loop MPC in a predictive generalized plant framework.

The problem formulation in model predictive controller does not explicitly contain future disturbances or measurements that are available for estimation. Although a Kalman filter uses the disturbance statistics and available measurements, it considers these signals only in the past. It is evident that estimation and control must be *un*-separated into a true stochastic control problem if the true solution to the stochastic closed-loop MPC problem must be found.

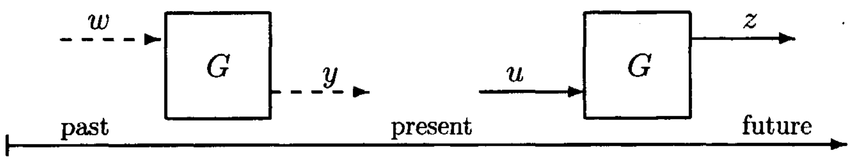


Figure 3.6: The classical decomposition of LQG control. The estimation problem considers disturbances and measured outputs only in past (dotted); the control law considers deterministic inputs and performance outputs only in future (solid).

3.5 Conic optimization

Convex optimization has a long history and there is a wealth of theory both for finite dimensional problems (Rockafellar, 1970) as well as infinite dimensional problems (Luenberger, 1969). Since the rediscovery of the interior point method in linear programming, (Karmarkar, 1984; Roos *et al.*, 1997), there has been a tremendous boost in optimization software that has decimated the gap between algorithms treating generalized inequalities (defined using nonstandard convex cones) and standard solvers for linear and quadratic programs defined on the standard positive cone $\mathbf{R}^n_+$ in the $\mathbf{R}^n$. Although one can dispute whether interior point methods are the best choice for linear programs, for generalized linear programming the primal-dual interior point approach seems to be the best alternative to find a solution in polynomial time. Due to the development of powerful algorithms on the products of convex cones (Wright, 1997; Sturm, 1999) the theory of generalized linear optimization has come to life to produce a vivid environment for modelling and optimization.

3.5.1 Conic programming

Conic programming is a collective term for a family of optimization problems of the form

$$\min_{\substack{x \succeq 0 \\ Ax = b}} \quad \langle c, x \rangle \tag{3.14}$$

where the generalized inequality $\succeq$ gives a (partial) ordering in X by means of a convex cone K. Despite the possible nonlinearity of the inequality constraints induced by this cone, the conic program (3.14) has the appearance of a linear program and it shares many duality properties. These generalized inequalities introduce a large versatility in constraint modelling and enable the numerical treatment of a large amount of convex nonlinear problems, and it will soon be clear how they can be used in the predictive control problem. The most important aspects of conic optimization are summarized below, largely taken from the reference text by Ben-Tal and Nemirovski (2001). In what follows, K will be a closed, convex, pointed cone with a nonempty interior in some finite dimensional real Hilbert space X. Such cones define generalized inequalities denoted by $\succeq$ for non-strict inequalities

$$x \succeq y \Leftrightarrow x - y \succeq 0 \Leftrightarrow x - y \in K$$

and $\succ$ for strict inequalities

$$x \succ y \Leftrightarrow x - y \succ 0 \Leftrightarrow x - y \in \text{int}(K).$$

Next to the cone K, a dual cone K^* is defined as the set

$$K^* := \{s \in \mathbf{X} : \langle s, x \rangle \geq 0 \text{ for all } x \in K\} \tag{3.15}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on X. K^* is again a pointed cone with a nonempty interior.

Definition 2 *Conic Optimization Problem (Nesterov and Todd, 1998).* Let K be a closed nonempty convex pointed cone in a finite dimensional real Hilbert space $\mathbf{X}$ and let K^* be the dual cone. The primal and dual conic optimization problem are defined as

$$\text{(Primal)} \quad p^\star = \inf_{\substack{x \in K \\ Ax = b}} \langle c, x \rangle \qquad \text{(Dual)} \quad d^\star = \sup_{\substack{y \in Y, s \in K^* \\ A^* y + s = c}} \langle b, y \rangle \tag{3.16}$$

where $A^* : \mathbf{Y} \to \mathbf{X}$ is the adjoint of the linear operator of $A : \mathbf{X} \to \mathbf{Y}$, $b \in \mathbf{Y}$ and $c \in \mathbf{X}$. □

The basic relation between the primal and dual problem are summarized in the following theorem.

Theorem 3 *Strong duality,(Nesterov and Todd, 1998).* Suppose the feasible sets of the primal and dual optimization problems

$$F^P := \{x \in \text{int}(K) : Ax = b\} \text{ and} \tag{3.17}$$

$$F^D := \{(y, s) \in Y \times \text{int}(K^*) : A^* y + s = c\} \tag{3.18}$$

are both non-empty. Then there exists solutions $x^\star$ and $(y^\star, s^\star)$ to both (Primal) and (Dual) respectively and their optimal values $p^\star$ and $d^\star$ are equal (zero duality gap)

$$p^\star = \langle c, x^\star \rangle = \langle b, y^\star \rangle = d^\star.$$

For any primal feasible x and dual feasible (y, s) the duality gap equals

$$\langle c, x \rangle - \langle b, y \rangle = \langle s, x \rangle$$

and at the optimum complementary slackness holds

$$\langle s^\star, x^\star \rangle = 0.$$

□

3.5.2 Examples of conic linear programs

Important conic programs defined on three different cones are linear and quadratic programming, second-order cone programming and semi-definite programming. A large number of applications using these optimization models are given in Ben-Tal and Nemirovski (2001) and Lobo *et al.* (1998). Linear and quadratic programming is used for feedforward optimization, second-order cone programming is used in the simultaneous feedforward/feedback optimization and semi-definite programming is used for the stationary solution all to be discussed in the subsequent chapters. A number of examples of these optimization problems are given below.

Linear programming. For LP's $\mathbf{X} = \mathbf{R}^n$, $c \in \mathbf{R}^n$ and $\langle c, x\rangle = c^T x$, the standard inner product on $\mathbf{R}^n$. If the standard positive cone is chosen

$$\mathbf{R}^n_+ = \{x \in R^n : x \geq 0\} = \{x \in \mathbf{R}^n : x_i \geq 0 \text{ for all } i = 1, .., n\},$$

one obtains the ordinary primal and dual linear programs

$$\text{(P-LP)} \quad \inf_{\substack{x \in \mathbf{R}^n_+ \\ Ax = b}} \; c^T x \qquad \text{(D-LP)} \quad \sup_{\substack{y \in \mathbf{R}^m, s \in \mathbf{R}^n_+ \\ A^T y + s = c}} \; b^T y$$

where $A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^m$.

Second-Order Cone Programming. For SOCP's $\mathbf{X} = \mathbf{R}^n$, $c \in \mathbf{R}^n$, and $\langle c, x\rangle = c^T x$. The second-order cone (sometimes referred to as the Lorentz or ice-cream cone) is given as

$$\begin{aligned} \mathbf{L}^n &= \{(y, z) \in \mathbf{R} \times \mathbf{R}^{n-1} : y \geq \|z\|\} \\ &= \{x \in \mathbf{R}^n : x_1 \geq \sqrt{x_2^2 + \cdots + x_{n-1}^2}\} \end{aligned}$$

which is self-dual $L^n = (L^n)_d$. The equality constraints are given by $\mathcal{A} : \mathbf{R}^n \to \mathbf{R}^m$, and $b \in \mathbf{R}^m$ and is, as for LP's, identified with a matrix $A \in \mathbf{R}^{m \times n}$. This gives the primal and dual problems

$$\text{(P-SOCP)} \quad \inf_{\substack{x \in \mathbf{L}^n \\ Ax = b}} \; c^T x \qquad \text{(D-SOCP)} \quad \sup_{\substack{y \in \mathbf{R}^m, s \in \mathbf{L}^n \\ A^T y + s = c}} \; b^T y \tag{3.19}$$

Second-order cone programs can be used to model a variety of important problems, see for instance the many examples in (Lobo *et al.*, 1998; Ben-Tal and Nemirovski, 2001). Second-order cone constraints are often encountered in the following format. Consider (D-SOCP) and partition the problem as in the cone constraint

$$s := (s_1, s_2) \in \mathbf{L}^n \Leftrightarrow s_1 \geq \|s_2\|, \quad c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad A = \begin{pmatrix} a_1 & A_2 \end{pmatrix} \tag{3.20}$$

where $s_1 := c_1 - a_1^T y$ en $s_2 := c_2 - A_2^T y$ and $y \in \mathbf{R}^m, s_1, c_1, \in \mathbf{R}, c_2 \in \mathbf{R}^{n-1}, a_1 \in \mathbf{R}^n, A_2 \in \mathbf{R}^{m \times (n-1)}$. Then the constraint (3.20) reads as

$$\|c_2 - A_2^T y\|_2 \leq c_1 - a_1^T y$$

which is of the structural form (primal notation)

$$\|Ax - b\|_2 \leq c^T x - d. \tag{3.21}$$

Another small but useful trick is the transformation of an objective function given by the squared Euclidean norm to a second order cone constraint. Let the objective be given as

$$f(x) = x^T x, \quad x \in \mathbf{R}^n.$$

Introduce a scalar variable γ such that the objective can be converted into constraint $f(x) \leq \gamma$ by replacing the objective f of the optimization with γ to be minimized. For any scalar γ we have

$$\gamma = \frac{(\gamma+1)^2}{4} - \frac{(\gamma-1)^2}{4}$$

and from this rule, the following second-order cone constraint can be constructed

$$x^T x \leq \gamma \Leftrightarrow x^T x + \frac{(\gamma-1)^2}{4} \leq \frac{(\gamma+1)^2}{4} \Leftrightarrow \left\| \begin{matrix} x \\ \frac{(\gamma-1)}{2} \end{matrix} \right\| \leq \frac{(\gamma+1)}{2} \tag{3.22}$$

By letting the new set of optimization variables include $\gamma \in \mathbf{R}$ and setting the new objective $f'(\gamma, x) = \gamma$, we observe that the Euclidean norm constraint (3.22) is indeed in the format given by (3.21) and hence a cone constraint.

Semi-Definite Programming. For SDP's, $\mathbf{X} = S^n$, let the space of square symmetric matrices in $\mathbf{R}^{n\times n}$ by

$$\mathbf{S}^n := \{M \in \mathbf{R}^{n\times n} : \quad M = M^T\},$$

and let the objective be given by the (Frobenius) inner-product $\langle C, X\rangle = \operatorname{tr} CX$, $C \in S^n$, the inner product on S^n. The cone is chosen as the set of positive definite symmetric matrices

$$\mathbf{S}^n_+ = \{M \in \mathbf{S}^n : M = M^T,\ x^T M x \geq 0 \text{ for all } x \in \mathbf{R}^n\}$$

which is again self-dual, $S^n_+ = (S^n_+)^*$. Let $\mathbf{Y} = \mathbf{R}^m$, $b \in \mathbf{R}^m$ and $\mathcal{A} : S^n \to \mathbf{R}^m$ is given as

$$\mathcal{A}X = \begin{pmatrix} \operatorname{tr} A_1 X & \cdots & \operatorname{tr} A_m X \end{pmatrix}^T \in \mathbf{R}^m$$

where each matrix $A_i \in S^n$. The dual mapping $\mathcal{A}^* : \mathbf{R}^m \to S^n$ of $\mathcal{A}$ is given by

$$\mathcal{A}^* y = A_1 y_1 + \cdots + A_m y_m$$

Then, a semi-definite optimization primal and dual problems are defined as

$$\begin{array}{llll} \text{(P-SDP)} & \inf\limits_{X \in \mathbf{S}^n_+} \operatorname{tr} CX & \text{(D-SDP)} & \sup\limits_{y \in \mathbf{R}^m, S \in \mathbf{S}^n_+} b^T y \\ & \begin{pmatrix}\operatorname{tr} A_1 X & \cdots & \operatorname{tr} A_m X\end{pmatrix}^T = b & & A_1 y_1 + \cdots + A_m y_m + S = C \end{array} \tag{3.23}$$

In Chapter 6 we will extensively use SDP's for LTI controller design. Another important aspect of SDP's is that they include SOCP's. Suppose $x \in \mathbf{L^n}$, then

$$x_1 \geq \|x_2\| \Leftrightarrow x_1^2 \geq \|x_2\|^2 \Leftrightarrow x_1^2 - x_2^T x_2 \geq 0.$$

Exploiting the Schur complement, (Horn and Johnson, 1999; Zhou *et al.*, 1996) one can directly show that this latter inequality is given in $\mathbf{S}^n_+$ as

$$\begin{pmatrix} x_1 & x_2^T \\ x_2 & x_1 I \end{pmatrix} \succeq 0$$

which shows that any SOCP constraint is also a SDP constraint. In turn, one can embed these in other SDP problems if this provides more freedom in modelling the control problem.

3.6 Stochastic constraint handling

Let us now assume that the controlled variables $\mathbf{z}_i(\xi)$ are subject to bounds that can be modelled as linear inequalities. This gives the ability to capture actuator saturation, model validity ranges, rate constraints, product specification ranges and the ability to leave out undesired operating regions from the feasible set of operation. More concretely, the output $\mathbf{z}(\xi)$ should be contained in the polytope defined by m linear inequality constraints

$$\mathcal{P} = \{\zeta \in \mathbf{R}^{Nn_z} : \; h_j^T \zeta \leq g_j, \text{ for all } j = 1, .., m\}$$

with probability larger than some user-chosen level α. Since the Gaussian disturbances are unbounded there is no guarantee that $\mathbf{z}(\xi) \in \mathcal{P}$ for all $\xi \in \Omega$. Hence, one is forced to take a decision on how the constraints are to be satisfied. Two common probability constraints over the set $\mathcal{P}$ in stochastic programming are

1) **individual chance constraints** in which each individual constraint must be met separately with a user-specified level of certainty α_j

$$P\left\{\xi \in \Omega : h_j^T \mathbf{z}(\xi) \leq g_j\right\} \geq \alpha_j \text{ for all } j = 1, .., m \tag{3.24}$$

2) **joint chance constraints** in which all constraints have to be met simultaneously with a user-specified level of certainty α

$$P\left\{\xi \in \Omega : h_j^T \mathbf{z}(\xi) \leq g_j \text{ for all } j = 1, .., m\right\} \geq \alpha \tag{3.25}$$

Both types of constraints are important constraint models for engineering systems, yet joint chance constraints seem to be more applicable to the model predictive control problem as shown in the next example.

Example 4 *Difference between individual and joint chance constraints.* Consider the following constraint on a stochastic scalar control input $u(\xi)$ of some process system

$$u^- \leq u(\xi) \leq u^+.$$

Let us enforce these constraints in terms of an equivalent deterministic joint change constraint

$$P\left(u^- \leq u(\xi) \leq u^+\right) \geq \alpha$$

such that these constraints can be added to an optimization program and alternatively by the individual chance constraints

$$P\left(u(\xi) \leq u^+\right) \geq \alpha, \quad P\left(u^- \leq u(\xi)\right) \geq \alpha.$$

The control input has a scalar Gaussian distribution function $u(\xi) \sim N(\hat{u}, \sigma)$ and we choose the variance σ^2 and mean $\hat{u}$ such that both individual chance constraints are satisfied. Then, for $\alpha = 0.95$ this implies that the joint chance constraint

$$P\left(u^- \leq u(\xi) \leq u^+\right) \geq 2\alpha - 1$$

gives the certainty level $2\alpha - 1 = 0.9$. Hence, the certainty level of the separate constraints do not give guarantee the simultaneous satisfaction of both constraints. When the number of individual constraints is large and the random vector is of a high dimension (typically the case for process systems), this analysis is very tedious and therefore not very attractive. This phenomenon does not occur when the joint chance constraints are enforced. The following relation between the constraint sets

$$\{\xi \in \Omega \;:\; u^- \leq u(\xi) \leq u^+\} \subset \{\xi \in \Omega \;:\; -\infty < u(\xi) \leq u^+\}$$

implies that

$$P\left(\, u^- \leq u(\xi) \leq u^+\right) \leq P\left(-\infty < u(\xi) \leq u^+\right).$$

and the same holds for the upper bounds. Hence, the simultaneous chance constraints are necessarily satisfied when the simultaneous chance constraint is thereby ensuring a safe operation. □

Joint chance constraints are used throughout this thesis instead of individual ones, but there are in principle no limitations on choosing either of them, or clustering sets of joint chance constraints in parallel with individual chance constraints and so on. Unfortunately, joint chance constraints are hard to take into account in optimization problems, let alone in controller synthesis, where the probability density function depends on the actual choice of controller. Calculation of the probability content of some structured set is usually possible for nice geometric bodies such as half-spaces and ellipsoids, see for instance Ruben (1960). In the analysis part, numerical integration techniques can be deployed to compute the probability mass of a polytope. In such computations it is assumed that the covariance matrix Z and the expected value of the controlled variables $\hat{\mathbf{z}}$ are known. However, because a general closed formula does not exist for the integral of a normal distribution over general polytopes, it is unknown how to proceed in solving the control synthesis problem. In the next two sections, it is revealed how to exploit a relaxation keep the computations efficient and meaningful in controller design.

3.6.1 Enforcing individual chance constraints

Contrary to joint chance constraints, handling individual chance constraints is straightforward due to the scalar nature of the probability density function under consideration. Enforcing such a bound is equivalent to adding a second-order cone constraint on the optimization problem. In Lobo *et al.* (1998) the authors explain how such a translation is performed and this procedure is repeated below for completeness. Individual chance constraints are constraints of the type

$$P\left(\, h_j^T \mathbf{z}(\xi) \leq g_j\right) \geq \alpha$$

which means that the probability that the random variable $\mathbf{z}(\xi)$ takes its value in the negative half-space is larger than α. For sufficiently large α, this constraint is a second-order cone constraint. Let Z be the covariance matrix of $\mathbf{z}(\xi)$

$$Z = E(\mathbf{z}(\xi) - \hat{\mathbf{z}})(\mathbf{z}(\xi) - \hat{\mathbf{z}})^T.$$

and suppose $\alpha \geq \frac{1}{2}$. Define

$$p(\xi) = h_j^T \mathbf{z}(\xi) \in \mathbf{R}$$

which has a scalar normal probability density with average $h_j^T \hat{\mathbf{z}}$ and variance

$$\sigma^2 = E(p(\xi) - \hat{p})(p(\xi) - \hat{p})^T = h_j^T E(\mathbf{z}(\xi) - \hat{\mathbf{z}})(\mathbf{z}(\xi) - \hat{\mathbf{z}})^T h_j = h_j^T Z h_j. \qquad (3.26)$$

To make the back-off computations easier, it is convenient to normalize the stochastic process $p(\xi)$ by introducing

$$p_n(\xi) := \frac{p(\xi) - \hat{p}}{\sigma},$$

then $p_n(\xi)$ is Gaussian with unit variance and zero mean. Then

$$p(\xi) \leq g_j \Leftrightarrow \frac{p(\xi) - \hat{p}}{\sigma} \leq \frac{g_j - \hat{p}}{\sigma} \Leftrightarrow p_n(\xi) \leq \frac{g_j - \hat{p}}{\sigma}$$

Therefore the constraint is replaced by

$$F_G(\frac{g_j - \hat{p}}{\sigma}) \geq \alpha,$$

where F_G is the standard one-dimensional Gaussian distribution function

$$F_G(x) = \frac{1}{2\pi} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt.$$

The constraint is satisfied if

$$\frac{g_j - \hat{p}}{\sigma} \geq F_G^{-1}(\alpha) \quad \text{or equivalently} \quad \hat{p} + F_G^{-1}(\alpha)\sqrt{h_j^T Z h_j} \leq g_j. \qquad (3.27)$$

Equation (3.27) explains the simple idea for keeping back-off to the constraints. For instance, let $\alpha = .99$ then the constraint is satisfied if one takes approximately

$$F_G^{-1}(\alpha) \approx 3$$

back-off to the constraint. The standard deviation of the process follows from equation (3.26) by factorizing the covariance matrix $Z = F_Z F_Z^T$

$$\hat{p} + F_G^{-1}(\alpha)\sqrt{h_j^T Z h_j} = \hat{p} + F_G^{-1}(\alpha)\|h_j^T F_Z\|_2 \leq g_j \qquad (3.28)$$

which is a **linear** constraint on the variable $\hat{p}$ for any **fixed** matrix F_Z. But, if Z is allowed to vary in an optimization problem (which will be the case in the following chapters on closed-loop optimization), the constraint is a convex second order cone constraint for $\alpha \geq \frac{1}{2}$, (that is $F_G^{-1}(\alpha) \geq 0$).

Remark 5 The lower bound $\alpha \geq \frac{1}{2}$ to keep the constraints convex is no limitation in application. Let us allow $\alpha = \frac{1}{2}$, then mathematically the expected value of the stochastic process $p(\xi)$ is allowed to be on the constraints. The half-spaces defined by $h_j^T \mathbf{z}(\xi) \leq g_j$ then divide the Gaussian probability density function in two symmetrical parts (50% certainty). Consequently, no back-off is taken to the constraints as in standard MPC, that is

$$h_j^T \hat{\mathbf{z}} \leq g_j, \quad j = 1, \ldots, m. \tag{3.29}$$

If the bound α is chosen even lower, $\alpha < \frac{1}{2}$, then for fixed standard deviation σ it follows that $F_G^{-1}(\alpha)\sigma < 0$. This leads to a **constraint relaxation**

$$\hat{p} \leq g_j + |F_G^{-1}(\alpha)\sigma|$$

where the expected value is even allowed to lie outside of the original constraint region which is not sensible. □

3.6.2 An ellipsoidal relaxation of joint chance constraints

A similar set up exists in the multidimensional case but, as mentioned earlier, a clever relaxation is used such that we can efficiently deal with several constraints simultaneously. For sound engineering reasons an ellipsoidal technique is introduced to solve the problem at hand. The mathematical advantage in using ellipsoids is the regular shape and its natural occurrence in the shape of high dimensional confidence intervals of the Gaussian probability density function and, as will soon be clear, it makes the discussion of remark 5 obsolete. The probability constraint (3.25) is equivalent to the following integral constraint

$$\frac{1}{\sqrt{(2\pi)^{n_z} \det(Z)}} \int_{\mathcal{P}} e^{-\frac{1}{2}(\zeta - \hat{z})^T Z^{-1}(\zeta - \hat{z})} d\zeta \geq \alpha, \tag{3.30}$$

where $\hat{\mathbf{z}} = E\mathbf{z}(\xi)$ is the mathematical expectation of the performance variable and Z is the covariance matrix of the stochastic process $\mathbf{z}(\xi)$

$$Z = E(\mathbf{z}(\xi) - \hat{\mathbf{z}})(\mathbf{z}(\xi) - \hat{\mathbf{z}})^T,$$

and $\mathcal{P} := \{\zeta \ : \ H^T\zeta \leq g\} \subset \mathbf{R}^m$ the polytope as above. Let us continue from the integral constraint (3.30) and introduce the ellipsoidal relaxation. Using the ellipsoidal approximation technique one proceed as follows, define the ellipsoid

$$\mathcal{E}_r = \{\zeta : \ \zeta^T Z^{-1} \zeta \leq r^2\}$$

with some radius r still to be chosen. If we can make sure that

$$\hat{\mathbf{z}} + \mathcal{E}_r \subset \mathcal{P}, \tag{3.31}$$

we infer that $P(\mathbf{z} \in \mathcal{P}) \geq \alpha$ is implied by

$$I_c = \frac{1}{\sqrt{(2\pi)^{n_z} \det(Z)}} \int_{\hat{\mathbf{z}} + \mathcal{E}_r} e^{-\frac{1}{2}(\zeta - \hat{\mathbf{z}})^T Z^{-1}(\zeta - \hat{\mathbf{z}})} \, d\zeta \geq \alpha.$$

Given $\alpha \in [0,1]$, the smallest r can be computed such that this latter inequality is satisfied as follows. Observe that the integral I_c can be simplified by bringing it to standard form by the substitution

$$v = Z^{-\frac{1}{2}}(\zeta - \hat{\mathbf{z}})$$

such that $\sqrt{\det(Z)}$ is cancelled. This leaves us with the condition

$$\frac{1}{(2\pi)^{\frac{2}{n_z}}} \int_{\|v\|^2 \leq r^2} e^{-\frac{1}{2}\|v\|^2} dv \geq \alpha.$$

As it turns out, this integral can be reduced to a one-dimensional integral by standard techniques, (Papoulis, 1965), leading to the condition

$$I = \frac{1}{2^{n_z/2}\Gamma(n_z/2)} \int_0^{r^2} \chi^{\frac{n_z}{2}-1} e^{-\frac{\chi}{2}} d\chi \geq \alpha.$$

Hence, r is chosen such that $I = \alpha$. Note that for a given probability α, the necessary confidence radius r increases with output dimension n_z. For fixed r it remains to replace the constraint $P(\mathbf{z} \in \mathcal{P}) \geq \alpha$ in our optimization problem by the stronger constraint $\hat{\mathbf{z}} + \mathcal{E}_r \subset \mathcal{P}$.

Ellipsoidal techniques in literature Applications of ellipsoidal approximation techniques to control problems and their stochastic interpretations in the research field on actuator saturation exist in literature, (Boyd and Vandenberghe, 2002; Hindi and Boyd, 1998; Gökcek *et al.*, 2001). However, these results are focussed on either finding the optimal shape of the ellipsoid for a *fixed center*, or finding the optimal center of the ellipsoid for a *fixed shape*. We want to stress here that the simultaneous optimization over the shape as well as the center of the ellipsoid is not a standard problem. The optimal reference trajectory depends on our ability to shape the variance of the performance output vector $\mathbf{z}$.

Practical justification of the ellipsoidal relaxation. One might question the justification of an ellipsoidal relaxation on mathematical grounds only but there are good engineering arguments as well. We have assumed the second-order statistics of the disturbances to be known, but in practise they must be estimated from data. Then, the part of the density function that can be determined with reliability is the central region of the distribution, while the 'tails' of the distribution may be quite inaccurate due to the low frequency of occurrence in measurement data. From a practical perspective, using an ellipsoidal relaxation means that the certainty of constraint satisfaction needs to be ensured by the region of the probability density function centered around the mathematical expectation. Hence, it seems fair to use the first two moments of the density function only and leave all higher order information outside the problem formulation. Future variants of the procedure mentioned here might be to consider robust versions of the chance constraints in which the mean and covariance matrix of the disturbances are allowed to vary in a whole set containing more than just a single point or one could include the problem of identifying the second-order statistics.

3.6.3 Enforcing joint chance constraints

There is a very elegant way to ensure that the ellipsoid $\mathcal{E}$ is contained in the polytope $\mathcal{P}$ by reducing the infinite dimensional constraint (that each point on the border $\partial\mathcal{E}$ of $\mathcal{E}$ is feasible) to a finite number of constraints no larger than the original number of constraints m. The trick is to compute the worst-case vectors in a stochastic sense and then to enforce these worst-case vectors to be feasible with respect to the constraints. The calculation of the back-off to the constraints is computed via the solution of a basic optimization problem. The solution to this specific problem is the major building block for the remainder of this thesis and therefore the solution is given explicitly below.

Theorem 6 Consider the following nonlinear optimization problem consisting of a linear objective and an ellipsoidal feasible set

$$\max_{x\in\mathbf{R}^n} \quad c^T x \quad | \quad x^T X^{-1} x \leq 1, \quad X > 0. \tag{3.32}$$

The solution to this optimization problem is given in closed form by

$$x^\star = \frac{Xc}{\sqrt{c^T Xc}} \text{ and } c^T x^\star = \sqrt{c^T Xc} \tag{3.33}$$

Proof. (Ben-Tal and Nemirovski, 2001). □

A geometric interpretation is given in figure 3.7. The hyperplane defined by $c^T x = \gamma$ is translated over the vector $x^\star \in \partial\mathcal{E} := \{x \;:\; x^T X^{-1} x = 1\}$ such that it coincides with the tangent plane to the ellipsoid $\mathcal{E}$. The optimal objective value $\gamma^\star$ then equals $c^T x^\star$.

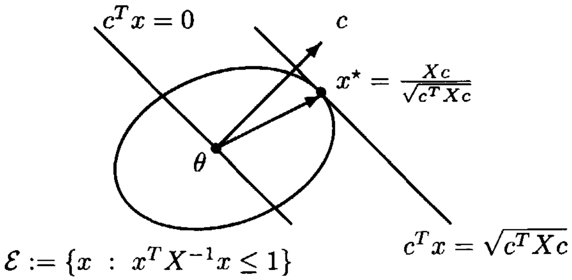


Figure 3.7: Geometric interpretation of Theorem 6.

By making the following substitutions in Theorem 6

$$X = r^2 Z, \; c = h_j, \; x = \delta\mathbf{z},$$

the use in the back-off calculation becomes clear. The worst-case vectors $\delta\mathbf{z}_j^\star$ and minimal amount of back-off $\nu_j^\star$ are given for each separate hyperplane with normal h_j by

$$\delta\mathbf{z}_j^\star = r\frac{Zh_j}{\sqrt{h_j^T Z h_j}} \text{ and } \nu_j^\star = h_j^T \delta\mathbf{z}_j^\star = r\sqrt{h_j^T Z h_j}.$$

Condition (3.31) above is ensured by obeying the following conditions

$$h_j^T \mathbf{z}^\star = h_j^T \hat{\mathbf{z}} + h_j^T \delta\mathbf{z}^\star = r\sqrt{h_j^T Z h_j} + h_j^T \hat{\mathbf{z}} \leq g_j \quad \forall j = 1, \ldots, m.$$

The geometric interpretation of these conditions is that the distance of the center of the ellipsoid $\hat{\mathbf{z}}$ to the j^{th} constraint is larger than the minimal required back-off $\nu_j^\star$, see Figure 3.8 where the situation is sketched for the half-spaces $\mathcal{H}_1, \mathcal{H}_2$. $\delta\mathbf{z}_j^\star$ is the 'worst-case' vector such that $\mathbf{z}^r + \delta\mathbf{z}_j^\star$ is contained in the hyperplane defining the half-space $\mathcal{H}_j$. The important observation here is that condition (3.31) is equivalent to requiring that the ellipsoid lies in the intersection of the half-spaces defined by $\mathcal{H}_j := \{\zeta : h_j^T \zeta \leq g_j\}$.

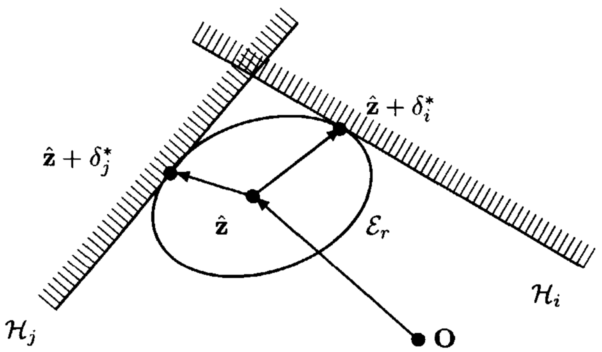


Figure 3.8: Ellipsoid in intersection half-spaces

Example 7 *Circle in box problem.* In this example we will illustrate the relaxation technique by computing the back-off or safety margins to a set of inequality constraints for a stochastic variable with unit variance. Consider the following optimization problem

$$\max_{y \in \mathbf{R}^2} \quad y_1 + y_2 \quad | \quad y + \mathcal{E} \subset \mathcal{P}.$$

where the ellipsoid $\mathcal{E}$ and the box $\mathcal{P}$ are defined as

$$\mathcal{E} := \{x \in \mathbf{R}^2 \ : \ 2\|x\|_2 \leq 1\} \text{ and } \mathcal{P} := \{x \in \mathbf{R}^2 \ : \ \|x\|_\infty \leq 1\}$$

The $\|\cdot\|_\infty$ constraint is reformulated using linear inequality constraints and the columns h_j of the corresponding constraint matrix are given by

$$h_j \in \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}\right\}.$$

The *covariance matrix* of the ellipsoid $X = \frac{1}{4}I$ such that the back-offs follow from

$$\nu_j^\star = \sqrt{\frac{1}{4}\|h_j\|_2^2} = \frac{1}{2}\|h_j\| = \frac{1}{2}$$

for $j = 1,..,4$. The righthand side of the inequality constraints is given by $g_j = 1$ for all j, hence the inequalities with back-off are given by

$$\nu_j^\star + h_j^T y \leq g_j \Leftrightarrow \|\frac{1}{2}h_j\|_2 + h_j^T y \leq g_j$$

Notice in particular the second-order cone structure of the constraint: norm+linear. After reduction with the back-off, the reduced righthand side is obtained as $\tilde{g}_j = \frac{1}{2}$ for all j

$$h_j^T y \leq \tilde{g}_j := g_j - \|\frac{1}{2}h_j\|_2$$

Hence the optimization problem reduces to

$$\max_y \quad y_1 + y_2 \quad | \quad \|y\|_\infty \leq \frac{1}{2}$$

which has the optimal solution $(y^\star)^T = (\frac{1}{2}, \frac{1}{2})$. See figure 3.9 for a sketch of the solution. $\square$

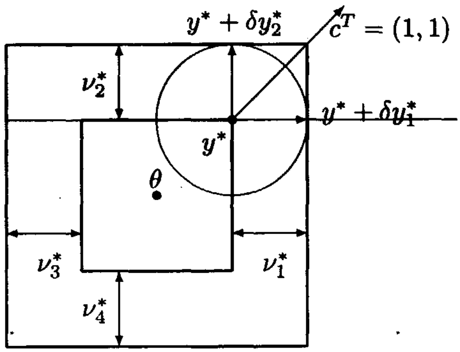


Figure 3.9: Optimal solution for circle in box problem

Remark 8 *Singularity of the variance matrix Z.* In general, it may happen that the variance matrix Z is singular in which case the components of the stochastic vector $\mathbf{z}(\xi)$ are correlated. Below it is shown that this has no consequence for the computation of the back-off. Indeed, suppose that Z is singular, then the description using the inverse of Z is then not applicable, instead, one should factor $Z = F_Z F_Z^T$, (which is well defined since Z is positive semi-definite) and use the description

$$\mathcal{E} := \{\mathbf{z}^r + F_Z\zeta \ : \ \|\zeta\|_2 \leq 1\}.$$

Then, $\mathcal{E}$ is contained in the linear manifold $V := \mathbf{z}^r + \operatorname{im} F_Z$. The factors of the matrix Z are computed via a singular value decomposition

$$Z = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma & O \\ O & O \end{pmatrix} \begin{pmatrix} U_1^T \\ U_2^T \end{pmatrix} = U_1 \Sigma U_1^T.$$

U_1 is used to compute a parameter transformation such that the transformed process has orthogonal elements

$$\mathbf{y}(\xi) := \frac{1}{r}\Sigma^{-\frac{1}{2}} U_1^T \mathbf{z}(\xi), \quad E\mathbf{y}(\xi)\mathbf{y}(\xi)^T = \frac{1}{r^2}\Sigma^{-\frac{1}{2}} U_1^T E\mathbf{z}(\xi)\mathbf{z}(\xi)^T U_1 \Sigma^{-\frac{1}{2}} = \frac{1}{r^2} I > 0.$$

The inequality constraint is given then by

$$\hat{\mathbf{z}} + rU_1\Sigma^{\frac{1}{2}}\mathcal{E}^Y \subset \mathcal{P}, \qquad \text{where} \qquad \mathcal{E}^Y := \{\zeta \ : \ \zeta^T\zeta \le 1\}$$

and the technical requirement amounts to adding the constraints

$$\max_{y\in\mathcal{E}^Y} h_j^T\hat{\mathbf{z}} + rh_j^T U_1\Sigma^{\frac{1}{2}} y \le g_j$$
$$h_j^T\hat{\mathbf{z}} + \max_{y^Ty\le 1} rh_j^T U_1\Sigma^{\frac{1}{2}} y \le g_j.$$

to the optimization problem. The back-off to the constraint then follow from

$$\nu_j = \max_{y^Ty\le 1} rh_j^T U_1 \Sigma^{\frac{1}{2}} y = \sqrt{rh_j^T U_1 \Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}} U_1^T h_j r} = r\sqrt{h_j^T Z h_j}.$$

Hence, ν_j is computed as if Z were nonsingular, therefore it is not necessary to make a distinction between the singular and nonsingular case. □

3.7 Adding bounded disturbances

Another important class of disturbances is the class of bounded disturbances. The interest in dealing with (persistent) bounded disturbances originates from our *signal-based* view on robust control for uncertain nonlinear dynamical systems. Consider again the nonlinear dynamical system

$$0 = f(\dot{\bar{x}}, \bar{x}, \bar{v}, \bar{u}, \bar{d}) \tag{3.34}$$

depending on an unknown scalar parameter $\bar{d} \in [\bar{d}^r - d^*, \bar{d}^r + d^*] \subset \mathbb{R}$ and consider the corresponding reference trajectories satisfying

$$0 = f(\dot{\bar{x}}^r, \bar{x}^r, \bar{v}^r, \bar{u}^r, \bar{d}^r).$$

The sensitivity s is defined as usual as the trajectory that satisfies the linearized dynamics along the reference trajectory

$$0 = \partial_{\dot{\bar{x}}} f|_0\, \dot{s} + \partial_{\bar{x}} f|_0\, s + \partial_{\bar{v}} f|_0\, v + \partial_{\bar{u}} f|_0\, u + \partial_{\bar{d}} f|_0\, d, \quad s(0) = 0$$

Then, the model uncertainty manifests itself, in a first order approximation, as a *persistent* and *bounded* disturbance

$$\bar{x} = \bar{x}^r + \bar{x}^c \simeq \bar{x}^r + x^c, \text{ where } x^c = sd = s(d + \tilde{d}^r)$$

When we sample the solution s at the time instants $t = t_i,\ i = 1, .., N$ we obtain the transfer matrices mapping the bounded disturbances to the state vector

$$G_{xd}^T = \begin{pmatrix} \mathsf{s}(t_1)^T & \cdots & \mathsf{s}(t_n)^T \end{pmatrix}.$$

Bounded disturbances are attractive because it is easier to analyze whether there there exists a control law that keeps the process feasible with respect to the constraints than in the stochastic case. The problem of handling constraints for bounded disturbances is very well defined by requiring that there exist feasible control sequences for all disturbances in the bounded set that keep the process within its constriants. This mathematical convenience is inspiring, yet in real systems one is confronted with both bounded and stochastic disturbances, hence stochastic disturbances cannot be ignored, especially not in inequality constrained systems. Because a systematic approach to constraint handling in the Gaussian stochastic case was given in the previous subsection, the worst-case approach to bounded disturbances can easily be incorporated to handle the combination of both.

Definition 9 *Bounded disturbances.* A bounded disturbance $d \in \mathbf{R}_d^n$ is an affine function of an unknown vector v that takes its value in a convex set V which is generated by finitely many n_v extreme points v_i^*, that is

$$d = d^r + Av \tag{3.35}$$

for some reference vector d^r and matrix A of appropriate dimensions where v is any member of V

$$v \in \operatorname{co}\{v_1^*, \quad \cdots, \quad v_{n_v}^*\} =: V. \tag{3.36}$$

□

Alternatively, it may be convenient to work with the image of V under A instead of the set V itself. It is not difficult to see if that if the (extreme) points d_j^* are defined as $d_j^* := Av_j^*$ and the set $D := co\{d_1^*, \quad \cdots, \quad d_{n_v}^*\}$ then any $v \in V$ defines a bounded disturbance

$$d - d^r = Av = A\sum_{j=1}^{n_v} \lambda_j v_j^* = \sum_{j=1}^{n_v} \lambda_j A v_j^* = \sum_{j=1}^{n_v} \lambda_j d_j^* \in D$$

and conversely for any $d - d^r \in D$ one has

$$d - d^r = \sum_{j=1}^{n_v} \lambda_j d_j^* = \sum_{j=1}^{n_v} \lambda_j A v_j^* = A\sum_{j=1}^{n_v} \lambda_j v_j^* = Av$$

for some $v \in V$. Hence one can directly work with the set $D = A(V)$. Note that the bounded disturbances d have *not* been given a stochastic interpretation by taking a uniform distribution over D. This is undesirable in the definition of the probability constraints which were introduced above. Namely, in absence of stochastic disturbances, we want the probability constraints to drop out of the problem formulation and deal with bounded disturbances in a worst-case setting. Secondly, one may have good modelling reasons to choose a nominal value of the parameter vector unequal to its mathematical expectation $\hat{d}$, for instance $d^r = d_i^*$ for some i.

Handling bounded and stochastic disturbances simultaneously.

The definition of the bounded and stochastic disturbances and the way they are handled separately, do not reveal how the constraints are to be enforced when both type of disturbances are encountered simultaneously. First consider the way in which the disturbances enter the dynamical system

$$\begin{aligned}\mathbf{y}(\xi, d) &= G_{yx}x_0(\xi) + G_{yu}\mathbf{u}(\xi) + G_{yw}\mathbf{w}(\xi) + G_{yd}d \\ \mathbf{z}(\xi, d) &= G_{zx}x_0(\xi) + G_{zu}\mathbf{u}(\xi) + G_{zw}\mathbf{w}(\xi) + G_{zd}d.\end{aligned} \tag{3.37}$$

Then, (at least) two alternatives to constraint handling can be identified

1) the set of disturbance realizations ξ for which the process $\mathbf{z}(\xi, d)$ remains feasible for all parameters $d \in D$ has a probability larger than α

$$P\left(\left\{\xi : H^T\mathbf{z}(\xi, d) \leq g, \forall d \in D\right\}\right) \geq \alpha. \tag{3.38}$$

2) for each $d \in D$, the set of disturbance realizations ξ for which $\mathbf{z}(\xi, d)$ remains feasible has a probability larger than α

$$P\left(\left\{\xi : H^T\mathbf{z}(\xi, d) \leq g\right\}\right) \geq \alpha, \ \forall d \in D \tag{3.39}$$

It is clear that the second option is less conservative than the first one. There is no need to require that there exists a single set of disturbance realizations with sufficient probability mass for all values of the bounded parameter d. Consequently, from an applied perspective, the second option provides the same level of safety to constraint violation with less constraints on the reference trajectory and feedback map to be optimized. Therefore option 2) is preferred for our framework. For this option, the following result has been obtained which reduces the infinite number of constraints in (3.39) to a finite number.

Theorem 10 *Bounded and stochastic disturbances.* Let d be a vector taking values in $D := \text{co}\left\{d_1^*, ..., d_{n_e}^*\right\} \subseteq \mathbf{R}^{n_d}$. Let $\mathbf{z} : D \to R^{n_z}$ be an affine function. Let $H \in \mathbb{R}^{n_z \times n_c}$, $g \in \mathbb{R}^{n_c}$, $Z \in \mathbb{R}^{n_z \times n_z}, Z \geq 0$. Then, the following two constraints are equivalent.

i) $r\sqrt{h_j^T Z h_j} + h_j^T \hat{\mathbf{z}}(d) \leq g_j \ \forall d \in D, j = 1, ..., n_c$

$$ii)\ r\sqrt{h_j^T Z h_j} + h_j^T \hat{\mathbf{z}}(d_i^*) \leq g_j\ \forall i = 1, ..., n_e, j = 1, ..., n_c$$

where h_j is the j^{th} column of H and g_j is the j^{th} element of g.
Proof. $(i) \Rightarrow (ii)$ is immediate. $(ii) \Rightarrow (i)$. Pick any $d \in D$, then $d = \sum_{i=1}^{n_e} \lambda_i d_i^*$ for some $\lambda_i \geq 0$, $\sum_{i=1}^{n_e} \lambda_i = 1$. As $\lambda_i \geq 0$, multiplication of (ii) with λ_i and summation over $i = 1, ..., n_e$ gives

$$\sum_{i=1}^{n_e} \lambda_i r\sqrt{h_j^T Z h_j} + \sum_{i=1}^{n_e} \lambda_i h_j^T \hat{\mathbf{z}}(d_i^*) \leq \sum_{i=1}^{n_e} \lambda_i g_j$$
$$\forall i = 1, ..., n_e, j = 1, ..., n_c$$

As $\sum_{i=1}^{n_e} \lambda_i = 1$ and $\hat{\mathbf{z}}(d)$ is affine we end up with

$$r\sqrt{h_j^T Z h_j} + h_j^T \hat{\mathbf{z}}(\sum_{i=1}^{n_e} \lambda_i d_i^*) \leq g_j\ \forall j = 1, ..., n_c$$

Together with the relation $d = \sum_{i=1}^{n_e} \lambda_i d_i^*$ this implies

$$r\sqrt{h_j^T Z h_j} + h_j^T \hat{\mathbf{z}}(d) \leq g_j\ \forall\ d \in D, j = 1, ..., n_c$$

which completes the proof. $\square$

The important observation from theorem 10 is that the original constraint

$$P\left(\{\mathbf{w} : H^T \mathbf{z}(\mathbf{w}, d) \leq g\}\right) \geq \alpha,\ \forall d \in D$$

is implied by the *finite!* number of constraints

$$r\sqrt{h_j^T Z h_j} + h_j^T \hat{\mathbf{z}}(d_i^*) \leq g_j\ \forall i = 1, ..., n_e, j = 1, ..., n_c$$

which allows us to handle these constraints using a numerical solver.

3.8 Chapter summary

In this chapter the generalized plant framework has been adapted to predictive control including stochastic disturbances and dynamic optimization, which provides a generic set-up for advanced control problems. The main aspects in the generalized plant are the inputs and outputs which relate directly to the problem formulation: performance outputs (or controlled variables) are used for the objective function and the inequality constraints, exogenous disturbances are a robustness measure for the back-off to the constraints and control inputs (or manipulated variables) and measured outputs (or process variables) determine the possibility for feedback control. Future disturbances and measured outputs are explicitly introduced in the dynamic optimization basic to predictive controllers. To deal with inequality constraints in a stochastic process environment, a safety margin to the constraints is required which is often referred to as back-off. Several ways of stochastic constraint handling in *open-loop* were discussed, in particular, to enforce simultaneous chance constraints, an ellipsoidal relaxation technique was shown to give convex second-order cone constraints. It was also shown how the combination of stochastic and bounded disturbances can be handled.

4 Stochastic Closed-loop Model Predictive Control

In this chapter the generalized plant is embedded in a feedforward/feedback control structure. The feedforward and feedback control actions are optimized simultaneously and it is shown that this problem is convex provided that the control problem is transformed by one of two separate techniques. The technical contribution of this chapter is to provide these transformation techniques in relation to the control structure.

4.1 Introduction

In many control strategies, *input* feedforward and *disturbance* feedforward enable high performance in terms of reference tracking and disturbance rejection. Both types of feedforward trajectories can be injected into our generalized plant, because of the entries for reference trajectories on both the control and disturbance input (Figure 3.2). Disturbance feedforward is dictated from outside the system boundary (for instance by a supervisory control system) and is therefore assumed to be given. This leaves the input feedforward to be determined by the control system by means of dynamic optimization. These *reference* trajectories for the manipulated variables fix the reference trajectories for the controlled variables and the measured outputs as well and it seems natural to compare these reference trajectories to the actual output measurements that become available in time.

This brings us to the second task of the control system, namely feedback control which is indispensable to provide the closed-loop system robustness against unpredicted disturbances and model uncertainties. Feedback control is even more necessary in process systems because the inequality constraints need to be satisfied. The key issue is to keep back-off to constraints based on the *closed-loop* worst-case trajectories which are significantly less conservative than the open-loop counterparts as for instance discussed by Campo and Morari (1987). In open-loop dynamic prediction, the uncertainty with respect to the system evolution tends to grow, which yields large or even unbounded trajectory envelopes that fail to be feasible from a certain time onwards. It is clear that if the full envelope of possible future trajectories must be kept in the feasible region, the room for optimization is small, and therefore this approach is not very attractive. To keep this envelope as small as possible, we should take into account that in the future measurements will become available for corrective action. In this chapter, we will show how to compute the best linear time-varying controller minimizing these safety margins. In the case of Gaussian disturbances, the proper interpretation of worst-case disturbances is again related to the confidence ellipsoids that are shaped by feedback control. The problem of finding the optimal feedforward and feedback control action simultaneously will be called closed-loop model predictive control (CLMPC). The *analysis* problem for a fixed controller is relatively simple, because it is an open-loop dynamic optimization problem, but it does not reveal how to do actual *synthesis* of controllers. How to solve this synthesis problem for its global optimal solution is revealed in this chapter. Let us first explore why the use of feedback control to minimize back-off is economically an appealing idea.

Example 11 *The benefit of control for inequality constrained process systems.* Consider a discrete linear time invariant stochastic system

$$\begin{pmatrix} x_{k+1}(\xi) \\ z_k(\xi) \\ y_k(\xi) \end{pmatrix} = \begin{pmatrix} A & B^w & B \\ C^z & O & D^z \\ C & D^w & O \end{pmatrix} \begin{pmatrix} x_k(\xi) \\ w_k(\xi) \\ u_k(\xi) \end{pmatrix}$$

where $x(\xi), y(\xi), u(\xi), w(\xi)$ ar the states, outputs, inputs and zero-mean white noise sequence with covariance matrix $W = Ew_k(\xi)w_k(\xi)^T > 0$. Then, the stationary open-loop covariance matrices

$$P := \lim_{k\to\infty} Ex_kx_k^T, \quad Z := \lim_{k\to\infty} Ez_kz_k^T$$

are computed via the Lyapunov equation

$$P = APA^T + B^wWB^{wT}, \quad Z = CPC^T$$

which easily leads to unbounded solutions if the unstable part of A is controllable from the disturbance input w. In that case, there is not much hope that the inequality constraints will be satisfied with success. If the system is pre-compensated using static output feedback

$$u_k = Ny_k, \quad N \in \mathbf{R}^{n_y \times n_u}$$

we have the following relations for the variance matrix of the performance outputs

$$Z = (C^z + D^zNC)P(C^z + D^zNC)^T + (D^zND^w)W(D^zND^w)^T \tag{4.1}$$

where P solves the matrix equation

$$P = (A + BNC)P(A + BNC)^T + (B^w + BND^w)W(B^w + BND^w)^T. \tag{4.2}$$

Because the variance matrix Z is a function of the feedback control law, the feedback gain N can be used to shape this covariance matrix such that the constraints are possibly easier to satisfy and that the constraints can be approached more closely without increased risk of violation. Low variance normal to a constraint means that the constraint can be approached more closely and more profit can be obtained from the plant. How much a controller may increase the profit can locally be estimated from a very simple dual calculation. Suppose one fixes a controller N in the problem above. For this specific controller the variance matrix $Z(N)$ is fixed and hence so are the back-offs to the constraints

$$\nu_j(N) = r\sqrt{h_j^T Z(N) h_j}.$$

Then, a linear program in which we optimize some linear objective function (for instance: maximal feed, minimal energy consumption) while keeping back-off to the constraints

$$\min_{\nu_j + h_j^T x \le g_j,\ j=1,\dots,m} \quad c^T x$$

has the associated dual linear program

$$\max_{\substack{\sum_{j=1}^m \lambda_j h_j = c \\ \lambda_j \ge 0}} \quad \sum_{j=1}^m \lambda_j (\nu_j - g_j)^T.$$

At the optimal solutions $(x^\star, \lambda^\star)$ of the primal and dual problems respectively, one can estimate how much reduced variance will increase the economic profit. From sensitivity theory it follows that under a regularity condition (Luenberger, 1973) a small reduction in variance $\Delta\nu \le 0$ will increase the profit to the level of

$$c^T x^\star - \lambda^\star \Delta\nu.$$

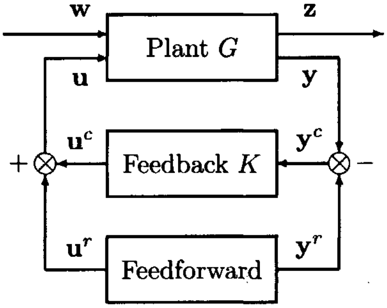
This shows in which direction the profit increase is relatively high by investigating the dual solution and this brings the question of how N can be chosen to minimize the variance in those directions. □

4.2 The closed-loop model predictive control problem

In this section the closed-loop MPC problem is formulated. The general idea is to implement a controller on the generalized plant as shown schematically in figure 4.1. This control architecture and its motivation is discussed in the tutorial overview by Athans (1971) in relation to LQG control design. The structure itself is still very useful if the LQG controller is replaced by a state estimator and a model predictive controller. In process control community it is sometimes referred to as delta-mode control emphasizing that it compensates deviations from optimized reference trajectories by additive feedback. This feedback action is then determined by minimizing the quadratic open-loop objective function

$$\sum_j (\hat{\mathbf{z}} - \mathbf{z}^r)^T Q (\hat{\mathbf{z}} - \mathbf{z}^r).$$

The set-up that is followed here parallels this configuration in its feedforward and additive feedback structure, but the technical form of the solution is completely different. Contrary to the standard interpretation, we are optimizing the feedforward reference signals (within the possibilities of feedback control) and hence not fixing them a priori, and contrary to the MPC literature we use a closed-loop solution to the error minimization.



$\mathbf{w}$: *disturbance inputs*
$\mathbf{z}$: *performance outputs*
$\mathbf{u}$: *control inputs*
$\mathbf{y}$: *measured outputs*
$(\cdot)^c$: *control error*
$(\cdot)^r$: *reference value*

Figure 4.1: The generalized plant for predictive control

4.2.1 Introducing the controller

Recall the generalized plant input-output structure with explicit entries for the reference signals presented in figure 4.2 The idea is to use the reference signal $\mathbf{u}^r$ as a feedforward signal to react to influences that relocate the economic optimum of the plant, typically changes in the references for the disturbances, such as upstream feed changes, scheduled grade changes, but also changes in the definition of the constraints and economic objectives etc. Feedback on the other hand is used to keep the system feasible with respect to the constraints under stochastic disturbances. For any stochastic process (signal) $\mathbf{s}(\xi)$, the control error (subscript c) is defined as

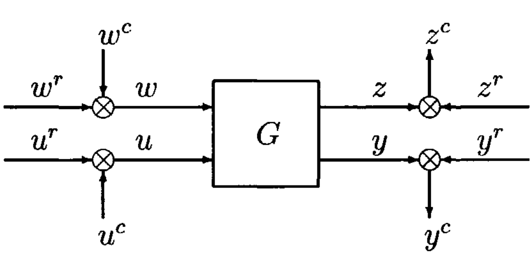


Figure 4.2: Generalized plant with reference signals.

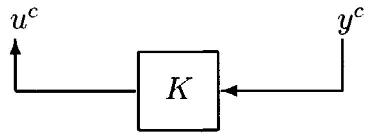


Figure 4.3: The feedback map added to the plant.

the difference between the actual process and its reference value

$$\mathbf{s}^c(\xi) := \mathbf{s}(\xi) - \mathbf{s}^r(\xi).$$

The control error for the manipulated variables and the measured outputs are defined as

$$\mathbf{u}^c(\xi) := \mathbf{u}(\xi) - \mathbf{u}^r, \quad \mathbf{y}^c(\xi) := \mathbf{y}(\xi) - \mathbf{y}^r$$

and the control input signal is computed via control law

$$\mathbf{u}^c(\xi) = K\mathbf{y}^c(\xi).$$

Note that at this point, there is no knowledge on the actual initial condition since no measurements have been processed yet. In the linear case this means that the estimate of the initial condition is the origin, while in the nonlinear case, the best estimate of the initial condition equals the reference value. Also note that as a consequence, the controller to be chosen does not depend on the initial condition itself. How to proceed once measurements become available is subject of Chapter 7. Schematically, this corresponds to adding the block depicted in figure 4.3, to the plant set-up in figure 4.2. For any outcome ξ_0, the realizations of the input process $\mathbf{u}(\xi_0)$ and output process $\mathbf{y}(\xi_0)$ are vectors in $\mathbf{R}^{nn_u}$ and $\mathbf{R}^{nn_y}$ respectively where n is the length of the horizon. The controller $K \in \mathbf{R}^{nn_u \times nn_y}$ is then a lower block triangular matrix

$$K := \begin{pmatrix} K^{11} & O & \cdots & O \\ K^{21} & K^{22} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ K^{n1} & K^{n2} & \cdots & K^{nn} \end{pmatrix}$$

where each block $K_{ij} \in \mathbf{R}^{n_u \times n_y}$ is a matrix of input-output dimension. Define the set of all admissible controllers as

$$\mathbf{K}_k = \{ \sum_{i=1}^{(n-k)} \sum_{j=1}^{i} E_i K^{ij} E_j^T \; : \; K^{ij} \in \mathbb{R}^{n_u \times n_y} \}$$

where each $E_i = (O, \ldots, O, I, O, \ldots, O)^T$ is a matrix vector with an identity matrix on the i^{th} spot. Then, the controller constraint is formalized by adding

$$K \in \mathbf{K}_0 \tag{4.3}$$

to the optimization problem. This feedback map has the algebraic properties of a static output feedback (as discussed in example 11), but it represents a linear time-varying system. Notice that no internal structure besides the lower block triangularity is enforced; K may be any time-varying system. For control problems with inequality constraints it is logical that the controller K is a time-varying system because the active set of constraints varies with time in the prediction horizon along specific feedforward trajectories. The structural constraint that K must be lower block triangular follows from the requirement that the feedback law must be non-anticipative to be realizable.

Definition 12 *Non-anticipating, (Water and Willems, 1981).* A control law $K : Y \to U$ is non-anticipating if for all $\mathbf{y}_1, \mathbf{y}_2 \in Y$ satisfying $\mathbf{y}_1(k) = \mathbf{y}_2(k),\ k \leq n$ implies $(K\mathbf{y}_1)(k) = (K\mathbf{y}_2)(k)$ for all $k \leq n$. □

It is immediate that $K \in \mathbf{K}_0$ is non-anticipating because of its triangular structure. At any future point t_i in time, the controller is allowed to use measurements up to time t_i to compute the control values $\mathbf{u}(t_i)$.

4.2.2 The closed-loop plant-controller interconnection

With this definition of the control law and the set of admissible controllers we are now ready to connect the controller to the generalized plant. Recall that the open-loop dynamics are given by the equations

$$\mathbf{y}(\xi) = G_{yx}x_0(\xi) + G_{yu}\mathbf{u}(\xi) + G_{yw}\mathbf{w}(\xi) \tag{4.4}$$
$$\mathbf{z}(\xi) = G_{zx}x_0(\xi) + G_{zu}\mathbf{u}(\xi) + G_{zw}\mathbf{w}(\xi) \tag{4.5}$$

and that alongside the actual stochastic processes, the deterministic reference trajectories were defined by

$$\mathbf{y}^r = G_{yx}x_0^r + G_{yu}\mathbf{u}^r + G_{yw}\mathbf{w}^r \tag{4.6}$$
$$\mathbf{z}^r = G_{zx}x_0^r + G_{zu}\mathbf{u}^r + G_{zw}\mathbf{w}^r. \tag{4.7}$$

The system matrices lower block triangular by causality (just as the controller is) and therefore the matrices $G_{yu}, G_{zu}, G_{yw}, G_{zw}$ are all lower block triangular matrices in $\mathbf{R}^{nn_y \times nn_u}$, $\mathbf{R}^{nn_z \times nn_u}$, $\mathbf{R}^{nn_y \times nn_w}$, $\mathbf{R}^{nn_z \times nn_w}$ respectively. Subtraction of the stochastic processes and the deterministic reference signals give, by linearity, the expression for the control errors

$$\mathbf{y}^c(\xi) = G_{yx}x_0^c(\xi) + G_{yu}\mathbf{u}^c(\xi) + G_{yw}\mathbf{w}^c(\xi) \tag{4.8}$$
$$\mathbf{z}^c(\xi) = G_{zx}x_0^c(\xi) + G_{zu}\mathbf{u}^c(\xi) + G_{zw}\mathbf{w}^c(\xi). \tag{4.9}$$

Then, introduce the feedback law as defined in the previous subsection

$$\mathbf{u}^c(\xi) = K\mathbf{y}^c(\xi) \tag{4.10}$$

to obtain for the tracking error in the future (measured) output

$$\begin{aligned} \mathbf{y}^c(\xi) &= G_{yx}x_0^c(\xi) + G_{yu}K\mathbf{y}^c(\xi) + G_{yw}\mathbf{w}^c(\xi) \\ (I - G_{yu}K)\mathbf{y}^c(\xi) &= G_{yx}x_0^c(\xi) + G_{yw}\mathbf{w}^c(\xi). \end{aligned} \tag{4.11}$$

For now, $(I - G_{yu}K)$ is assumed to be invertible (which is a well-posedness condition on the closed-loop preventing algebraic loops) such that

$$\mathbf{y}^c(\xi) = (I - G_{yu}K)^{-1}G_{yx}x_0^c(\xi) + (I - G_{yu}K)^{-1}G_{yw}\mathbf{w}^c(\xi). \tag{4.12}$$

Consequently, substitution of (4.12) into the control law (4.10) leads to the perturbation in the input

$$\mathbf{u}^c(\xi) = K(I - G_{yu}K)^{-1}G_{yx}x_0^c(\xi) + K(I - G_{yu}K)^{-1}G_{yw}\mathbf{w}^c(\xi).$$

This then reveals that the measured and performance outputs obey the following closed-loop system equations

$$\begin{aligned} \mathbf{y}^c(\xi) &= (G_{yx} + G_{yu}K(I - G_{yu}K)^{-1}G_{yx})x_0^c(\xi) + \\ &\quad + (G_{yw} + G_{yu}K(I - G_{yu}K)^{-1}G_{yw})\mathbf{w}^c(\xi) \\ \mathbf{z}^c(\xi) &= (G_{zx} + G_{zu}K(I - G_{yu}K)^{-1}G_{yx})x_0^c(\xi) + \\ &\quad + (G_{zw} + G_{zu}K(I - G_{yu}K)^{-1}G_{yw})\mathbf{w}^c(\xi) \end{aligned}$$

which are the standard linear fractional transformations. The problem amounts to finding good values for this feedback K to reduce back-off to constraints that are economically interesting as in the introductory example 11 in the beginning of this chapter.. The terms

$$K(I - G_{yu}K)^{-1}$$

are typical in (complementary) sensitivity functions. This reveals how the closed-loop sensitivity of the system to disturbances $\mathbf{w}^c(\xi)$ and uncertain initial conditions $x_0^c(\xi)$ can be shaped to our improve the stochastic response processes. As a technical difficulty, we must deal with the nonlinearity in how the sensitivity functions depend on the controller parameters as in any feedback design method. In frequency domain design techniques one aims at relating the desired properties of the sensitivity functions

$$G_{yu}K(I - G_{yu}K)^{-1} \quad \text{and} \quad (I - G_{yu}K)^{-1}$$

to the open-loop interconnection between the plant and the controller

$$G_{yu}K$$

by means of graphical representations (Bode, Nyquist, Nichols). It seems to be difficult to extend such rules to deduce closed-loop properties of lifted systems. Moreover, an algorithm is needed to find globally optimal controller, such that engineering insight on specific applications is not of any help in finding a the numerical value of the solution.

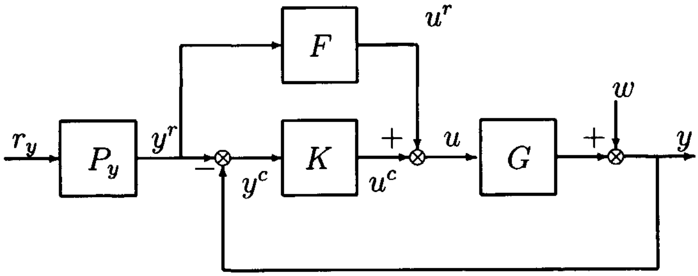


Figure 4.4: Classical 2 degrees-of-freedom feedback/feedforward design. G: open-loop plant, F: feedforward filter K: controller, P_y: pre-filter.

4.2.3 Feedforward and feedback in predictive control

The optimal controller is determined by the economy of the plant via the feedforward trajectory and it turns out that the feedforward/feedback structure we have adopted has a very favorable property. Simultaneous optimization of the feedforward signal and the feedback controller generally leads to bilinear products of the optimization parameters involved. As it turns out, these bilinear terms are eliminated by exploiting the control structure of figure 4.1. Then, this leads to a convex parameterization of the controls without any need for approximation. This is considered to be a powerful aspect and an important cornerstone in our approach. In classical feedforward design, a dynamical system F is sought that filters the reference signal and adds this to the plant input, (figure 4.4). A typical control objective is to minimize the control error $y^c(\xi)$ in the measured output, which is given in terms of transfer matrices by

$$y^c = y^r - y = (I + GK)^{-1}(I - GF)y^r - (I + GK)^{-1}w$$

where for historical reasons a different sign convention is used for the tracking error. If the feedforward filter F is chosen as a stable approximate inverse of the plant dynamics G, for instance via solving a model matching problem

$$\min_{F \in \mathcal{RH}_\infty} \quad \|I - GF\|_\infty.$$

Full decoupling of disturbance rejection and reference tracking is achieved if a stable proper inverse of G exists in which case the model matching error is zero. For strictly proper physical systems this leads to unacceptable high gains in the high frequency range and for systems with unstable zeros this leads to unstable inverses. In process control practice, there is an alternative to find the input u^r matching the output y^r because contrary to high bandwidth control systems, there is usually sufficient time to select a reference trajectory for the measured output from the image of G by means of dynamic optimization. That is, one optimizes u^r and sets $y^r = Gu^r$ instead of filtering y^r with approximate inverse plant dynamics. In that case, direct optimization over all possible reference trajectories for the input u^r is possible as the plant dynamics are not inverted. Because $y^r = Gu^r$ it follows that

$$y^c = (I + GK)^{-1}y^r - (I + GK)^{-1}Gu^r - (I + GK)^{-1}w = -(I + GK)^{-1}w.$$

The schematic picture that corresponds to this situation is depicted in figure 4.5 and is equivalent to the variational control configuration of figure 4.1!

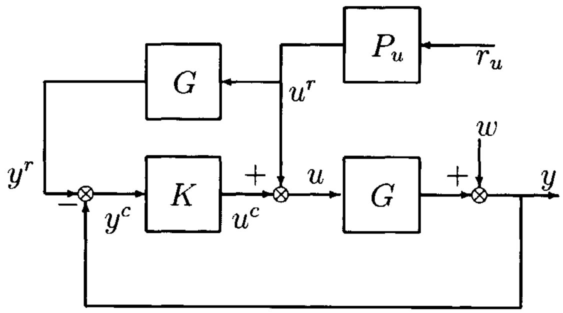


Figure 4.5: Predictive 2 degrees-of-freedom feedback/feedforward design. G: open-loop plant, K: controller, P_u: pre-filter (dynamic optimization).

Thus the variational control scheme decouples reference tracking from disturbance rejection and thereby removes the bilinearity. As will soon be clear, this is one of the crucial properties that allows us to solve for a reference trajectory u^r and the controller K in a single step.

4.2.4 The closed-loop model predictive control problem

All necessary ingredients to formulate the closed-loop MPC problem now available and briefly summarized below to give a compact problem formulation. Let $\mathbf{K}_0$ be the admissible set of control laws such that

$$\mathbf{K}_0 = \{\sum_{i=1}^{n}\sum_{j=1}^{i} E_i K^{ij} E_j^T \; : \; K^{ij} \in \mathbb{R}^{n_u \times n_y}\}$$

where the subscript 0 refers to the initial solution at time zero. Let $\mathcal{U} := \mathbf{R}^{Nn_u}$ is the admissible set of reference control signals for the dynamical system given by

$$\begin{aligned}\mathbf{y}(\xi) &= G_{yx}x_0(\xi) + G_{yu}\mathbf{u}(\xi) + G_{yw}\mathbf{w}(\xi)\\ \mathbf{z}(\xi) &= G_{zx}x_0(\xi) + G_{zu}\mathbf{u}(\xi) + G_{zw}\mathbf{w}(\xi)\end{aligned}$$

where $\xi \in \Omega$ is the stochastic element with Gaussian distribution. The covariance matrix of the uncertain initial condition and disturbances are

$$P = E(x_0(\xi) - x_0^r)(x_0(\xi) - x_0^r)^T = F_P F_P^T,\; W := E(\mathbf{w}(\xi) - \mathbf{w}^r)(\mathbf{w}(\xi) - \mathbf{w}^r)^T = F_W F_W^T$$

which are assumed to be known and fixed a priori. Alongside the stochastic system the reference system is defined as

$$\begin{aligned}\mathbf{y}^r &= G_{yx}x_0^r + G_{yu}\mathbf{u}^r + G_{yw}\mathbf{w}^r\\ \mathbf{z}^r &= G_{zx}x_0^r + G_{zu}\mathbf{u}^r + G_{zw}\mathbf{w}^r\end{aligned}$$

The inequality constraints on the process variables are given by the polytope $\mathcal{P}$ defined as

$$\mathcal{P} := \{\zeta \ : \ H^T\zeta \leq g\} = \{\zeta \ : \ h_j^T\zeta \leq g_j \text{ for } i = 1, \ldots, m\}.$$

and let for any $K \in \mathbf{K}_0$ a feedback law be defined and implemented as

$$\mathbf{u}(\xi) = \mathbf{u}^r + K(\mathbf{y}(\xi) - \mathbf{y}^r).$$

Let the performance output $\mathbf{z}(\xi)$ have the expected value and covariance matrix

$$E\mathbf{z}(\xi) = \mathbf{z}^r, \quad \text{such that} \quad Z = E(\mathbf{z}(\xi) - \mathbf{z}^r)(\mathbf{z}(\xi) - \mathbf{z}^r)^T$$

where Z is factored as in (4.14). Define for notational convenience the closed-loop transfer matrices

$$\begin{aligned} G_{zx}^K &:= G_{zx} + G_{zu}K(I - G_{yu}K)^{-1}G_{yx} \\ G_{zw}^K &:= G_{zw} + G_{zu}K(I - G_{yu}K)^{-1}G_{yw} \end{aligned}$$

then the covariance matrix Z is expressed in terms of the covariance matrices of the disturbances $\mathbf{w}(\xi)$ and initial conditions $x_0(\xi)$ as well as the controller parameters via

$$\begin{aligned} Z(K) &= \begin{pmatrix} G_{zx}^K & G_{zw}^K \end{pmatrix} E \begin{pmatrix} x_0^c \\ \mathbf{w}^c \end{pmatrix} \begin{pmatrix} x_0^c \\ \mathbf{w}^c \end{pmatrix}^T \begin{pmatrix} G_{zx}^K & G_{zw}^K \end{pmatrix}^T \\ &= \begin{pmatrix} G_{zx}^K & G_{zw}^K \end{pmatrix} \begin{pmatrix} P_0 & O \\ O & W \end{pmatrix} \begin{pmatrix} G_{zx}^K & G_{zw}^K \end{pmatrix}^T \end{aligned} \tag{4.13}$$

The back-off or safety margin to the constraints is given by

$$\nu_j(K) = r\sqrt{h_j^T Z(K) h_j}$$

where r is related to the confidence ellipsoids, see Section 3.6.3. This is the square root of a nonlinear function of K which reveals the problem of optimizing the back-off directly. Fortunately, the covariance matrix $Z(K)$ is easily factored and therefore, the square-root nonlinearity can immediately be removed as follows

$$\begin{aligned} \nu_j(K) &= r\sqrt{h_j^T Z(K) h_j} \\ &= r\sqrt{h_j^T \begin{pmatrix} G_{zx}^K F_P & G_{zw}^K F_W \end{pmatrix} \begin{pmatrix} G_{zx}^K F_P & G_{zw}^K F_W \end{pmatrix}^T h_j} \\ &= r\left\| \begin{pmatrix} G_{zx}^K F_P & G_{zw}^K F_W \end{pmatrix}^T h_j \right\|_2. \end{aligned} \tag{4.14}$$

This reveals that the back-off term for each constraint is obtained as the 2-norm of a vector and it remains to remove the non-convex way the controller parameters enter this vector via G_{zk}^K. How to actually do so is part of the solution to the closed-loop MPC problem defined below.

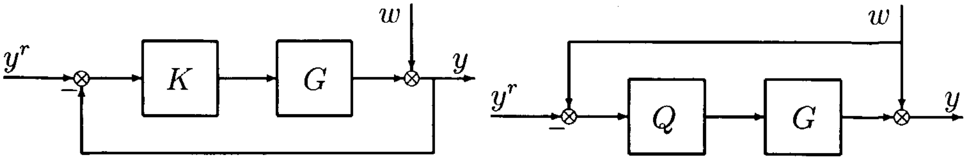


Figure 4.6: Direct design in K vs. analytical design in Q.

Definition 13 *Closed-loop stochastic model predictive control.* With everything as above, let f be a convex function. Then, the closed-loop model MPC problem is defined as

$$\textbf{(CLMPC)} \qquad \min_{\mathbf{u}^r \in \mathbf{R}^{nn_u}, \nu \in \mathbf{R}^m, K \in \mathbf{K}_0} \quad f(\mathbf{z}^r)$$
$$\mathbf{z}^r = G_{zx}x_0^r + G_{zu}\mathbf{u}^r + G_{zw}\mathbf{w}^r \tag{4.15}$$
$$r\| \begin{pmatrix} G_{zx}^K F_P & G_{zw}^K F_W \end{pmatrix}^T h_j \|_2 + h_j^T \mathbf{z}^r \le g_j$$

□

4.3 A Q-parameterization approach

The closed-loop MPC problem can be rendered convex by using a parameter transformation to remove the main nonlinearity in the back-off function. As discussed above, the main issue is to parameterize the term

$$K(I - G_{yu}K)^{-1} \tag{4.16}$$

in a convex way, such that an algorithm can find the global optimal controller parameters. The way to do so is to introduce the so-called Youla parameter Q which renders the closed-loop system affine in these transformed controller parameter. This direct approach is known by several names such as the Q-parameterization in control literature, (Zames, 1981), Internal Model Control (IMC) in process control literature, (Garcia and Morari, 1982), but it can be traced back to early analytical feedback design methods, (Newton *et al.*, 1957).

4.3.1 A Q-parameterization or Internal Model Control solution

The basic idea in the Q-parameterization is the following. Consider a basic single-input single-output classical control set-up shown in 4.6. The transfer functions from the reference signal y^r and the disturbance input w to the measured output $y(\xi)$ are given by

$$y(\xi) = (I + GK)^{-1}GKy^r + (I - (I + GK)^{-1}GK)w(\xi) \tag{4.17}$$

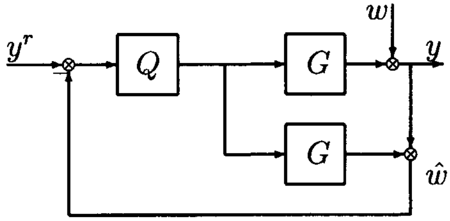


Figure 4.7: Internal Model Control.

where indeed the sensitivity and the complementary sensitivity functions

$$S = (I + GK)^{-1}, \quad T = (I + GK)^{-1}GK$$

depend nonlinearly on the controller parameters in K. The simple trick to get a convenient parameterization of these functions is to exploit the parameter transformation

$$Q = K(I + GK)^{-1} \quad \text{and conversely} \quad K = Q(I - GQ)^{-1}$$

to renders the closed-loop system affine in the design parameter Q. Then by substitution of Q for K, the closed-loop transfer functions are given by

$$y(\xi) = GQy^r + (I - GQ)w(\xi) \tag{4.18}$$

leading to the block-diagram given in figure 4.6. If the original plant G is stable, then searching the controller Q over the set of stable dynamical systems guarantees the closed-loop to be stable since then GQ is stable. Due to the specific use of feedforward in the control set-up as discussed in subsection 4.2.3, the tracking problem and the disturbance rejection problem are separated into a feedforward and feedback problem. Then, the tracking term in the closed-loop systems (4.17), (4.18) disappears and the Youla parameter Q is chosen only to reject the effect of the disturbance w.

In process control, the analytical design method is known as internal model control, (Garcia and Morari, 1982), where it is used to systematically tune model predictive controllers for open-loop stable systems. The name follows from the observation that the disturbance w in the scheme of Figure 4.7 is unknown but can be reconstructed by substraction of the predicted output Gu from the measured output y, ($w = y - Gu$). The resulting block diagram then contains the model of the plant in parallel to the plant itself.

4.3.2 The Q-parameterization in LTV predictive control

It is not difficult to see how application of the Q-parameterization to the closed-loop prediction problem leads to a convex optimization problem. In fact, the only interpretation of the Youla parameter that is needed to arrive at the desired results is that it is a transformation rendering the closed-loop system affine in the controller parameters.

Definition 14 *The Youla parameter Q.* Fix any non-anticipative controller $K \in \mathbf{K}_0$, and let $G_{yu} \in \mathbf{R}^{nn_y \times nn_u}$ be the causal system matrix mapping the control inputs to the measured outputs. The Youla parameter $Q \in \mathbf{R}^{nn_u \times nn_y}$ is defined as

$$Q := K(I - G_{yu}K)^{-1}. \tag{4.19}$$

□

In this definition, Q is simply a matrix with the property that it is lower-block triangular. This follows from the fact that I, G_{yu} and K are all lower block triangular and products and inverses of lower-block triangular matrices are again lower block triangular. A key question is whether the above equation can be put in explicit form such that a one-to-one mapping is obtained. Two structural requirements on the controller that have to be met is that the K is non-anticipative and that the closed-loop is well-posed. Enforcing the causality constraint on K is no technical difficulty as $K := (I + QG_{yu})^{-1}Q$ is a lower block triangular matrix if and only if $Q := K(I - G_{yu}K)^{-1}$ is lower block triangular matrix. Then, causality of K is obtained by enforcing all upper diagonal blocks of Q to be zero

$$E_k^K Q E_l \equiv 0 \text{ for all } l \geq k+1 \geq 1.$$

Secondly, the requirement that $I + QG_{yu}$ is non-singular a well-posedness condition for the lifted system that prevents the closed-loop system from being ill-posed due to algebraic loops. It is immediate that $I + QG_{yu}$ is non-singular if and only if all diagonal blocks

$$E_i^T(I - QG_{yu})E_i$$

are non-singular, which follows from the fact that both the Youla parameter Q and the plant G are lower-block triangular. From an optimization perspective, this constrained can be enforced by requiring that

$$\det(E_i^T(I - QG_{yu})E_i) \neq 0, \quad \text{for all } i,$$

but this is a non-convex constraint on the optimization problem destroying all efforts to render the CLMPC problem convex. An alternative solution may be to enforce the constraint

$$E_i^T QG_{yu}E_i = 0, \quad \text{for all } i,$$

leading to equality constraints on the optimization problem. In our set-up these problems are circumvented by assuming that the plant has no feedthrough from the inputs to the measured outputs.

4.3.3 A snapshot solution to the CLMPC problem

With the preparations made in the previous section, the CLMPC problem can now be solved by rendering the optimization problem convex, which allows us to globally solve the problem efficiently in polynomial time using modern optimization algorithms. The main result on the Q-parameterization is given in the following theorem.

Theorem 15 The closed-loop model predictive control problem is rendered convex via the Q-parameterization.

Proof. The only part of CLMPC that is not convex is the way the covariance matrix Z depends on the controller parameters. Recall that the back-off term is given by

$$\nu_j(K) = r\| \begin{pmatrix} G^K_{zx}F_P & G^K_{zw}F_W \end{pmatrix}^T h_j\|_2.$$

With the aid of the Youla parameter

$$Q = K(I - G_{yu}K)^{-1}$$

one obtains the affine parameterization of the closed-loop system

$$\begin{aligned} G^K_{zx} &:= G_{zx} + G_{zu}K(I - G_{yu}K)^{-1}G_{yx} = G_{zx} + G_{zu}QG_{yx} := G^Q_{zx} \\ G^K_{zw} &:= G_{zw} + G_{zu}K(I - G_{yu}K)^{-1}G_{yw} = G_{zw} + G_{zu}QG_{yw} := G^Q_{zw}. \end{aligned}$$

Then, the equations describing the closed-loop

$$\begin{aligned} \mathbf{y}^c(\xi) &= (G_{yx} + G_{yu}K(I - G_{yu}K)^{-1}G_{yx})x^c_0(\xi) + \\ &+ (G_{yw} + G_{yu}K(I - G_{yu}K)^{-1}G_{yw})\mathbf{w}^c(\xi) \\ \mathbf{z}^c(\xi) &= (G_{zx} + G_{zu}K(I - G_{yu}K)^{-1}G_{yx})x^c_0(\xi) + \\ &+ (G_{zw} + G_{zu}K(I - G_{yu}K)^{-1}G_{yw})\mathbf{w}^c(\xi) \end{aligned}$$

are returned in terms of Q as

$$\mathbf{y}^c(\xi) = (G_{yx} + G_{yu}QG_{yx})x^c_0(\xi) + (G_{yw} + G_{yu}QG_{yw})\mathbf{w}^c(\xi) \tag{4.20}$$

$$\mathbf{z}^c(\xi) = (G_{zx} + G_{zu}QG_{yx})x^c_0(\xi) + (G_{zw} + G_{zu}QG_{yw})\mathbf{w}^c(\xi) \tag{4.21}$$

This renders the back-off terms convex and reveals that CLMPC is equivalent to the following optimization problem

$$\begin{aligned} &\min_{\mathbf{u}^r \in \mathbf{R}^{n n_u}, \nu \in \mathbf{R}^m, Q \in \mathbf{K}_0} \quad f(\mathbf{z}^r) \\ &\nu_j(Q) + h_j^T \mathbf{z}^r \leq g_j, \quad j = 1, \ldots, m \\ &\mathbf{z}^r = G_{zx}x^r_0 + G_{zu}\mathbf{u}^r + G_{zw}\mathbf{w}^r \\ \nu_j(Q) &= r\|h_j^T(G_{zx} + G_{zu}QG_{yx})F_P + h_j^T(G_{zw} + G_{zu}QG_{yw})F_W\|_2 \end{aligned} \tag{4.22}$$

where the norm-constraints are second-order cone constraints, (Lobo *et al.*, 1998; Nesterov and Todd, 1998), hence, both the objective and the constraints are all convex. □

4.3.4 Adding bounded disturbances

An advantage as well as a disadvantage of the Q-parameterization approach is that no internal structure is assumed or enforced, in the sense that it is observer based.

The unknown structure is a disadvantage if the internal structure is needed to generate solutions recursively (as will be shown in Chapter 7). However, it does provide a fairly general procedure that works for non-Gaussian disturbances as well. As an illustration of this advantage, the case of bounded disturbances will be discussed. In fact, a combination of Gaussian and bounded disturbances is considered which yields a solution for bounded disturbances only as a special case. Consider the following extended system with bounded disturbances $d \in \mathbf{R}^{n_d}$ as well as stochastic disturbances

$$\begin{aligned}\mathbf{y}(\xi, d) &= G_{yx}x_0(\xi) + G_{yu}\mathbf{u}(\xi) + G_{yw}\mathbf{w}(\xi) + G_{yd}d \\ \mathbf{z}(\xi, d) &= G_{zx}x_0(\xi) + G_{zu}\mathbf{u}(\xi) + G_{zw}\mathbf{w}(\xi) + G_{zd}d\end{aligned}$$

where G_{yd} is an impulse response function mapping the parameter d to a signal. Alongside the stochastic system, the reference system is defined as

$$\begin{aligned}\mathbf{y}^r &= G_{yx}x_0^r + G_{yu}\mathbf{u}^r + G_{yw}\mathbf{w}^r + G_{yd}d^r \\ \mathbf{z}^r &= G_{zx}x_0^r + G_{zu}\mathbf{u}^r + G_{zw}\mathbf{w}^r + G_{zd}d^r.\end{aligned}$$

By the definition of bounded disturbances (definition 9), there exists a reference value d^r such that D is generated by convex hull of the set of extreme points $\{d_i^*\}_i$

$$d - d^r \in D := \mathrm{co}\{d_1^*, \ldots, d_{n_e}^*\}.$$

The feedback law is formally adapted to the bounded disturbance case as

$$\mathbf{u}(\xi, d) - \mathbf{u}^r = K(\mathbf{y}(\xi, d) - \mathbf{y}^r)$$

and one can directly write down the closed-loop performance outputs in analogy with the previous results using the Youla parameter

$$\begin{aligned}\mathbf{z}(\xi, d) - \mathbf{z}^r &= (G_{zx} + G_{zu}QG_{yx})(x_0(\xi) - x_0^r) \\ &\quad + (G_{zw} + G_{zu}QG_{zw})(\mathbf{w}(\xi) - \mathbf{w}^r) + (G_{zd} + G_{zu}QG_{yu})(d - d^r).\end{aligned} \tag{4.23}$$

This brings us to the point where we must define how the inequality constraints are to be satisfied. The bounded disturbances are considered in a worst-case setting such that in absence of stochastic disturbances, the performance output must be feasible with respect to the inequality constraints for all values of d, see also the discussion in section 4.3.4. The constraints on the performance variables are given by

$$P\left(\{\xi :\ H^T\mathbf{z}(\xi, d) \le g\}\right) \ge \alpha,\ \forall d \in D \tag{4.24}$$

which by theorem 10 can be reduced to the finite number of constraints in the optimization problem

$$r\sqrt{h_j^T Z h_j} + h_j^T\hat{\mathbf{z}}(d_i^*) \le g_j\ \forall i = 1, ..., n_e, j = 1, ..., n_c. \tag{4.25}$$

The calculations on stochastics concerning variance, expected value etc. do not change by the presence of the bounded disturbance d as d has no stochastic interpretation, e.g. using a uniform probability density function.

In practical terms, the change to the inequality constraints (4.25) means that each constraint is repeated for each extreme point d_i^*. Suppose that n_d deterministic uncertainties d_i are defined such that each parameter varies in some interval $[\underline{d}_i, \overline{d}_i]$. The uncertain parameter vector d then lies in a hypercube in $\mathbf{R}^{n_d}$ and the number of extreme points (corners of the hypercube) grows exponentially as 2^{n_d}, so one must be careful with its use. A way to circumvent this combinatorial explosion in the number of constraints is to consider the different but perhaps alternative stochastic formulation in which a bounded disturbance is modelled via step with stochastic amplitude

$$\mathbf{w}(\xi) = (1, \ldots, 1)^T \xi.$$

Adding a number these stochastic step shaped disturbances has the desirable effect that the number of constraints grows linearly with the number of disturbances instead of combinatorially.

4.3.5 A Kronecker implementation

Several algorithms can be devised to solve convex problems including cutting plane algorithms, but the fastest method to solve second-order cone programs currently appears to be the primal-dual interior point method, (Boyd and Vandenberghe, 2002). Both commercial software (Andersen *et al.*, 2000) as well as non-commercial packages (Sturm, 1999) are available to solve second-order cone programs, not to mention all semi-definite programming codes. In order to use these methods, the constraints must be formulated in the general second-order cone form as

$$Ax - b \in \mathbf{L}_+^n \tag{4.26}$$

where $x \in \mathbf{R}^n$ is some large vector. The formulation in (4.22) has both matrices and vectors as free optimization variables and must therefore be translated into the format of (4.26) for software implementation. The convex formulation of the closed-loop MPC problem shows that the back-off ν_j is the norm of a vector y_j where

$$y_j = h_j^T \begin{pmatrix} (G_{zx} + G_{zu}QG_{yx})F_P & (G_{zw} + G_{zu}QG_{yw})F_W \end{pmatrix}.$$

Introduce the following system related matrices

$$\begin{aligned} S_0 &= \begin{pmatrix} G_{zx}F_P & G_{zw}F_W \end{pmatrix}, \quad S_1 = G_{zu}, \\ S_2 &= \begin{pmatrix} G_{yx}F_P & G_{yw}F_W \end{pmatrix} \end{aligned} \tag{4.27}$$

then

$$y_j = h_j^T(S_0 + S_1QS_2) = h_j^T S_0 + h_j^T S_1 Q S_2$$

Using the following basic relation in Kronecker algebra (Brewer, 1978) for arbitrary matrix products

$$\mathbf{vec}(AXB) = (B^T \otimes A)\mathbf{vec}(X)$$

it follows from $q := \mathbf{vec}(Q)$ that

$$\mathbf{vec}(y_j) = \mathbf{vec}(h_j^T S_0) + \mathbf{vec}(h_j^T S_1 Q S_2) = S_0^T h_j + S_2^T \otimes (h_j^T S_1) q. \tag{4.28}$$

To guarantee that the controller K is lower block triangular, it is necessary and sufficient to constrain the Youla parameter to be lower block triangular. This is achieved by constructing Q by placing full blocks $Q^{ij} \in \mathbf{R}^{n_u \times n_y}$ on the lower block triangular spots

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{i} E_i Q^{ij} E_j^T = \sum_{i \geq j \geq 1} E_i Q^{ij} E_j^T \tag{4.29}$$

where $E_i^T = \begin{pmatrix} O & \cdots & O & I & O & \cdots & O \end{pmatrix}$. Vectorization of equation (4.29) gives the following expression for $\mathbf{q}$

$$\begin{aligned} \mathbf{q} = \mathbf{vec}(Q) &= \mathbf{vec}\Big(\sum_{i \geq j \geq 1} E_i Q^{ij} E_j^T \Big) = \\ &= \sum_{i \geq j \geq 1} \mathbf{vec}(E_i Q^{ij} E_j^T) = \sum_{i \geq j \geq 1} (E_j \otimes E_i) \mathbf{vec}(Q^{ij}) \end{aligned}$$

Then define $p_{ij} = \mathbf{vec}(Q^{ij})$,

$$U := \begin{pmatrix} E_1 \otimes E_1 & E_1 \otimes E_2 & E_2 \otimes E_2 & \cdots \end{pmatrix},$$

and stack the free parameters in a long vector

$$\mathbf{p}^T = \begin{pmatrix} p_{11}^T & p_{12}^T & p_{22}^T & \cdots \end{pmatrix}$$

such that

$$\mathbf{q} = U\mathbf{p}.$$

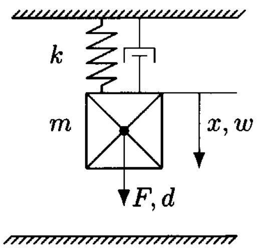
Substitution into (4.28) gives the vectorized back-off formula

$$\nu_j(\mathbf{p}) = r \| S_0^T h_j + S_2^T \otimes (h_j^T S_1) U \mathbf{p} \|_2 .$$

As a result, all variables appear as vectors in the optimization problem such that the problem is in the right format

$$\begin{array}{cc} \min\limits_{\mathbf{p}, \mathbf{z}^r} & f(\mathbf{z}^r) \\ r \| S_0^T h_j + S_2^T \otimes (h_j^T S_1) U \mathbf{p} \|_2 + h_j^T \mathbf{z}^r \leq g_j, & j = 1, \ldots, m. \end{array} \tag{4.30}$$

and conversion to (4.26) is immediately obtained. Problem (4.30) can be solved numerically in the vectorized format. The translation to the format (4.26) is straightforward as was explained in section 3.5 and is not explicitly presented here.



Performance outputs		
z_1	x	Displacement
z_2	$\dot{x}$	Velocity
Measured outputs		
y	x	Displacement
Inputs		
u	F	Force
Disturbances		
w	–	Process noise
d	–	Force disturbance
Miscellaneous		
–	k	Stiffness spring
–	m	Mass

Figure 4.8: The mass-damper-spring example

4.4 A mechanical example

In this section we aim at visualizing the solution to the closed-loop MPC problem. The mechanical example consists of a mass-damper-spring system, (figure 4.8), chosen intentionally for its 2nd order dynamics, which allows a full display of the system behavior in the state space. Because of the linear dynamics, the problem is convex such that the global optimum is found. The LTI dynamics of the system are given by

$$\left(\frac{x_{k+1}(\xi)}{y_k(\xi)}\right) = \left(\begin{array}{cc|ccc|c} .9756 & .0965 & .0316 & 0 & 0 & .0489 \\ -.4825 & .9225 & 0 & .0316 & 0 & .9649 \\ \hline 1 & 0 & 0 & 0 & .01 & 0 \end{array}\right) \left(\begin{array}{c} x_k(\xi) \\ \hline w_k(\xi) \\ \hline u_k(\xi) \end{array}\right)$$

This system is stable with eigenvalues $0.9490 \pm 0.2141i$ and it is both observable and controllable. The constraints are formulated such that the displacement, the velocity and the applied force are bounded in all directions. To be specific, let the performance output contain all states and all inputs

$$\mathbf{z}(\xi) = \begin{pmatrix} \mathbf{x}(\xi) \\ \mathbf{u}(\xi) \end{pmatrix}, \quad Ew_k(\xi)w_k(\xi) = I, \quad Ex_0(\xi)x_0(\xi)^T = P_0$$

where $P_0 = AP_0A^T + B^wB^{wT}$. Let the linear inequalities $H^Tz \leq g$ be defined using

$$\begin{aligned} H &= \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \\ g^T &= \begin{pmatrix} x_{max} & -x_{min} & v_{max} & -v_{min} & u_{max} & -u_{min} \end{pmatrix} \\ &= (1,\ -1,\ 1,\ -1,\ .5,\ -.5) \end{aligned} \tag{4.31}$$

We wish to maximize the vertical deflection of the mass such that we choose a linear objective defined by

$$c^T = \begin{pmatrix} -1 & 0 & 0 \end{pmatrix}.$$

Further, we choose a certainty level $\alpha = .97$ corresponding to a radius $r = 3$ of the certainty ellipsoid. The optimal transition from the origin to the steady state is plotted in figure 4.9. In the upper two plots one sees the transition for the position and velocity. In the lower left plot the force input is plotted with error bars representing the projection of the uncertainty ellipsoids onto the input space. In the lower right plot the result in plotted in the state space also with the projected uncertainty ellipsoids. The thick solid lines represent the inequality constraints, while the uncertainty ellipsoids visualize the back-off needed to avoid constraint violation.

To illustrate the solution for the bounded disturbance case as discussed in section 4.3.4, we slightly change the problem formulation. The stochastic process disturbances on the position and the velocity are replaced by a deterministic but unknown force bias d of $\pm 30\%$ of the maximal force that can be exerted. For illustration purposes, the covariance of the measurement noise are amplified by a factor 10 such that the control system must make a sensible choice between cancellation of the force bias and amplification of measurement noise. The system is given by

$$\left(\begin{array}{c} x_{k+1} \\ \hline y_k \end{array}\right) = \left(\begin{array}{cc|cc|c} .9756 & .0965 & .0489 & 0 & .0489 \\ -.4825 & .9225 & .9649 & 0 & .9649 \\ \hline 1 & 0 & 0 & 0.032 & 0 \end{array}\right) \left(\begin{array}{c} x_k \\ \hline d \\ w_k \\ \hline u_k \end{array}\right)$$

with $d \in [-0.15, 0.15]$. The result of the optimization is plotted in figure 4.10. The top two plots show the transition in the position of the system. The marked line shows the reference trajectory for the nominal value of the deterministic disturbance while the solid lines represent the evolution of the system for the extreme points of the bias. Notice the nice convergence of the envelope of extremal trajectories to the reference trajectory. This integral action of the controller is seen in the middle two plots where both the extremal trajectories of the inputs are plotted which differ in the steady state exactly ± 0.15 from the reference value. Around these extremal trajectories we have plotted the error bars that visualize the contribution of the feedback of measurement noise to the uncertainty in the loop. In the lower two plots, the extremal trajectories are plotted in the state space.

4.5 Feedback of the innovation sequence

The Q-parameterization approach has the flaw that if the open-loop system is unstable, the Youla parameter Q must have unstable zeros on the precise location of the open-loop unstable poles. In that case, stability of GQ is still necessary but no longer sufficient. This may lead to an unstable zero in the resulting controller K coinciding with an unstable plant pole rendering the design useless due to unstable hidden modes (Youla *et al.*, 1976a; Youla *et al.*, 1976b). To avoid these hidden modes, internal stability of the closed-loop must be guaranteed (Zames and Francis, 1983). Then, a necessary and sufficient conditions are that both GQ and

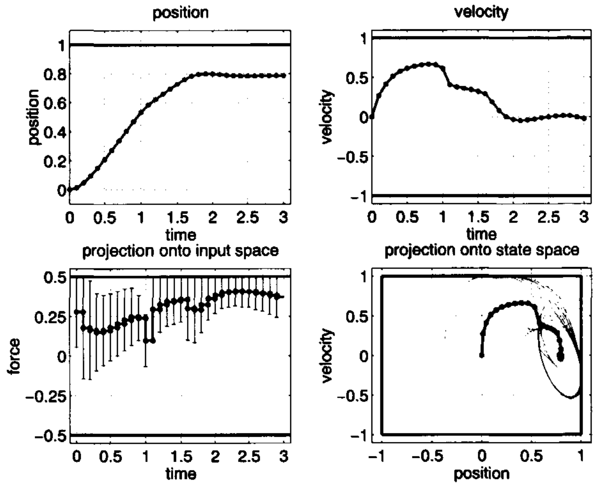


Figure 4.9: Optimal result for the mechanical system.

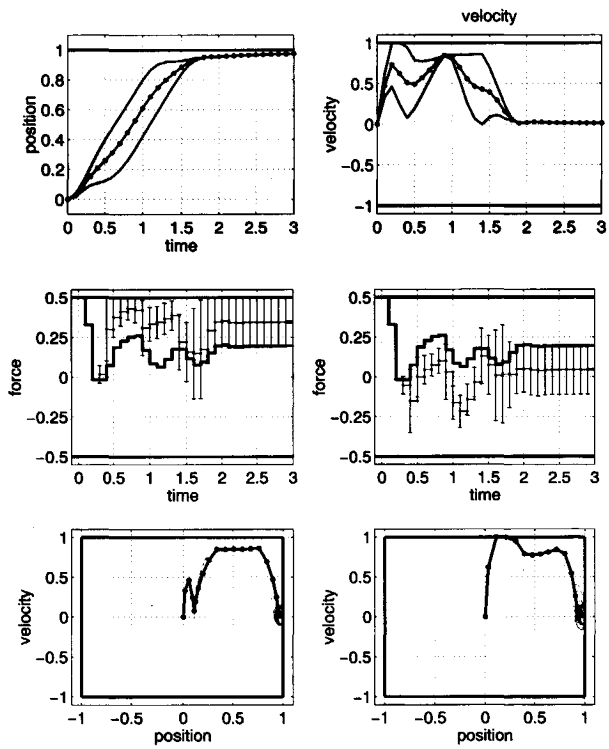


Figure 4.10: Optimal result for the bounded disturbance case with measurement noise.

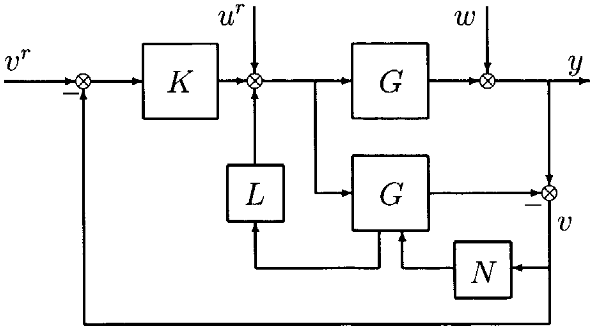


Figure 4.11: The Youla-Kučera parameterization via state and innovations feedback. A closed-loop predictive alternative to open-loop internal model control.

$(I - GQ)G$ are stable. A constructive way to obtain such a controller Q is to stabilize the system with an initial controller leading to a stable closed-loop system $\bar{G}$. Then, by replacing G by $\bar{G}$, the original procedure can be applied again (Kučera, 1974). This approach leads to a parameterization of all stabilizing controllers referred to as the Youla-Kučera parameterization in the sense of (Desoer *et al.*, 1980; Nett *et al.*, 1984; Maciejowski, 1994), see figure 4.11.

The standard state-space method to apply the Youla-Kučera parameterization is to stabilize the system with an observer and a state feedback gain. The output error v is then used in an additive feedback loop through a controller K. Many structural properties in linear time-invariant systems naturally extend to linear time-varying systems, including possible stability problems in using the Youla parameterization. In the forthcoming sections it will turn out that in the case of lifted systems, there is no need to fully fix the initial controller contrary to the standard state-space procedure. In fact, the observer is chosen fixed as a Kalman predictor in line with the Gaussian disturbances, while the state feedback gain L is optimized simultaneously with K. This seems to be a generalization of the parameterization and therefore the scheme of figure 4.11 is addressed as state and innovations feedback. We will be very explicit on how to design these state feedbacks in line with the Youla parameter. The specifics will be discussed in detail in Chapters 6 and 7 on the stationary results and the receding horizon implementation.

A second drawback of the Q-parameterization approach is that it is not clear what the internal structure of the solutions is, in particular how the controller depends on the problem data. In unconstrained stochastic control theory, the separation theorem states that the optimal solution to the LQG problem is divided in an optimal estimator and a separate control problem that depends only on the most recent state estimate. This structure allows recursive treatment of measurement data, a property of prime importance in the receding horizon implementation and for this reason it is also important to shift from output feedback to innovations feedback. This alternative solution is an observer-based controller leading to a separation principle for constrained predictive control as will be shown in Chapter 7. This separation principle is best understood if the estimator is in its error dynamics form. Then, the estimation error is independent of the control input. In the case of LQG control, the estimator is a Kalman filter (Kwakernaak and Sivan, 1972) that pre-whitens the output measurements such that the so-called innovations sequence is obtained (Kailath, 1968). The innovations sequence is then used in the feedback control law instead of the process measurements as in figure 4.11. Below, these steps will be presented in detail.

4.5.1 Lifting an observer-based controllers

In what follows we will use lifted systems describing the behavior of the Kalman filter over a future horizon. In this subsection we introduce the idea for the standard LQG case. Consider the following discrete time LTI system

$$\begin{pmatrix} x_{k+1}(\xi) \\ y_k(\xi) \end{pmatrix} = \begin{pmatrix} A \\ C \end{pmatrix} x_k(\xi) + \begin{pmatrix} B \\ O \end{pmatrix} u_k(\xi) + \begin{pmatrix} B^w \\ D^w \end{pmatrix} w_k(\xi) \tag{4.32}$$

where the disturbance w_k is a member of a discrete time white noise sequence and let us start with the standard observer-based control structure as presented by Maciejowski (1994). The optimal stationary a priori estimator is given by the Kalman predictor (Lewis, 1986). As explained in section 3.5, the optimal state estimator is given by the conditional expectation of the state vector at time t_k

$$\hat{x}_k(\xi) = E\left(x_k(\xi) \mid y_0(\xi), \dots, y_{k-1}(\xi)\right)$$

where the notation $\hat{x}_{k|k-1}$ is used to denote the estimate of the state x at time k using all measurements up to time $k-1$. The state estimate is the conditional expectation evaluated at the partial realization $y_0, \dots, y_\nu$ of the stochastic process $\mathbf{y}(\xi)$

$$\hat{x}_k = E\left(x_k(\xi) \mid y_0(\xi) = y_0, \dots, y_k(\xi) = y_k\right).$$

and this can be written as a recursive dynamical system as

$$\hat{x}_{k+1}(\xi) = (A - NC)\hat{x}_k(\xi) + Bu_k + Ny_k(\xi) \tag{4.33}$$

It is also well known that the LQG optimal control sequence in the case of partial state measurements is given by state-feedback of the state estimate, where the gain matrix is equivalent to its deterministic LQR counterpart (by the certainty equivalence property of LQG control)

$$u_k(\xi) = F\hat{x}_k(\xi). \tag{4.34}$$

The LQG controller can therefore be written as

$$\begin{pmatrix} \hat{x}_{k+1}(\xi) \\ u_k(\xi) \end{pmatrix} = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \begin{pmatrix} \hat{x}_k(\xi) \\ y_k(\xi) \end{pmatrix} \tag{4.35}$$

again starting from a zero initial condition. The controller parameters are given by

$$\left(\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right) = \left(\begin{array}{c|c} A - NC + BF & N \\ \hline F & O \end{array}\right) \tag{4.36}$$

In this stationary case, the feedback gains are computed via the solution of the control and estimation Riccati equations. If the Kalman predictor is lifted over a finite time horizon we obtain

$$\begin{pmatrix} \hat{x}_0(\xi) \\ \hat{x}_1(\xi) \\ \vdots \\ \hat{x}_n(\xi) \end{pmatrix} = \begin{pmatrix} I \\ A_c \\ \vdots \\ A_c^n \end{pmatrix} \hat{x}_0(\xi) + \begin{pmatrix} O & O & \cdots & O \\ B_c & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ A_c^{n-1}B_c & A_c^{n-2}B_c & \cdots & O \end{pmatrix} \begin{pmatrix} y_0(\xi) \\ y_1(\xi) \\ \vdots \\ y_n(\xi) \end{pmatrix}.$$

and the controller output (control input) is then given by the sequence

$$\begin{pmatrix} u_0(\xi) \\ u_1(\xi) \\ \vdots \\ u_n(\xi) \end{pmatrix} = \begin{pmatrix} C_c \\ C_cA_c \\ \vdots \\ C_cA_c^n \end{pmatrix} \hat{x}_0(\xi) + \begin{pmatrix} D_c & O & \cdots & O \\ C_cB_c & D_c & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ C_cA_c^{n-1}B_c & C_cA_c^{n-2}B_c & \cdots & D_c \end{pmatrix} \begin{pmatrix} y_0(\xi) \\ y_1(\xi) \\ \vdots \\ y_n(\xi) \end{pmatrix} \tag{4.37}$$

To observe the link with the closed-loop MPC problem, introduce the reference signals for the control inputs and states

$$(u_k^r, x_k^r)$$

to which the system must be controlled by means of the state feedback

$$u_k(\xi) = F\hat{x}_k(\xi) = Fx_k^r + F(\hat{x}_k(\xi) - x_k^r) = u_k^r + F(\hat{x}_k(\xi) - x_k^r)$$

where the deterministic control law is defined as

$$u_k^r = Fx_k^r.$$

In general, the initial controller state is unknown, $(s_0 = s_0^r)$, such that the lifted controller dynamics over n samples is given by

$$\begin{pmatrix} u_0(\xi) - u_0^r \\ u_1(\xi) - u_1^r \\ \vdots \\ u_n(\xi) - u_n^r \end{pmatrix} = \begin{pmatrix} D_c & O & \cdots & O \\ C_cB_c & D_c & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ C_cA_c^{n-1}B_c & C_cA_c^{n-2}B_c & \cdots & D_c \end{pmatrix} \begin{pmatrix} y_0(\xi) - y_0^r \\ y_1(\xi) - y_2^r \\ \vdots \\ y_n(\xi) - y_n^r \end{pmatrix}.$$

The controller state is the state estimate (minus the reference value) but it plays no distinct role itself. If we let the dynamics in the observer-based controller be time-varying, then it is a small step to generalize to the controller configuration given by

$$\begin{pmatrix} u_0(\xi) - u_0^r \\ u_1(\xi) - u_1^r \\ \vdots \\ u_n(\xi) - u_n^r \end{pmatrix} = \begin{pmatrix} K_{00} & O & \cdots & O \\ K_{10} & K_{11} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ K_{n0} & K_{n1} & \cdots & K_{nn} \end{pmatrix} \begin{pmatrix} y_0(\xi) - y_0^r \\ y_1(\xi) - y_2^r \\ \vdots \\ y_n(\xi) - y_n^r \end{pmatrix}.$$

which is then free of any predefined structure as in the observer-based controller, (4.37).

4.5.2 From generic to observer-based controllers

The steps above can be reversed to obtain from a generic output feedback controller in lifted form, an equivalent innovations feedback controller in lifted form that is by construction observer-based. We now continue with the more general LTV case in which we consider the system

$$\begin{pmatrix} x_{k+1} \\ y_k \end{pmatrix} = \begin{pmatrix} A_k \\ C_k \end{pmatrix} x_k + \begin{pmatrix} B_k \\ O \end{pmatrix} u_k + \begin{pmatrix} B_k^w \\ D_k^w \end{pmatrix} w_k$$

derived along a reference trajectory which we can (without loss of generality) assume to be zero because they are deterministic and do not enter the feedback path. In a later stage we can then add these trajectories again. Contrary to the stationary counterpart discussed above, we now turn to the finite horizon observer, (which consists of time varying-feedback gains even if the original system is time-invariant, (Rhodes, 1971)). As suggested above, the observer is put in its error dynamics form, where the state of the observer is the error between the process state and its estimate

$$e_k(\xi) := x_k(\xi) - \hat{x}_k(\xi).$$

This dynamical system is given in recursive form by

$$\begin{aligned} e_{k+1}(\xi) &= (A_k - N_k C_k) e_k(\xi) + (B_k^w - N_k D_k^w) w_k(\xi) \\ &= A_k^e e_k(\xi) + B_k^e w_k(\xi) \end{aligned} \tag{4.38}$$

where

$$\Phi_{k,j}^e = A_{k-1}^e A_{k-2}^e \cdots A_j^e, \quad \Phi_{j,j}^e = I, \; A_k^e = A_k - N_k C_k, \text{ and } B_k^e = B_k^w - N_k D_k^w.$$

In its lifted form it has the representation

$$\mathbf{e}_0(\xi) = G_{ee} e_0(\xi) + G_{ew} \mathbf{w}_0(\xi)$$

or in terms of the system matrices $(\Phi_{k,l}^e, B_k^e)$

$$\begin{pmatrix} e_0(\xi) \\ e_1(\xi) \\ \vdots \\ e_n(\xi) \end{pmatrix} = \begin{pmatrix} I \\ \Phi_{1,0}^e \\ \vdots \\ \Phi_{n,0}^e \end{pmatrix} e_0(\xi) + \begin{pmatrix} O & O & \cdots & O \\ B_0^e & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n,1}^e B_0^e & \Phi_{n,2}^e B_1^e & \cdots & O \end{pmatrix} \begin{pmatrix} w_0(\xi) \\ w_1(\xi) \\ \vdots \\ w_n(\xi) \end{pmatrix}. \tag{4.39}$$

The innovation sequence, (Kailath, 1968), is by definition given by the expressions

$$\begin{aligned} v_k(\xi) :=& y_k(\xi) - \hat{y}_k(\xi) \\ =& C_k\left(x_k(\xi) - \hat{x}_k(\xi)\right) + D_k^w w_k(\xi) \\ =& C_k e_k(\xi) + D_k^w w_k(\xi). \end{aligned} \tag{4.40}$$

It should be noted here that we have two representations of the innovations sequence. The definition expressed in terms of the measured and estimated outputs gives us the actual numerical value for feedback, while the second representation using the unknown estimation error and unknown process disturbance provides us with the covariance matrix needed to compute the back-off. Putting the innovations sequence in its lifted form we obtain

$$\mathbf{v}_0(\xi) = G_{ve} e_0(\xi) + G_{vw}\mathbf{w}_0(\xi) \tag{4.41}$$

or in terms of the system matrices $(\Phi_{k,l}^e, B_k^e, C_k, D_k^w)$

$$\begin{pmatrix} v_0(\xi) \\ v_1(\xi) \\ \vdots \\ v_n(\xi) \end{pmatrix} = \begin{pmatrix} C_0 \\ C_1\Phi_{1,0}^e \\ \vdots \\ C_n\Phi_{n,0}^e \end{pmatrix} e_0(\xi) + \begin{pmatrix} D_0^w & O & \cdots & O \\ C_1 B_0^e & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ C_n\Phi_{n,1}^e B_0^e & C_n\Phi_{n,2}^e B_1^e & \cdots & D_n^w \end{pmatrix} \begin{pmatrix} w_0(\xi) \\ w_1(\xi) \\ \vdots \\ w_n(\xi) \end{pmatrix}$$

Then, instead of using an output feedback law as in the Q-parameterization approach, the control sequence is determined by feedback of the tracking error in the innovations sequence: (4.41), that is

$$\mathbf{u}_0^c(\xi) = \mathbf{u}_0(\xi) - \mathbf{u}_0^r = K_v(\mathbf{v}_0(\xi) - \mathbf{v}_0^r(\xi)) = K_v\mathbf{v}_0^c(\xi), \quad K_v \in \mathbf{K}_0. \tag{4.42}$$

One obtains expressions for $\mathbf{e}^c, \mathbf{v}^c$ by subtraction of the reference values from the observer equations, such that by linearity we obtain

$$\begin{aligned} \mathbf{e}_0^c(\xi) &= G_{ee} e_0^c(\xi) + G_{ew}\mathbf{w}_0^c(\xi) \\ \mathbf{v}_0^c(\xi) &= G_{ve} e_0^c(\xi) + G_{vw}\mathbf{w}_0^c(\xi). \end{aligned}$$

These equations represent an observer in error dynamics form for the variational dynamical system with state vector $\mathbf{x}_0^c(\xi) = \mathbf{x}_0(\xi) - \mathbf{x}_0^r$. Then, the following result shows that this would solve the CLMPC problem as well.

Theorem 16 Let us fix the observer gains equal to the Kalman filter gains N_k. Then, the closed-loop MPC problem is convex in the innovations feedback controller K_v.

Proof. Upon substitution of (4.41) into (4.42), we observe that the control sequence is given by

$$\mathbf{u}_0^c(\xi) = K_v G_{ve} e_0^c(\xi) + K_v G_{vw}\mathbf{w}_0^c(\xi)$$

The crucial observation is that, contrary to the output feedback case, the input is independent of itself since there is no transfer between the input u and the estimation error e nor the innovations v. As as result, no inversion of the closed-loop dynamics is needed and one immediately writes for the control error in the performance output

$$\mathbf{z}_0^c(\xi) = G_{zx}x_0^c(\xi) + G_{zu}\mathbf{u}_0^c(\xi) + G_{zw}\mathbf{w}_0^c(\xi).$$

The initial condition can be decomposed into its estimate and the estimation error

$$x_0^c(\xi) = \hat{x}_0^c(\xi) + e_0^c(\xi)$$

such that the performance output can equally be given by

$$\mathbf{z}_0^c(\xi) = G_{zx}\hat{x}_0^c(\xi) + G_{zx}e_0^c(\xi) + G_{zu}\mathbf{u}_0^c(\xi) + G_{zw}\mathbf{w}_0^c(\xi).$$

Because no measurements are available yet, it follows that

$$\hat{x}_0^c(\xi) \equiv 0.$$

It is then immediate that the control error in the performance output is given by

$$\mathbf{z}_0^c(\xi) = (G_{zx} + G_{zu}K_vG_{ve})e_0^c(\xi) + (G_{zw} + G_{zu}K_vG_{vw})\mathbf{w}_0^c(\xi)$$

The variance matrix of this performance output is given as

$$\begin{aligned} E\mathbf{z}_0^c(\xi)\mathbf{z}_0^c(\xi)^T &= \begin{pmatrix} G_{zx} + G_{zu}K_vG_{ve} & G_{zw} + G_{zu}K_vG_{vw} \end{pmatrix} \times \\ & E\begin{pmatrix} e_0^c(\xi) \\ \mathbf{w}_0^c(\xi) \end{pmatrix}\begin{pmatrix} e_0^c(\xi) \\ \mathbf{w}_0^c(\xi) \end{pmatrix}^T \begin{pmatrix} G_{zx} + G_{zu}K_vG_{ve} & G_{zw} + G_{zu}K_vG_{vw} \end{pmatrix}^T \\ &= \begin{pmatrix} G_{zx} + G_{zu}K_vG_{ve} & G_{zw} + G_{zu}K_vG_{vw} \end{pmatrix}\begin{pmatrix} P_0 & O \\ O & W \end{pmatrix}\begin{pmatrix} \star & \star \end{pmatrix}^T \end{aligned}$$

which directly gives a factor that depends on the controller K_v in an affine and hence convex way. □

The importance of theorem 16 is that **CLMPC** is *also* rendered convex using innovations feedback, however, this latter solution has the additional desirable property that the solution has an observer structure, which is important in deriving the receding horizon implementation.

Remark 17 In the case that the state estimate is non-zero we cannot discard it. As will be shown in Chapter 7, we must in that case include a state feedback term in the input as

$$\mathbf{u}_0^c(\xi) = L_0\hat{x}_0^c + K_vG_{ve}e_0^c(\xi) + K_vG_{vw}\mathbf{w}_0^c(\xi)$$

where $L_0 \in \mathbf{R}^{nn_u \times n_x}$. Then the covariance matrix Z must be computed on the basis of the following representation of the performance output

$$\mathbf{z}_0^c(\xi) = (G_{zx} + G_{zu}L_0)\hat{x}_0^c(\xi) + (G_{zx} + G_{zu}K_vG_{ve})e_0^c(\xi) + (G_{zw} + G_{zu}K_vG_{vw})\mathbf{w}_0^c(\xi)$$

which is a little more involved since the joint covariance matrix of $\hat{x}_0^c, e_0^c(\xi), \mathbf{w}_0^c(\xi)$ must then be computed. Note however that this still produces a convex optimization problem such that all conclusions carry over to the more general case. □

4.5.3 Optimality of the innovations feedback approach

Switching from output feedback to innovations feedback enforces an observer based-structure in the solution, and no performance is lost in doing so because every controller that is found via output feedback can also be constructed from innovations feedback. To show this, we return to the definition of the innovation sequence for the variational system

$$v_k^c(\xi) := y_k^c(\xi) - \hat{y}_k^c(\xi). \tag{4.43}$$

The observer system lifted over a future time horizon is given by

$$\begin{pmatrix} \hat{x}_0^c(\xi) \\ \hat{x}_1^c(\xi) \\ \vdots \\ \hat{x}_n^c(\xi) \end{pmatrix} = \begin{pmatrix} I \\ \Phi_{1,0}^e \\ \vdots \\ \Phi_{n,0}^e \end{pmatrix} \hat{x}_0^c(\xi) + \begin{pmatrix} O & O & \cdots & O \\ B_0 & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n,1}^e B_0 & \Phi_{n,2}^e B_1 & \cdots & O \end{pmatrix} \begin{pmatrix} u_0^c(\xi) \\ u_1^c(\xi) \\ \vdots \\ u_n^c(\xi) \end{pmatrix} + \begin{pmatrix} O & O & \cdots & O \\ N_0 & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n,1}^e N_0 & \Phi_{n,2}^e N_1 & \cdots & O \end{pmatrix} \begin{pmatrix} y_0^c(\xi) \\ y_1^c(\xi) \\ \vdots \\ y_n^c(\xi) \end{pmatrix} \tag{4.44}$$

where the transition matrix $\Phi_{k,j}^e$ for the observer system mapping $\hat{x}_j$ to $\hat{x}_k$ is given for $k > j$ by

$$\Phi_{k,j}^e = A_{k-1}^e A_{k-2}^e \cdots A_j^e, \quad \Phi_{j,j}^e = I, \quad \text{where} \quad A_k^e = A_k - N_k C_k.$$

The estimated outputs in (4.43) follow immediately from (4.44) by multiplying the a priori state estimate with the sequence $\{C_k\}_k$

$$\begin{pmatrix} \hat{y}_0^c(\xi) \\ \hat{y}_1^c(\xi) \\ \vdots \\ \hat{y}_n^c(\xi) \end{pmatrix} = \begin{pmatrix} C_0 \\ C_1\Phi_{1,0}^e \\ \vdots \\ C_n\Phi_{n,0}^e \end{pmatrix} \hat{x}_0^c(\xi) + \begin{pmatrix} O & O & \cdots & O \\ C_1 B_0 & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ C_n\Phi_{n,1}^e B_0 & C_n\Phi_{n,2}^e B_1 & \cdots & O \end{pmatrix} \begin{pmatrix} u_0^c(\xi) \\ u_1^c(\xi) \\ \vdots \\ u_n^c(\xi) \end{pmatrix} + \begin{pmatrix} O & O & \cdots & O \\ C_1 N_0 & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ C_n\Phi_{n,1}^e N_0 & C_n\Phi_{n,2}^e N_1 & \cdots & O \end{pmatrix} \begin{pmatrix} y_0^c(\xi) \\ y_1^c(\xi) \\ \vdots \\ y_n^c(\xi) \end{pmatrix} \tag{4.45}$$

Then, subtraction of the estimated output process (4.45) from the output process $\mathbf{y}(\xi)$ leads to the alternative expression, (compare to (4.40)), for the innovations sequence $\mathbf{v}(\xi)$. The lifted form is given by

$$\mathbf{v}_0^c(\xi) = G_{vx}\hat{x}_0^c(\xi) + G_{vu}\mathbf{u}_0^c(\xi) + G_{vy}\mathbf{y}_0^c(\xi) \tag{4.46}$$

where the individual transfer matrices are given by

$$G_{vx} = -\begin{pmatrix} C_0 \\ C_0\Phi^e_{1,0} \\ \vdots \\ C_n\Phi^e_{n,0} \end{pmatrix} \quad G_{vu} = -\begin{pmatrix} O & O & \cdots & O \\ C_1 B_0 & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ C_n\Phi^e_{n,1}B_0 & C_n\Phi^e_{n,2}B_1 & \cdots & O \end{pmatrix} \tag{4.47}$$

$$G_{vy} = \begin{pmatrix} I & O & \cdots & O \\ -C_1 N_0 & I & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ -C_n\Phi^e_{n,1}N_0 & -C_n\Phi^e_{n,2}N_1 & \cdots & I \end{pmatrix}. \tag{4.48}$$

This brings us to the following simple observation on optimality of the structural choice of feedback of the innovations sequence.

Theorem 18 For any output feedback controller

$$\mathbf{u}_0^c(\xi) = K_y \mathbf{y}_0^c(\xi)$$

there exists an equivalent innovations feedback controller

$$\mathbf{u}_0^c(\xi) = K_v \mathbf{v}_0^c(\xi)$$

and the converse also holds true.

Proof. The important observation is that the system transfer matrix

$$G_{vy}$$

in (4.48) is invertible due to the lower block triangular structure and the identity matrices on the block-diagonal. Fix any controller K_v, then from equation (4.46) it follows after some manipulation

$$\mathbf{v}_0^c(\xi) = (I - G_{vu}K_v)^{-1} G_{vy} \mathbf{y}_0^c(\xi)$$

Hence, the equivalent feedback is obtained by using the output feedback controller

$$K_y = K_v(I - G_{vu}K_v)^{-1}G_{vy} \tag{4.49}$$

where $(I - G_{vu}K)^{-1}$ is guaranteed to be invertible. Conversely, fix any controller K_y, then by comparable manipulations of (4.46) it follows that an equivalent feedback is obtained by using the innovations feedback

$$K_v = K_y(G_{vy} + G_{vu}K_y)^{-1} \tag{4.50}$$

where the inverse $(G_{vy} + G_{vu}K_y)^{-1}$ is also guaranteed to exist. $\square$

From theorem 18, it follows that the output feedback controller can be interchanged with the innovations feedback controller using equations (4.49) or (4.50). The key point is that the measured output sequence and the innovations sequence carry the same information since either sequence can be constructed from the other as long as *all* elements are stored (observe the subscript 0 on the signals). The continuation of the solution in a receding horizon fashion is another matter that is discussed in chapter 7.

Remark 19 *Recursive disturbance models in estimation.* The recursive procedure to construct the observer implies that the disturbance models, as discussed in section 3.3.1, need to be in recursive format as well. We shall typically be concerned with bias or persistent disturbance models that are modelled recursively by

$$d_{k+1}(\xi) = d_k(\xi), \quad d_0(\xi) = \xi, \quad w_k(\xi) = d(\xi) \tag{4.51}$$

with uncertain initial condition. The state-space is expanded to include the additional dynamics governed by the state vector d_k

$$x_k^a(\xi) = \begin{pmatrix} x_k(\xi) \\ d_k(\xi) \end{pmatrix} \tag{4.52}$$

Then, an observer is constructed for this augmented system with state vector x^a to find the innovations sequence. □

4.6 The LTV approach to nonlinear dynamic optimization

The derivations of the closed-loop properties are all based on linear time-varying dynamics. In this section, it is illustrated how the techniques are used for nonlinear systems. The philosophy is simple, namely to apply the tools developed here directly to smooth nonlinear systems. This approach is justified by stressing that feedback has a linearizing effect, as it forces the system to stay close to the reference trajectories (that *do* satisfy the nonlinear model equations), such that the LTV models remain valid. The nonlinear process model itself is however still used for integration of the model equations in open-loop prediction corresponding of the feedforward control action and to construct the LTV models for feedback control along these predicted trajectories.

4.6.1 The basic sequential optimization algorithm

The basic algorithm for sequential optimization that is used in this thesis is based on linearization of the optimization problem to find search directions. There are several possible extensions to this straightforward approach for improving on optimality, accuracy and rate of convergence, but such inquiries fall outside the scope of this thesis. In principle, our approach will be based on sequential quadratic programming when it comes to optimizing nonlinear systems, but we will ignore the second order derivatives in the system dynamics. Luenberger (1973) has been used as basic reference on general nonlinear optimization.

Consider a nonlinear optimization problem with twice differentiable real valued functions $f : \mathbf{R}^n \to \mathbf{R}$, $g_j : \mathbf{R}^n \to \mathbf{R}$ and $h_i : \mathbf{R}^n \to \mathbf{R}$

$$\begin{array}{lcc} \text{(NLP)} & \min\limits_{x} & f(x) \\ & h_i(x) = 0, \ i = 1, \ldots, p & \\ & g_j(x) \leq 0, \ j = 1, \ldots, m. & \end{array}$$

Given an initial guess x_l, one solves the linearized version of (NLP)

$$\begin{array}{lcc} \text{(LNLP)} & \min\limits_{d_l} & \partial f(x_l) d_l + \frac{1}{2} d_l^T B_l d_l \\ & h_i(x_l) + \partial h_i(x_l) d_l = 0, \ i = 1, \ldots, p & \\ & g_j(x_l) + \partial g_j(x_l) d_l \leq 0, \ j = 1, \ldots, m. & \end{array}$$

for some properly chosen B_l and we update the initial guess with

$$x_{l+1} = x_l + d_l.$$

Define the Lagrangian corresponding to (NLP) as

$$L(x, \mu, \lambda) = f(x) + \sum_{i=1}^{p} \mu_i h_i(x) + \sum_{j=1}^{m} \lambda_j g_j(x)$$

where $\mu \in \mathbf{R}^p$ and $\lambda \in \mathbf{R}^m$ are the Lagrange multipliers. The matrix B_l in (LNLP) is often set equal to the Hessian of the Lagrangian or an approximation thereof, that is

$$B_l = \partial_x^2 L(x_l, \mu_l, \lambda_l) = \partial^2 f(x_l) + \sum_{i=1}^{p} \mu_{l,i} \partial^2 h_i(x_l) + \sum_{j=1}^{m} \lambda_{l,j} \partial^2 g_j(x_l)$$

where the Lagrange multipliers are set to the Lagrange multipliers of (LNLP). In the basic algorithm used in this thesis we simply set

$$B_l = \partial^2 f(x_l).$$

We motivate this by the fact that in our setup the functions g_j are linear such that

$$\partial^2 g_j(x) \equiv 0$$

while the equality constraints consist of the nonlinear dynamics for which we will assume that the second-order effects do not dominate the solution of (LNLP). It is emphasized that there is a clear opportunity for improvement here, but these improvements fall outside our scope. Summarizing, we will solve (NLP) by sequentially solving the linearized problem

$$\begin{array}{lcc} \text{(LNLP)} & \min\limits_{d_l} & \partial f(x_l) d_l + \frac{1}{2} d_l^T \partial^2 f(x_l) d_l \\ & h_i(x_l) + \partial h_i(x_l) d_l = 0, \ i = 1, \ldots, p & \\ & g_j(x_l) + \partial g_j(x_l) d_l \leq 0, \ j = 1, \ldots, m. & \end{array}$$

Note that the closed-loop MPC problem is actually a second-order cone problem in which one must be careful in using derivatives. The general problem is of then of the form

$$
\begin{array}{lcc}
\text{(NLP)} & \displaystyle\min_{x} & f(x) \\
& h_i(x) = 0, \; i = 1, \ldots, p & \\
& g_j(x) \leq 0, \; j = 1, \ldots, m & \\
& \|A_j x + b_j\| \leq p_j^T x + q_j, \; j = 1, \ldots, m. &
\end{array}
$$

and (LNLP) is then defined as

$$
\begin{array}{lcc}
\text{(LNLP)} & \displaystyle\min_{d_l} & \partial f(x_l) d_l + \frac{1}{2} d_l^T \partial^2 f(x_l) d_l \\
& h_i(x_l) + \partial h_i(x_l) d_l = 0, \; i = 1, \ldots, p & \\
& g_j(x_l) + \partial g_j(x_l) d_l \leq 0, \; j = 1, \ldots, m & \\
& \|A_j x_l + A_j d_l + b_j\| \leq p_j^T x_l + p_j^T d_l + q_j, \; j = 1, \ldots, m &
\end{array}
$$

Remark 20 Consider the dynamic optimization problem

$$
\begin{array}{lcc}
\text{(DOP)} & \displaystyle\min_{x \in \mathbf{R}^N} & \displaystyle\sum_{k=0}^{N} J_k(x_k) \\
& h(x) = 0 & \\
& g(x) \leq 0 &
\end{array}
$$

where x is a discrete time signal, N is the length of the horizon and $h(x) = 0$ represents the constraint imposed by the discretized dynamics. If N is too large to handle in a single optimization, we solve this problem in a receding horizon or sliding window fashion using a reduced problem defined over a shorter horizon $n \ll N$. Suppose we are given an initial solution $x_{i,l}$ for the time instances $i = k, \ldots, k+n$. Then solve the linearized problem

$$
\begin{array}{lcc}
\text{(LDOP)}_k & \displaystyle\min_{d_l \in \mathbf{R}^n} & \displaystyle\sum_{i=k}^{k+n} \partial J_i(x_{i,l}) d_{i,l} + \frac{1}{2} d_{i,l}^T \partial^2 J_i(x_{i,l}) d_{i,l} \\
& h_l(x_l) + \partial h_l(x_l) d_l = 0 & \\
& g_l(x_l) + \partial g_l(x_l) d_l \leq 0 &
\end{array}
$$

where h_l, g_l, x_l are the restrictions of h, g, x to the smaller horizon. Then we use the update law

$$x_{k+i,l+1} = x_{k+i,l} + d_{i,l}, \quad i = 1, \ldots, n$$

and shift the time horizon by one sample. □

Remark 21 In dynamic optimization with quadratic weighting matrices one often encounters objectives of the type

$$f(\bar{\mathbf{u}}) = \bar{\mathbf{z}}(\bar{\mathbf{u}})^T Q \bar{\mathbf{z}}(\bar{\mathbf{u}}), \quad \bar{\mathbf{z}}(\bar{\mathbf{u}}) = h(\bar{\mathbf{u}}), \quad Q \succeq 0.$$

Then, since we assume that the second-order terms in the dynamics h are not dominating the solution to (LNLP) it follows that the update

$$\bar{\mathbf{u}}_{l+1} = \bar{\mathbf{u}}_l + \mathbf{u}_l$$

leads to the approximate update

$$\bar{\mathbf{z}}_{l+1} \simeq \bar{\mathbf{z}}_l + G_{zu}\mathbf{u}_l, \quad G_{zu} = \partial h(\bar{\mathbf{u}}_l).$$

It is immediate that the Hessian of the objective is approximately equal to

$$\partial^2 f(\bar{\mathbf{u}}) \simeq G_{zu}^T Q G_{zu}$$

which is quite an accurate guess (accurate up to first order) and it could even be used as initial guess in more profound quasi-Newton methods to update B_l. $\square$

4.6.2 Application to the closed-loop MPC problem

Consider the smooth nonlinear dynamical system as discussed earlier in Section 3.2

$$\begin{aligned} 0 &= f(\dot{\bar{x}}, \bar{x}, \bar{v}, \bar{u}, \bar{w}, \bar{d}), \qquad \bar{x}(0) = \bar{x}_0 \\ \bar{y} &= C_y^x \bar{x} + C_y^v \bar{v} + D_y^u \bar{u} + D_y^w \bar{w} + D_y^d \bar{d} \\ \bar{z} &= C_z^x \bar{x} + C_z^v \bar{v} + D_z^u \bar{u} + D_z^w \bar{w} + D_z^d \bar{d} \end{aligned}$$

for which we want to solve a nonlinear dynamic optimization problem

$$\text{(NDOP)} \qquad \begin{array}{lll} \min\limits_{\bar{u}^r, K} & \int_0^T c^T \bar{z}^r dt & \\ s.t. & 0 = f(\dot{\bar{x}}^r, \bar{x}^r, \bar{v}^r, \bar{u}^r, \bar{w}^r), & \bar{x}^r(0) = \bar{x}_0^r \\ & \bar{y}^r = C_y^x \bar{x}^r + C_y^v \bar{v}^r + D_y^u \bar{u}^r + D_y^w \bar{w}^r & \\ & \bar{z}^r = C_z^x \bar{x}^r + C_z^v \bar{v}^r + D_z^u \bar{u}^r + D_z^w \bar{w}^r & \\ & v_j + h_j^T \bar{z}(t_k) \le g_j, \ j = 1, \ldots, m, & \\ & r\sqrt{h_j^T Z(K) h_j} \le v_j & j = 1, \ldots, m \\ & Z = G_{zx}^K P {G_{zx}^K}^T + G_{zw}^K W {G_{zw}^K}^T & \\ & G_{zx}^K := G_{zx} + G_{zu} K (I - G_{yu} K)^{-1} G_{yx} & \\ & G_{zw}^K := G_{zw} + G_{zu} K (I - G_{yu} K)^{-1} G_{yw} & \end{array} \qquad (4.53)$$

Notice that we already incorporate the discrete time linear dynamics to compute the covariance matrix Z. We do so from the outset to avoid having to compute the actual covariance matrix Z from the nonlinear dynamic system. It is important to note that the covariance matrix in (NDOP) is computed using the following data

$$\begin{aligned} &Z = E\mathbf{z}^c(\xi)\mathbf{z}^c(\xi)^T \\ &P = E x_0^c(\xi) x_0^c(\xi)^T \quad \text{(fixed)} \\ &W = E\mathbf{w}^c(\xi)\mathbf{w}^c(\xi)^T \quad \text{(fixed)} \\ &\mathbf{y}^c(\xi) = G_{yx} x_0^c(\xi) + G_{yu}\mathbf{u}^c(\xi) + G_{yw}\mathbf{w}^c(\xi) \\ &\mathbf{z}^c(\xi) = G_{zx} x_0^c(\xi) + G_{zu}\mathbf{u}^c(\xi) + G_{zw}\mathbf{w}^c(\xi) \\ &\mathbf{u}^c(\xi) = K\mathbf{y}^c(\xi) \end{aligned}$$

Hence, although the problem formulation contains stochastic elements, the actual optimization is well defined in the sense that the actual realizations of the stochastic processes that are involved are unimportant for the solution of the feedfoward trajectory $\mathbf{u}^r$ and feedback controller K. Secondly, we do not enforce the path constraints on each time instant t in the interval $[0, T]$, instead we will enforce them on the sample time instances $t_k = t_0 + kT_s, \quad k = 1, \ldots, n$ only where $nT_s = T$. A similar trick is applied to the objective which is replaced by a Riemann sum. For the coexistence of the discrete time and continuous time system, introduce the discretization of the signals

$$\begin{aligned}\bar{x}^c(t,\xi) &:= \bar{x}(t,\xi) - \bar{x}_l^r(t),\\ \bar{u}^c(k,\xi) &:= \bar{u}(k,\xi) - \bar{u}_l^r(k),\\ \bar{y}^c(t_k,\xi) &:= \bar{y}(t_k,\xi) - \bar{y}_l^r(t_k)\end{aligned}$$

where $\bar{x}(t,\xi)$ is a continuous time stochastic process, $\bar{y}(t_k,\xi)$ is a sampled continuous time stochastic process, and $\bar{u}(k,\xi)$ is a discrete time process applied to the plant using a zero-order-hold sampling device.

Suppose we have an initial guess for the reference trajectories given in the $l^t h$ iteration of some dynamic optimization algorithm that satisfies the nonlinear dynamics

$$\begin{aligned}0 &= f(\dot{\bar{x}}_l^r, \bar{x}_l^r, \bar{v}_l^r, \bar{u}_l^r, \bar{w}_l^r), \qquad \bar{x}_l^r(0) = \bar{x}_{l,0}^r\\ \bar{y}_l^r &= C_y^x \bar{x}_l^r + C_y^v \bar{v}_l^r + D_y^u \bar{u}_l^r + D_y^w \bar{w}_l^r\\ \bar{z}_l^r &= C_z^x \bar{x}_l^r + C_z^v \bar{v}_l^r + D_z^u \bar{u}_l^r + D_z^w \bar{w}_l^r\end{aligned} \tag{4.54}$$

The LTV feedback law is then used to control the error between the actual trajectories $(\bar{x}(\xi), \bar{y}(\xi))$ and the reference $(\bar{x}^r, \bar{y}^r)$ after direct injection of the feedforward $\bar{u}^r$) (recall the original control architecture discussed in (Athans, 1971)). Let us stack all signals as before and analyze the situation from a nonlinear programming perspective. Image that in the l^{th} iteration of some sequential optimization algorithm iterating on problem (NDOP) the solutions

$$\bar{\mathbf{u}}_l^r, \quad \bar{\mathbf{w}}_l^r, \quad \bar{\mathbf{y}}_l^r, \quad \bar{\mathbf{z}}_l^r$$

for the reference signals of the generalized plant have been found. We seek updates on the trajectories to find the next improved trajectory one iteration later. For the input this leads to the update law

$$\bar{\mathbf{u}}_{l+1}^r = \bar{\mathbf{u}}_l^r + \mathbf{u}^r$$

where $\mathbf{u}^r$ is the update computed via the (sub)optimization in the sequential optimization algorithm and as before we are given the update on the reference signal w^r. In many cases, the update w^r is zero, but it is kept here for generality. For these updates, we seek trajectories x^r, v^r, y^r, z^r satisfying the linearized dynamics

$$\begin{aligned}0 &= \partial_{\dot{\bar{x}}} f|_l \, \dot{x}^r + \partial_{\bar{x}} f|_l \, x^r + \partial_{\bar{v}} f|_l \, v^r + \partial_{\bar{u}} f|_l \, u^r + \partial_{\bar{w}} f|_l \, w^r, \quad x^r(0) = x_0^r\\ y^r &= C_y^x x^r + C_y^v v^r + D_y^u u^r + D_y^w w^r\\ z^r &= C_z^x x^r + C_z^v v^r + D_z^u u^r + D_z^w w^r\end{aligned}$$

where

$$\partial_\star f|_l = \partial_\star f(\bar{x}_l^r, \bar{v}_l^r, \bar{u}_l^r, \bar{w}_l^r).$$

As mentioned above, we discretize the problem formulation and only consider the system on the sample times. To do so, we compute the discrete time approximation as discussed in Section 3.2.2 via

$$\begin{aligned}\mathbf{y}^r &= G_{yx}x_0^r + G_{yu}\mathbf{u}^r + G_{yw}\mathbf{w}^r\\ \mathbf{z}^r &= G_{zx}x_0^r + G_{zu}\mathbf{u}^r + G_{zw}\mathbf{w}^r.\end{aligned}$$

Then, this leads to the first order approximations of the updates on the outputs

$$\bar{\mathbf{y}}_{l+1}^r \simeq \bar{\mathbf{y}}_l^r + \mathbf{y}^r \tag{4.55}$$

$$\bar{\mathbf{z}}_{l+1}^r \simeq \bar{\mathbf{z}}_l^r + \mathbf{z}^r \tag{4.56}$$

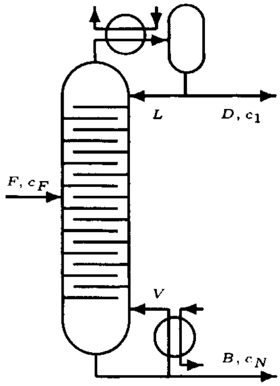
To compute these updates, we linearize the nonlinear optimization problem (4.53) to arrive at the following linearized dynamic optimization problem

$$\text{(LDOP)}\qquad \begin{array}{lcl} \min\limits_{u^r\in\mathbf{R}^{nnu},K\in\mathbf{K}_0} & T_s\sum_{k=1}^n c^T z_k^r & \\ s.t. & \mathbf{y}^r = G_{yx}x_0^r + G_{yu}\mathbf{u}^r + G_{yw}\mathbf{w}^r & \\ & \mathbf{z}^r = G_{zx}x_0^r + G_{zu}\mathbf{u}^r + G_{zw}\mathbf{w}^r & \\ & \nu_j + h_j^T\mathbf{z}^r \le g_j - h_j^T\bar{\mathbf{z}}_l^r & j=1,\ldots,m\\ & r\|\begin{pmatrix} G_{zx}^K F_P & G_{zw}^K F_W\end{pmatrix}^T h_j\|_2 \le \nu_j & j=1,\ldots,m\\ & Z = G_{zx}^K P {G_{zx}^K}^T + G_{zw}^K W {G_{zw}^K}^T & \\ & G_{zx}^K = G_{zx} + G_{zu}K(I-G_{yu}K)^{-1}G_{yx} & \\ & G_{zw}^K = G_{zw} + G_{zu}K(I-G_{yu}K)^{-1}G_{yw} & \end{array} \tag{4.57}$$

Once this subproblem is solved, we can update the outputs via equations (4.56) and (4.55). No sophisticated update mechanism are pursued at this point, but one can image more elaborated schemes including line searches and checks on actual optimality.

4.7 A distillation column under stochastic disturbances

The second example is a binary distillation column, sketched in figure 4.12, which is a small variation on the distillation example in (Ingham *et al.*, 1994). The column has twenty trays, however, note that the number of states is irrelevant for the complexity of the closed-loop model predictive control problem. The complexity is dominated by the number of inputs, outputs, constraints and the horizon length. The model



Performance outputs		
z_1	c_1	Top purity
z_2	c_N	Bottom impurity
Measured outputs		
y_1	c_1	Top purity
y_2	c_N	Bottom impurity
Inputs		
u_1	L	Reflux flow
u_2	V	Boil-up rate
Disturbances		
d_1	F	Feed flow
d_2	c_F	Feed composition
w_i	–	Measurement noise
Miscellaneous		
–	D	Distillate flow
–	L	Reflux flow
–	B	Bottom flow

Figure 4.12: Binary distillation example for load-change scenario

equations are given by

$$\begin{aligned}
M_C\dot{c}_1 &= Vv_2 - (L+D)c_1 \\
M\dot{c}_2 &= L(c_1 - c_2) + V(v_3 - v_2) \\
\vdots \quad & \vdots \quad \vdots \\
M\dot{c}_{n_f} &= L_1(c_{n_f-1} - c_{n_f}) + V_1(v_{n_f+1} - v_{n_f}) + \\
& \quad qF(c_F - c_{n_f}) + (1-q)F(v_F - v_{n_f}) \\
\vdots \quad & \vdots \quad \vdots \\
M_R\dot{c}_n &= L_1 c_n - Bc_n - V_1 v_n
\end{aligned}$$

M denotes the hold-up on the plates, M_R, M_C the hold-up in the reboiler and condensor respectively, V, L the vapor and liquid flows in the enriching section, V_1, L_1 the vapor and liquid flows in the stripping section, D, B the product and bottom flows respectively, n_f denotes the feed tray. The fraction of the light component in the vapor phase v to the fraction of the light component in the liquid phase c by thermodynamic equilibrium

$$v_i(c_i) = \frac{\alpha c_i}{(1 + (\alpha - 1)c_i)} \tag{4.58}$$

The stationary mass balance is given by

$$D = V - L, \quad W = L_1 + F - V_1, \quad V = V_1 + (1-q)F, \quad L_1 = L + qF$$

where q is the thermal quality of the feed. It is customary to control the reflux-ratio instead of the reflux flow, hence

$$R = \frac{L}{D} \Rightarrow L = \frac{R}{R+1}V, \quad D = \frac{1}{R+1}V.$$

The model parameters are given by $F = 100, R = 2, \alpha = 2.2, V_1 = 140, q = 0.9, n = 20, n_f = 10, M = 2, M_R = 5$. The disturbance model is given by

$$\tau \dot{x}_d = -x_d + w_1, \quad q = q_0 + x_d$$

where $\tau = 1.10^6$, and q_0 is the nominal value of q. This closely approximates a drifting disturbance with the frequency content of step shaped disturbance.

4.7.1 The control problem

The column is in some arbitrary initial condition and we seek to maximize the product flow D. In terms of the generalized plant setting, define the performance output z_k, the control inputs u_k, the measured outputs y_k and the disturbance inputs w_k as

$$\begin{aligned} &z_k^1 = c^1(t_k),\ z_k^2 = D(t_k),\ z_k^3 = c^{20}(t_k),\ z_k^4 = B(t_k),\ z_k^5 = u_k^1,\ z_k^6 = u_k^2,\\ &u_k^1 = V_k,\ u_k^2 = R_k,\ y_k^1 = c^1(t_k) + w_k^2,\ y_k^2 = c^{20}(t_k) + w_k^3 \end{aligned}$$

Both the control and disturbance inputs are assumed to be generated by a zero-order hold sampling device, while the outputs are assumed to be sampled at the time instances t_k. The covariance matrix of the disturbances is given by

$$W_k = \mathrm{diag}(10^{-3},\ 2.10^{-2}, 2.10^{-3}), P_0 = \mathrm{diag}(1.10^{-6}x_{ss}, 1.10^{-3})$$

where x_{ss} is the steady-state of the process (excluding the disturbance state). The constraints on the performance outputs are given by a set of upper and lower bounds $z_{\mathrm{lo}} \leq z_k \leq z_{\mathrm{up}} \quad \forall k$ where

$$z_{\mathrm{lo}} = (0.95, 20, 0.01, 0, 100, 1), \quad z_{\mathrm{up}} = (1, 100, 0.06, 200, 200, 4).$$

The constraint matrix H and vector of upper bounds g are given by

$$H = \begin{pmatrix} I \\ -I \end{pmatrix}, \ g = \begin{pmatrix} z_{\mathrm{up}} \\ -z_{\mathrm{lo}} \end{pmatrix}$$

After generating the lifted systems $G_{zx}, G_{zw}, G_{vx}, G_{vw}, G_{vx}, G_{vw}$ in the observer format along the initial reference trajectory defined by $\bar{x}_0^r, \bar{\mathbf{u}}_0^r, \bar{\mathbf{w}}_0^r$ one seeks the optimal update $\mathbf{u}^r$ on this control trajectory such that after optimization the reference trajectory is given as

$$\bar{\mathbf{u}}_1^r = \bar{\mathbf{u}}_0^r + \mathbf{u}^r$$

and the controller $\mathbf{u}^c = K\mathbf{v}^c$. This brings us to the (single stage) closed-loop MPC problem for the column

$$\textbf{(CP)} \qquad \min_{\mathbf{u}^r \in \mathbf{R}^{nn_u}, \nu \in \mathbf{R}^m, K \in \mathbf{K}} \qquad \sum_{k=0}^{N} f^T(z_r^0(k) + z^r(k))$$

$$\begin{aligned} r\nu_j + h_j^T\mathbf{u}^r &\leq g_j - h_j^T\bar{\mathbf{z}}_0^r \\ \mathbf{z}^r &= G_{zu}\mathbf{u}^r \\ K &= \textstyle\sum_{i \geq j \geq 1} E_i K_{ij} E_j^T \\ \|h_j^T \begin{pmatrix} G_{zx}F_P & G_{zw}F_W \end{pmatrix} + h_j^T G_{zu} K &\begin{pmatrix} G_{vx}F_P & G_{vw}F_W \end{pmatrix}\| \leq \nu_j \end{aligned}$$

where F_P, F_W are the factors of P_0, W respectively. The objective vector is given by $f^T = \begin{pmatrix} 0 & -1 & 0 & 0 \end{pmatrix}$.

4.7.2 The closed-loop results

The result of the optimization problem is plotted in figure 4.13. In each of the plots one of the performance outputs z_i is plotted over the full horizon. The feedforward exploits the relatively high top purity during the initial phase to generate a high product flow. The vapor flow is fully used only during the first sample and is immediately reduced in the second sample to avoid violation of the bottom impurity constraint. At the same time, the reflux ratio is kept at its lowest possible level (active constraint) for as long as possible without the top-purity violating its specification. Once the top purity and bottom impurity are at their constraints, the reflux ratio is turned up, and the boil-up rate down to steady-state operation level, $D \approx 52$. No more production is possible with the limitations on the specs and the fixed feed.

The action of the feedback controller is represented via the confidence intervals in the plots. The error bars represent the 95% certainty intervals, which give a visual representation of the envelopes of most likely trajectories when applying the closed-loop control law K. Notice that contrary to standard MPC, the future inputs are also uncertain. This variance follows from the feedback of the measured outputs. To visualize the effect of feedback control in terms of system trajectories 100 open- and closed-loop simulations have been made. The open-loop results are plotted in figure 4.14. It is clear that in the open-loop scenario the constraints both on the top purity and bottom impurity are grossly violated. The closed-loop results are plotted in figure 4.15 and then the realizations precisely lie within the feasible region as predicted by the confidence intervals.

4.8 Chapter summary

In this chapter closed-loop model predictive control was introduced. Closed-loop MPC is a technique in which feedforward control and feedback control are optimized simultaneously in a variational or delta-mode control structure. Starting from the generalized plant framework, a controller was introduced that maximizes the process profit rate by minimizing the back-off to the constraints. The direct formulation of this control problem is non-convex, but it can be rendered convex by using either the Q-parameterization as in Internal Model Control or via an observer-based innovations feedback controller as in the Youla-Kučera parameterization of the closed-loop. This allows to efficiently solve CLMPC for its global optimum by using modern interior point primal dual algorithms. Sequential application of this convex sub-optimization problem is used to control nonlinear dynamical systems. Two small control examples were given to visualize the control and optimization techniques.

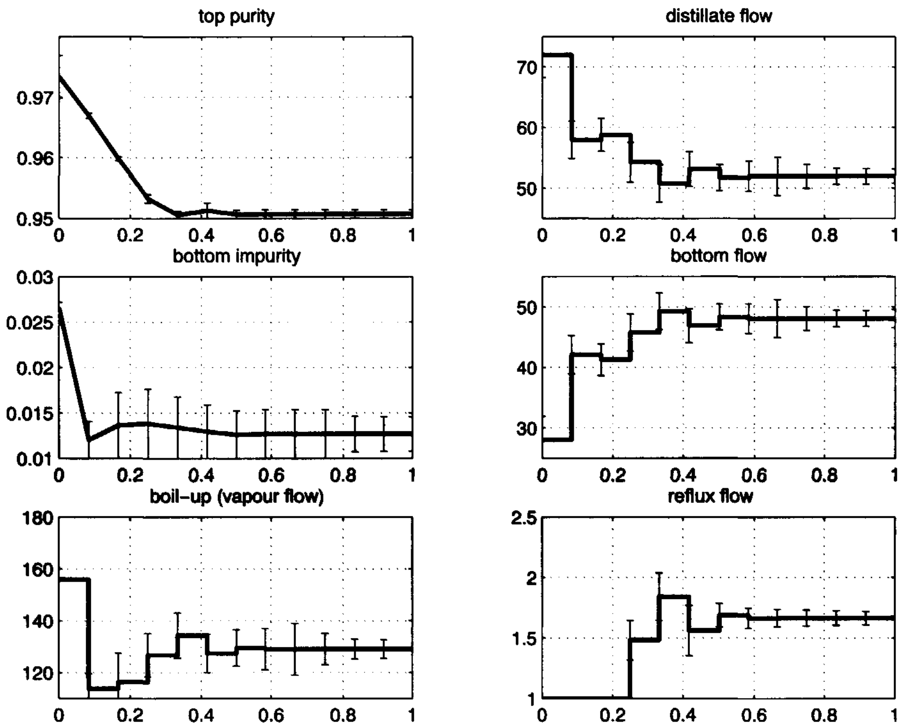


Figure 4.13: Optimal transition of column to new set-point. Solid-lines represent the reference feedforward trajectory. Error-bars represent uncertainty in the transition in closed-loop.

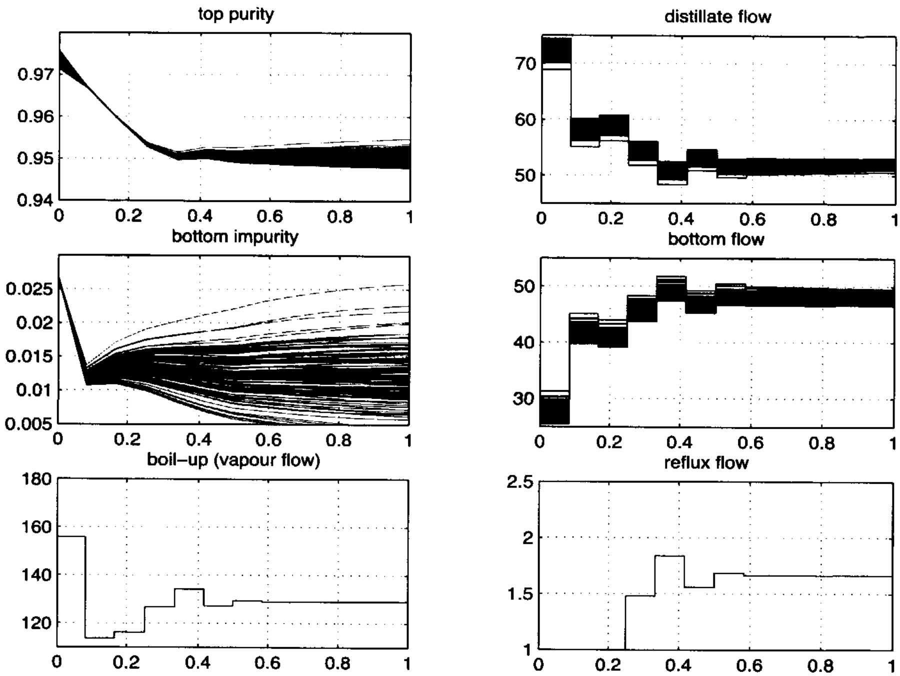


Figure 4.14: 100 open-loop realizations.

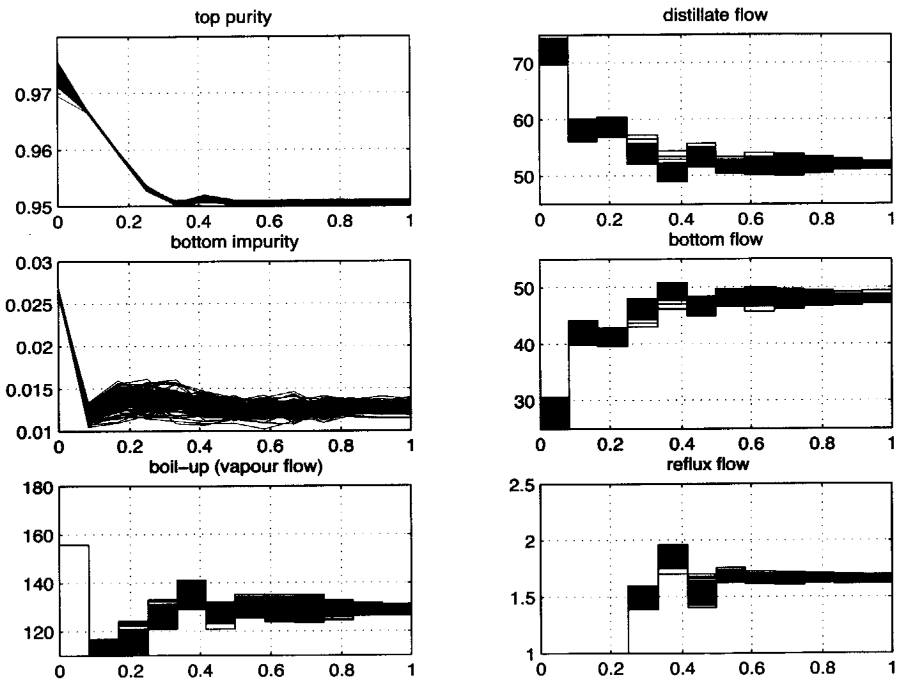


Figure 4.15: 100 closed-loop realizations.

5 Inequality Constrained Linear-Quadratic-Gaussian Control

The main contribution of this chapter is to define an inequality constrained finite horizon Linear-Quadratic-Gaussian (LQG) problem as a computationally cheap alternative to closed-loop MPC. As a by-product, it will be shown that the finite horizon LQG problem is partially dual to the closed-loop MPC problem. This provides new useful insights that can be exploited in numerical algorithms and is crucial in theoretical advances as well.

5.1 Introduction

The use of Kronecker algebra has the serious drawback that the dimension of the matrices involved in multiplication with the Youla parameter Q can become quite large, especially for real process systems which may have quite a number of inputs and outputs. As a result, building the problem may take quite some calculation time already, not to mention a possible huge computational burden in actually solving the problem. Furthermore, each second-order cone constraint

$$r\|h_j^T S_0 + h_j^T S_1 Q S_2\| + h_j^T \mathbf{z} \leq g_j$$

introduces a vector valued Lagrange multiplier. A large number of constraints give rise to many of these Lagrange multiplier vectors blowing up the dimension of the optimization problem in primal-dual algorithms. Despite the efficiency of current convex programming algorithms, even more efficient ways of calculating the solution must be developed to deal with the possibly large dynamical process systems. Two intuitive ideas are under consideration. The first idea is to reduce the number of constraints in the problem by estimating the number of active constraints in each iteration; another idea is to solve a suboptimal problem that is intimately related to the original problem. As it turns out, the finite horizon LQG problem (FHLQG) is a promising candidate for both ideas.

The LQG problem and solution are well known for linear systems, which can be exploited to derive properties of the CLMPC solution. The relation to the problem at hand clearly is the class of Gaussian disturbances and the observer structure of the LQG controller if the innovations approach in the solution of CLMPC is used. This motivates to investigate whether FHLQG control can be used to approximate the closed-loop MPC problem. The first step in this investigation is to derive the LQG solution directly using a lifted systems approach to finite horizon control. This idea is not new in literature, however, contrary to the existing contributions on finite horizon LQG by Furuta *et al.*(1993,1994,1995), we do *not* seek to provide an alternative derivation of the recursive Riccati/state-feedback solution, but instead we seek to compute the optimal output feedback controller K directly from the objective function and lifted system dynamics. Our approach leads via the optimality conditions to an alternative numerical solution and allows us to choose the quadratic weighting matrices corresponding to the CLMPC problem.

5.2 From finite horizon quadratic control to CLMPC

To understand the differences and interrelations between open- and closed-loop MPC, the Linear Quadratic Regulator (LQR) and the Linear Quadratic Gaussian (LQG) controller, the LQR and LQG problems are briefly discussed in the generalized plant set-up below.

5.2.1 The finite horizon Linear Quadratic Regulator

The LQR problem is a deterministic regulation problem that provides the optimal control sequence to steer the dynamic system from an arbitrary initial condition to the origin. Given the dynamics

$$\mathbf{z} = G_{zx}x_0 + G_{zu}\mathbf{u}$$

and an initial condition x_0, the LQR problems amounts to solving the following optimization problem

$$\textbf{(FHLQR)} \quad \min_{\mathbf{u}} \quad (S\mathbf{z} - s)^T R(S\mathbf{z} - s) \tag{5.1}$$

where R is some positive definite matrix, S is a data matrix and s is a data vector providing flexibility in modelling quadratic objective functions in the generalized plant framework.

As an example, consider a tracking problem in which the output $\mathbf{y}$ of some dynamical system must track some desired reference trajectory $\mathbf{y}^r$. To this end, the objective function

$$(\mathbf{y} - \mathbf{y}^r)^T R_y(\mathbf{y} - \mathbf{y}^r)$$

is minimized. To avoid high frequent behavior of the inputs $\mathbf{u}$, an additional penalty is put on the rate of change of $\mathbf{u}$, a common design approach in practical MPC applications (recall the discussion in Chapter 3). The use of relative weights provides the controller with integral action, since it allows the inputs to drift freely from any reference signal $\mathbf{u}^r$ such that $\mathbf{y} - \mathbf{y}^r \to 0$. To be specific, the term

$$(S_u\mathbf{u} - s_u)^T R_u (S_u\mathbf{u} - s_u)$$

is added to the objective function, where R_u is any positive definite penalty matrix for the rate of change of the inputs. It is often desirable to define weights on the rate of change of the inputs instead of the absolute values of the inputs. It is not difficult to see that S_u and s_u can be chosen to include such a scenario. Summarizing, let the performance outputs $\mathbf{z}$ contain $(\mathbf{y}, \mathbf{u})$ again, then with

$$\mathbf{z} = \begin{pmatrix} \mathbf{y} \\ \mathbf{u} \end{pmatrix}, \quad R = \begin{pmatrix} R_y & O \\ O & R_u \end{pmatrix}, \quad S_z = \begin{pmatrix} I & O \\ O & S_u \end{pmatrix}, \quad s_z = \begin{pmatrix} \mathbf{y}^r \\ s_u \end{pmatrix}$$

the **(FHLQR)** (5.1) is obtained.

5.2.2 A lifted perspective on the LQG problem

A direct extension of the **(FHLQR)** problem is obtained by addition of stochastic disturbances in the form of process and measurement noise, which brings us one step closer to closed-loop MPC problem. As before, the measured outputs and controlled variables are given by the following algebraic representation

$$\begin{aligned} \mathbf{y}(\xi) &= G_{yx}x_0(\xi) + G_{yu}\mathbf{u}(\xi) + G_{yw}\mathbf{w}(\xi) \\ \mathbf{z}(\xi) &= G_{zx}x_0(\xi) + G_{zu}\mathbf{u}(\xi) + G_{zw}\mathbf{w}(\xi) \end{aligned} \tag{5.2}$$

where all variables are now stochastic processes. Suppose we are given some reference trajectories $\mathbf{y}^r$ and $\mathbf{z}^r$ and suppose we are also given an initial condition x_0^p which is an estimate of the state. The same quadratic form of the objective function can be used by optimizing the mathematical expectation

$$E(S\mathbf{z}(\xi) - s)^T R(S\mathbf{z}(\xi) - s).$$

We adopt the following strategy to find a control sequence to minimize this objective. Imagine the tracking error build up as

$$\begin{aligned} \mathbf{y}^c(\xi) &= \mathbf{y}(\xi) - \mathbf{y}^r = (\mathbf{y}(\xi) - \mathbf{y}^p) + (\mathbf{y}^p - \mathbf{y}^r) \\ \mathbf{z}^c(\xi) &= \mathbf{z}(\xi) - \mathbf{z}^r = (\mathbf{z}(\xi) - \mathbf{z}^p) + (\mathbf{z}^p - \mathbf{z}^r) \end{aligned}$$

containing a deterministic part $\mathbf{z}^p - \mathbf{z}^r$ which we will reduce by feedforward and a stochastic part $\mathbf{z}(\xi) - \mathbf{z}^r$ which we will reduce with feedback. The predictions are determined by the dynamics

$$\begin{aligned} \mathbf{y}^p &= G_{yx}x_0^p + G_{yu}\mathbf{u}^p + G_{yw}\mathbf{w}^p \\ \mathbf{z}^p &= G_{zx}x_0^p + G_{zu}\mathbf{u}^p + G_{zw}\mathbf{w}^p. \end{aligned}$$

and $\mathbf{w}^p$ is a prediction of the future disturbances. Clearly, $\mathbf{z}^p - \mathbf{z}^r$ is a deterministic quantity and can be minimized by standard open-loop dynamic optimization. The effect of the disturbance will be reduced with a a feedback controller $K \in \mathbf{K}_0$ via the feedback law

$$\mathbf{u}(\xi) - \mathbf{u}^p = K(\mathbf{y}(\xi) - \mathbf{y}^p). \tag{5.3}$$

Note that the difference to the closed-loop MPC problem is that the feedback controller is defined on the future output prediction error $\mathbf{y}(\xi) - \mathbf{y}^p$ instead of the future tracking error $\mathbf{y}(\xi) - \mathbf{y}^r$ (the difference between reference and actual output). Thus the controller is designed to suppress the unknown disturbances while the newly introduced predictions $\mathbf{u}^p, \mathbf{y}^p$ are used to track any desired reference $\mathbf{u}^r, \mathbf{y}^r$ via a feedforward optimization. The numerical values of $\mathbf{u}^p, \mathbf{y}^p$ are related to the expected value (as in standard open-loop MPC) via

$$E\mathbf{u}(\xi) = \mathbf{u}^p, \quad E\mathbf{y}(\xi) = \mathbf{y}^p. \tag{5.4}$$

By combining equations (5.2),(5.3), (5.4) and with the aid of the Youla parameter Q we arrive at

$$\mathbf{z}(\xi) - \mathbf{z}^p = (G_{zx} + G_{zu}QG_{yx})(x_0(\xi) - x_0^p) + (G_{zw} + G_{zu}QG_{yw})(\mathbf{w}(\xi) - \mathbf{w}^p).$$

With these definitions, we can define the following optimization problem

$$\begin{array}{lcc}
\textbf{(FHLQG)} & \min\limits_{\mathbf{u}^p, Q} & E(S\mathbf{z}(\xi) - s)^T R(S\mathbf{z}(\xi) - s) \\
& \multicolumn{2}{c}{\mathbf{z}^p = G_{zx}x_0^p + G_{zu}\mathbf{u}^p + G_{zw}\mathbf{w}^p} \\
& \multicolumn{2}{c}{\mathbf{z}(\xi) - \mathbf{z}^p = (G_{zx} + G_{zu}QG_{yx})(x_0(\xi) - x_0^p) +} \\
& \multicolumn{2}{c}{(G_{zw} + G_{zu}QG_{yw})(\mathbf{w}(\xi) - \mathbf{w}^p).}
\end{array}$$

To evaluate the objective function, we expand the quadratic term

$$\begin{aligned}
&E(S\mathbf{z}^p + S(\mathbf{z}(\xi) - \mathbf{z}^p) - s)^T R(S\mathbf{z}^p + S(\mathbf{z}(\xi) - \mathbf{z}^p) - s) = \\
&E(S(\mathbf{z}(\xi) - \mathbf{z}^p))^T R((S\mathbf{z}(\xi) - \mathbf{z}^p)) + 2(S\mathbf{z}^p - s)^T RSE(\mathbf{z}(\xi) - \mathbf{z}^p) + (S\mathbf{z}^p - s)^T R(S\mathbf{z}^p - s).
\end{aligned} \tag{5.5}$$

and the trivial fact $E(\mathbf{z}(\xi) - \mathbf{z}^p) = 0$ from (5.4) implies that

$$(S\mathbf{z}^p - s)^T RSE(\mathbf{z}(\xi) - \mathbf{z}^p) = 0$$

drops out of the equation (5.5) and therefore the objective function simplifies to

$$E(S(\mathbf{z}(\xi) - \mathbf{z}^p))^T R(S(\mathbf{z}(\xi) - \mathbf{z}^p)) + (S\mathbf{z}^p - s)^T R(S\mathbf{z}^p - s)$$

and as a result, it splits up into two parts 1) a variance part that depends on the controller Q only and 2) a deterministic part which depends on the control signal $\mathbf{u}^p$ only. As a consequence, the FHLQG problem is split into a finite horizon minimal variance problem and a FHLQR problem

$$\min_{Q \in \mathbf{K}} \quad E(S(\mathbf{z}(\xi) - \mathbf{z}^p))^T R(S(\mathbf{z}(\xi) - \mathbf{z}^p)) + \min_{\mathbf{u}^p \in \mathbb{R}^{nn_u}} \quad (S\mathbf{z}^p - s)^T R(S\mathbf{z}^p - s). \tag{5.6}$$

which can be solved independently.

The second problem is equivalent to the FHLQR problem discussed in the previous subsection which has the desirable property that it admits an open-loop as well as a closed-loop state-feedback solution. The important property of the minimal variance solution is that the optimal controller Q has an internal observer/state-feedback structure (separation property), where the state-feedback gain is independent of the stochastic properties of the disturbances (Kwakernaak and Sivan, 1972). It depends only on the quadratic objective function and it coincides with the deterministic regulator solution (certainty equivalence property). Furthermore, the average solution is optimal for the specific sample paths of the stochastic state process as well. This separation property has led in the model predictive control literature to the approach in which the minimal variance problem is generally not considered and only the second deterministic regulation problem is solved under the addition of inequality constraints

$$\begin{aligned} \min_{\mathbf{u}^p} \quad & (S\mathbf{z}^p - s)^T R(S\mathbf{z}^p - s) \\ \mathbf{z}^p = G_{zx}x_0^p &+ G_{zu}\mathbf{u}^p + G_{zw}\mathbf{w}^p \\ h_j^T\mathbf{z}^p \leq g_j&,\ j = 1, \ldots, m \end{aligned} \tag{5.7}$$

where feedback is obtained by solving a similar problem every time sample. A better problem formulation is obtained if the disturbances are not discarded but instead used in the future the prediction. Then, the closed-loop problem is solved once for Q and $\mathbf{u}^p$ to generate future control moves for the whole horizon at once. The difficulty that we are facing is that the inequality constraints must be added to the problem formulation. Solving this problem is discussed next with the introduction of the constrained finite horizon LQG problem.

5.2.3 Inequality constrained FHLQG control

A logical extension to the finite horizon unconstrained LQR and LQG problems is a quadratic regulation problem that includes linear inequality constraints on the process variables

$$\mathbf{z}(\xi) \in \mathcal{P}, \quad \mathcal{P} := \{\zeta \ : \ h_j^T\zeta \leq g_j \text{ for } j = 1, \ldots, m\}.$$

One way of doing so is to simply add the inequality constraints to the FHLQG problem formulation and call this inequality constrained FHLQG. To converge towards the closed-loop MPC problem, the innovations feedback solution is adopted and we make the following substitutions

$$\mathbf{u}^p \mapsto \mathbf{u}^r,\ \mathbf{u}^p \mapsto \mathbf{u}^r,\ \mathbf{y}^p \mapsto \mathbf{y}^r,\ \mathbf{z}^p \mapsto \mathbf{z}^r.$$

The difference between the two approaches is that when there is a fixed reference trajectory $\mathbf{u}^r, \mathbf{y}^r$ we use $\mathbf{u}^p$ to minimize the error $\mathbf{y}^r - \mathbf{y}^p$, while we can also work directly with a free (to be optimized) reference trajectory $\mathbf{u}^r, \mathbf{y}^r$ in which $\mathbf{u}^p, \mathbf{y}^p$

become obsolete. From a mathematical point of view, we only need to consider on approach as the other approach is technically the same. From an application point of view, one is free to choose either method, and changing from one formulation to another is an easy task. As in the closed-loop MPC problem, the inequality constraints are enforced in a probabilistic sense

$$P\{\xi \in \Omega \ : \ h_j^T \mathbf{z}(\xi) \leq g_j \text{ for all } j\} \geq \alpha.$$

As before, it makes sense to approximate this constraint making use of an ellipsoidal relaxation (Z can be assumed non-singular)

$$\mathbf{z}^r + \mathcal{E}_r \subset \mathcal{P}, \quad \mathcal{E}_r = \{\zeta : \ \zeta^T Z^{-1} \zeta \leq r^2\}$$

leading to the optimization problem is given by

$$\begin{aligned} &\min_{\mathbf{u}^r, K} \qquad E(S\mathbf{z}(\xi) - s)^T R(S\mathbf{z}(\xi) - s) \\ \mathbf{z}^r &= G_{zx} x_0^r + G_{zu}\mathbf{u}^r + G_{zw}\mathbf{w}^r \\ r\sqrt{h_j^T Z h_j} &+ h_j^T \mathbf{z}^r \leq g_j \\ Z &= E\mathbf{z}^c(\xi)\mathbf{z}^c(\xi)^T \\ \mathbf{z}(\xi) &= \mathbf{z}^r + \mathbf{z}^c(\xi) \\ \mathbf{z}^c(\xi) &= (G_{zx} + G_{zu}KG_{vx})x_0^c(\xi) + (G_{zw} + G_{zu}KG_{vw})\mathbf{w}^c(\xi) \end{aligned}$$

Unfortunately, this problem is structurally similar to a closed-loop MPC and therefore again a second-order cone problem with equal computational complexity. To reduce this complexity, it makes sense to separate the constrained LQG problem into a minimum variance controller and an inequality constrained deterministic prediction problem. We start with the computation of the minimal variance controller and then proceed to the computation of the optimal trajectory using a standard open-loop MPC, where the back-off to the inequality constraints is determined by the using the minimum variance feedback controller of the first step. This approach will be called constrained finite horizon LQG or CFHLQG.

1) Subproblem CFHLQG^A. The first step is to solve for given F_R and S the minimal variance problem

$$\begin{aligned} &\min_{K \in \mathbf{K}} \qquad \operatorname{tr} S F_R F_Z F_Z^T F_R^T S^T \\ F_Z &= \big((G_{zx} + G_{zu}KG_{vx})F_P \quad (G_{zw} + G_{zu}KG_{vw})F_W\big) \end{aligned}$$

   After this problem is solved for the optimal feedback controller $K^\star$, the corresponding variance matrix $Z^\star$ follows from $F_Z F_Z^T$ and the back-off terms are found using the ellipsoidal relaxation as

$$\nu_j^\star = r\sqrt{h_j^T Z^\star h_j}. \tag{5.8}$$

2) Subproblem CFHLQG^B. In the second step, the optimal transition is computed by solving a deterministic optimization problem. The back-off to the

constraints follows from (5.8).

$$\begin{array}{ll} \min\limits_{\mathbf{u}^r} & (S\mathbf{z}^r - s)^T R(S\mathbf{z}^r - s) \\ \nu_j^\star + h_j^T \mathbf{z}^r \leq g_j, j = 1, \ldots, m & \\ \mathbf{z}^r = G_{zx} x_0^r + G_{zu}\mathbf{u}^r + G_{zw}\mathbf{w}^r & \end{array}$$

The computational complexity of this approach is relatively small. The subproblem CFHLQG^A is a matrix valued unconstrained least-squares problem that can be solved efficiently (see the next section). The subproblem CFHLQG^B is a quadratic program that can be solved using a standard QP solver and its computational cost is equal to that of the standard open-loop MPC. Note that the specific choice of controller in the first step, may lead to infeasibility of the second optimization problem is infeasible. In that case one resorts to constraint relaxation or softening as one would do in the standard MPC case

5.3 The optimality conditions of FHLQG

The additional computational cost to solve the constrained FHLQG problem, on top of open-loop MPC cost, consists of the numerical effort to solve the matrix valued least-squares problem. This problem can be solved by a Riccati recursion, but that would not shed any light on the structural relation to the CLMPC problem. The relation between FHLQG and CLMPC is obtained via the optimality conditions of both problems in the lifted domain, where, contrary to Furuta and Wongsaisuwan (1993,1994,1995), a direct matrix solution is used to compute the controller parameters. Let R be a symmetric positive definite matrix in the FHLQG objective function. Then R has a full rank Cholesky factorization $F_R F_R^T$ and the objective function can be reformulated as

$$\begin{aligned} E(\mathbf{z}^c(\xi)^T S^T R S \mathbf{z}^c(\xi)) &= \operatorname{tr} S F_R E \mathbf{z}^c(\xi) \mathbf{z}^c(\xi)^T F_R^T S^T \\ &= \operatorname{tr} S F_R Z F_R^T S^T \\ &= \operatorname{tr} S F_R F_Z F_Z^T F_R^T S^T. \end{aligned}$$

Because the tracking error dynamics are given by

$$\mathbf{z}^c(\xi) = (G_{zx} + G_{zu} K G_{vx}) x_0^c(\xi) + (G_{zw} + G_{zu} K G_{vw}) \mathbf{w}^c(\xi),$$

the factor F_Z of Z is given by

$$F_Z = \big((G_{zx} + G_{zu} K G_{vx}) F_P \quad (G_{zw} + G_{zu} K G_{vw}) F_W \big)$$

where F_P, F_W are the matrix factors of the variance matrices as before. Hence, the minimal variance problem is of the structural form

$$\min_{X \in \mathbf{K}} \quad \operatorname{tr}(AXB + C)(AXB + C)^T$$

or in Frobenius norm $\|\cdot\|_F$ notation

$$\min_{X \in \mathbf{K}} \quad \|AXB + C\|_F^2$$

The data matrices A, B, C (not related to the state-space system matrices!) and the free parameter X are defined as

$$\begin{aligned} A &= SF_R G_{zu} & B &= \begin{pmatrix} G_{yx}F_P & G_{yw}F_W \end{pmatrix} \\ C &= SF_R \begin{pmatrix} G_{zx}F_P & G_{zw}F_W \end{pmatrix} & X &= Q \end{aligned}$$

when using the Q-parameterization and as

$$\begin{aligned} A &= SF_R G_{zu} & B &= \begin{pmatrix} G_{ve}F_P & G_{vw}F_w \end{pmatrix} \\ C &= SF_R \begin{pmatrix} G_{zx}F_P & G_{zw}F_W \end{pmatrix} & X &= K \end{aligned}$$

when using the innovations feedback approach. Note that both approaches are fully exchangeable at this point. This problem can be solved after vectorization (using Kronecker algebra) by solving the corresponding normal equations numerically. Again, this would lead to a very high dimension of the matrices involved and is therefore not efficient. This is avoided in a direct solution using matrix manipulations without any vectorization.

5.3.1 The optimality conditions for unstructured controllers

In the first step to find a solution, the lower-block triangular structure of X is ignored to arrive at the first-order optimality condition of the minimum variance problem. The objective function has the following quadratic form

$$f(X) = \operatorname{tr}(AXB + C)(AXB + C)^T, \quad f : \mathbf{R}^{nn_u \times nn_y} \to \mathbf{R}$$

and for the problem of minimizing $f(X)$ we assume that A has full column rank and B full row rank. These regularity conditions are not that restrictive in practise. Then, since f is convex and differentiable we can set its gradient with respect to X to zero to find the first order necessary and sufficient optimality condition. We define the (Fréchet) derivative of f as in (Rudin, 1976; Luenberger, 1969). If there exists a matrix $M \in \mathbf{R}^{nn_u \times nn_y}$ such that for any perturbation X at X_0 we have

$$f(X_0 + X) - f(X_0) = \langle M, X \rangle + o(\|X\|_F)$$

then f is differentiable at X_0 and one writes

$$\partial_X f(X_0) = M.$$

The inner product is given by $\langle M, X \rangle = \operatorname{tr} M^T X$. Exploiting the definition we find

$$\begin{aligned} f(X_0 + X) &= \operatorname{tr}(A(X_0 + X)B + C)(A(X_0 + X)B + C)^T \\ &= \operatorname{tr}(AX_0B + C + AXB)(AX_0B + C + AXB)^T \\ &= \operatorname{tr}(AX_0B + C)(AX_0B + C)^T + \operatorname{tr}(AXB)(AX_0B + C)^T + \\ &+ \operatorname{tr}(AX_0B + C)(AXB)^T + \operatorname{tr} AXB(AXB)^T \end{aligned}$$

and hence after rearranging we find

$$f(X_0 + X) - f(X_0) = \operatorname{tr} 2(A^T(AX_0B + C)B^T)^T X + \|AXB\|_F^2$$

such that

$$M = 2A^T(AX_0B + C)B^T. \tag{5.9}$$

In some old papers by Athans and Schweppe (1965) and Geering (1976) M is referred to as the *matrix* gradient. The first-order optimality is condition is then formulated as follows. If for a given $X^\star$ we have that

$$\langle \partial_X f(X^\star), X \rangle = 0 \quad \text{for all } X \in \mathbf{R}^{nn_u \times nn_y} \tag{5.10}$$

then, by convexity of f, $X^\star$ is a global minimizer of f. Condition (5.10) requires that we must solve the matrix equation

$$A^T(AX^\star B + C)B^T = 0$$

for the unstructured $X^\star$. Since the problem is convex in X, the solution to the first-order optimality necessary condition is sufficient as well for finding the optimal solution and the solution follows immediately

$$X^\star = -(A^TA)^{-1}A^TCB^T(BB^T)^{-1}.$$

The inverses exist due to the rank conditions on the data A, B.

5.3.2 The optimality conditions for structured controllers

The solution for the unstructured controller is unfortunately not very useful since the controller anticipates future measurements thereby conflicting with physical realizability. In this section the previous analysis is extended to restrict our controllers to be lower-block triangular. Reconsider the objective function

$$f(X) = \operatorname{tr}(C + AXB)(C + AXB)^T, \quad f : \mathbf{R}^{nn_u \times nn_y} \to \mathbf{R}$$

and the problem of minimizing the $f(X)$ by choosing a lower block triangular matrix X_0 constructed as

$$X_0 = \sum_{i,j} E_i X_{0ij} E_j^T, \quad X_{0ij} \in \mathbf{R}^{n_u \times n_y} \tag{5.11}$$

where we use the short hand $\sum_{i,j} = \sum_{i=1}^{n} \sum_{j=1}^{i}$ and $E_i^T = (O, \ldots, O, I, O, \ldots, O)$. In this case, we seek the partial derivatives of f at X_0 for the structured block perturbations X_{ij}

$$X = \sum_{i,j} E_i X_{ij} E_j^T, \quad X_{ij} \in \mathbf{R}^{n_u \times n_y}. \tag{5.12}$$

Following the same steps as before one finds

$$\begin{aligned} f(X_0+X)-f(X_0) &= \langle M, X\rangle + o(\|X\|_F) \\ &= \langle M, \sum_{i,j} E_i X_{ij} E_j^T\rangle + o(\|X\|) \\ &= \sum_{i,j}\langle M_{ij}, X_{ij}\rangle + o(\|X\|_F) \end{aligned}$$

where we have introduced

$$M_{ij} = E_i^T M E_j$$

and used the relations

$$\begin{aligned} \langle M, E_i X_{ij} E_j^T\rangle &= \operatorname{tr} M^T E_i X_{ij} E_j^T = \operatorname{tr} E_j^T M^T E_i dX_{ij} \\ &= \operatorname{tr}(E_i^T M E_j)^T X_{ij} = \langle E_i^T M E_j, X_{ij}\rangle = \langle M_{ij}, X_{ij}\rangle. \end{aligned}$$

The partial derivatives of f at X_0 are then given by

$$\partial_{X_{ij}} f(X_0) = M_{ij}$$

and related to the derivative of f at X_0 via

$$\partial_{X_{ij}} f(X_0) = M_{ij} = E_i^T M E_j = E_i^T \partial_X f(X_0) E_j.$$

Reusing our formula (5.9) for M we find for the partial derivatives

$$M_{ij} = 2E_i^T A^T (A\sum_{i,j} E_i X_{ij} E_j^T B + C) B^T E_j$$

and the first order optimality condition is then given as follows. If a structured solution $X^\star$ satisfies

$$E_i^T A^T (A\sum_{i,j} E_i X_{ij}^\star E_j^T B + C) B^T E_j = 0 \quad \text{for } 1 \le i \le j \le n,$$

then it is a global minimizer of f. These equations form a coupled set of linear equations that has a more compact representation by observing that the condition is equivalent to requiring that the matrix

$$A^T (A(\sum_{i,j} E_i X_{ij}^\star E_j^T) B + C) B^T$$

is zero only on the lower block triangular. If we introduce the matrix Λ with an upper-block triangular form that is zero on the diagonal we can compactly write the first order optimality condition as follows. If there exists a Λ of the form

$$\Lambda = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E_i \Lambda_{ij} E_j^T \tag{5.13}$$

such that

$$A^TAX^\star BB^T + A^TCB^T = \Lambda, \tag{5.14}$$

then $X^\star$ is a global minimizer of f. Summarizing, to find the solution to the optimization problem, the set of equations (5.11),(5.13) and (5.14) need to be solved simultaneously.

5.3.3 A Cholesky solution

In this section we will solve equations (5.11),(5.13) and (5.14) for the structured controller X and the Lagrange multiplier Λ. The crucial property of the problem is that X is lower and Λ is upper block triangular, which suggest the following decoupling procedure. Compute the following Cholesky factorizations, (Horn and Johnson, 1999; Golub and van Loan, 1996), of the left and right factors of X

$$UU^T = A^TA$$

and

$$LL^T = BB^T$$

where U is upper triangular and L is lower triangular. Both factors can be obtained using the same algorithm by proper permutation. Since A is assumed to have full column rank and B has full row rank, these Cholesky factors exist and they are unique. As a consequence we arrive at

$$UU^TXLL^T + A^TCB^T = \Lambda.$$

Both U and L are nonsingular and their inverses, which are again upper and lower triangular respectively, can efficiently be computed. This can be exploited as follows. Define the matrix

$$\tilde{C} := -U^{-1}A^TCB^TL^{-T}$$

and partition $\tilde{C}$ in a lower block triangular matrix $\tilde{C}^l$ and a upper block triangular matrix $\tilde{C}^u$

$$\tilde{C}^l = \sum_{i=1}^{n}\sum_{j=1}^{i} E_i^T\tilde{C}E_j$$

and

$$\tilde{C}^u = \sum_{i=1}^{n}\sum_{j=i+1}^{n} E_i^T\tilde{C}E_j.$$

Notice that these matrices do not have any overlap in their nonzero blocks. Hence, the optimality condition, (5.14) reduces to

$$U^TXL - U^{-1}\Lambda L^{-T} = -U^{-1}A^TCB^TL^{-T} = \tilde{C}^l + \tilde{C}^u$$

This decomposes the problem into upper and lower triangular algebraic problems

$$U^T X L = \tilde{C}^l, \quad U^{-1} \Lambda L^{-T} = -\tilde{C}^u.$$

The structured solutions for X and Λ are obtained as

$$X = U^{-T} \tilde{C}^l L^{-1}, \quad \Lambda = -U \tilde{C}^u L^T$$

which are solved recursively for numerical efficiency.

Remark 22 *Solution in the case of singular data matrices* A limitation of the above mentioned technique to solve the optimality condition is that A and B must have full column and full row rank respectively for the (complete) Cholesky factor to exist. In principle, these conditions are pretty standard in LQG designs where one needs to solve the estimation and control Riccati equations. Suppose (W, V) are the process and measurement noise covariance matrices and (Q, R) are the state and input weights in the quadratic objective. A sufficient condition for the solution to the Riccati equation to exist is that the measurement noise V and the input weight R are both positive definite. If $V > 0$, then the full row rank condition on B is indeed satisfied. A similar argument can be uses to guarantee full column rank of A, since $R > 0$ implies that the performance channel has a feedtrough $\sqrt{R}$. However, not every performance channel z_j has a feed-through from the input u and consequently not every A is of full column rank. Due to the parameter Λ appearing on the right hand side of equation (5.14), it is not straightforward to solve the singular case, hence we turn to regularization methods by a perturbation of the product

$$A^T A + \varepsilon I$$

where $\varepsilon > 0$ is some small number to render it positive definite. Note that a Kronecker solution can always be devised since then the problem is a vector-valued quadratic program which is easily regularized by augmenting the objective function without distorting optimality of the original problem. This augmentation in the matrix case is more involved and the suggestion above is the easy way out, however, it does distort optimality in the sense that the solution to this new problem is no longer optimal for the original problem. □

5.4 The optimality conditions of CLMPC

The results of the previous section will now be used in deriving necessary optimality conditions of the closed-loop MPC problem. Recall that the CLMPC problem was defined as

$$\textbf{(CLMPC)} \qquad \begin{array}{cc} \min\limits_{\mathbf{u}_0^r \in \mathbf{R}^{nn_u}, \nu \in \mathbf{R}^m, K \in \mathbf{K}_0} & f(\mathbf{z}_0^r) \\ \mathbf{z}_0^r = G_{zx} x_0^r + G_{zu} \mathbf{u}_0^r + G_{zw} \mathbf{w}_0^r & \\ \nu_j + h_j^T \mathbf{z}_0^r \leq g_j, \; j = 1, \ldots, m & \\ Z(K) = G_{zx}^K P G_{zx}^{K\,T} + G_{zw}^K W G_{zw}^{K\,T} & \\ r\sqrt{h_j^T Z(K) h_j} \leq \nu_j, \; j = 1, \ldots, m & \end{array} \qquad (5.15)$$

Suppose we square the last constraint and assume for ease of presentation that $r = 1$ and we discard the dynamics and imagine we search directly over $\mathbf{z}_0^T$. To avoid confusion, we make the substitution

$$\mathbf{z}_0^T \mapsto \mathbf{p}.$$

This leads to the following nonlinear optimization problem (NOP) of the form

$$\text{(NOP)} \qquad \begin{array}{rl} \min\limits_{\mathbf{p}\in\mathbf{R}^{nn_p},\nu\in\mathbb{R}^m,X\in\mathbf{R}^{nn_u\times nn_y}} & f(\mathbf{p}) \\ \nu_j + h_j^T\mathbf{p} \le g_j, \quad j = 1,\dots,m & \\ X = \sum_{i,j} E_i X_{ij} E_j^T & \\ h_j^T(AXB + C)(AXB + C)^T h_j \le \nu_j^2, \quad j = 1,\dots,m & \end{array} \tag{5.16}$$

where the data matrices are given by

$$\begin{array}{ll} A = G_{zu} & B = \begin{pmatrix} G_{ve}F_P & G_{vw}F_W \end{pmatrix} \\ C = \begin{pmatrix} G_{zx}F_P & G_{zw}F_W \end{pmatrix} & X = K \end{array}$$

Note that this is a non-convex optimization problem due to the presence of ν_j and ν_j^2 in the inequality constraints. The reason for looking at this formulation of the CLMPC problem is that one can immediately recognize the quadratic form that also appears in the FHLQG problem. By exploiting Lagrangian duality we wish to go from constraints on the variance of the controlled variables to a control problem with a minimal variance objective for which we known the solution (LQG!). The Lagrangian function for (NOP) is defined as

$$\mathbf{L}(\mathbf{p}, X, \nu, \Lambda, \lambda, \eta) = f(\mathbf{p}) + \sum_{j=1}^m \lambda_j h_j^T \mathbf{p} + \sum_{j=1}^m (\lambda_j \nu_j - \eta_j \nu_j^2) - \sum_{j=1}^m \lambda_j g_j$$
$$+ \sum_{j=1}^m \eta_j h_j^T (AXB + C)(AXB + C)^T h_j + \operatorname{tr} \Lambda^T \Big(\sum_{i,j} E_i X_{ij} E_j^T - X\Big)$$

The Karush-Kuhn-Tucker first-order necessary optimality conditions (Luenberger, 1973) for (NOP) are given as follows. Suppose the solution $(\mathbf{p}^\star, X^\star, \nu_j^\star)$ is a local minimizer of (NOP) and at this feasible point a constraint qualification is satisfied. Then, there exist Lagrange multipliers

$$\lambda \in \mathbf{R}_+^m, \quad \mu \in \mathbf{R}_+^m, \quad \Lambda \in \mathbf{R}^{nn_u \times nn_y}$$

such that

$$\partial_{\mathbf{p},X,\nu} L(\mathbf{p}^\star, X^\star, \nu^\star, \Lambda, \lambda, \eta) = 0$$
$$\sum_{j=1}^m \lambda_j (\nu_j^\star + h_j^T \mathbf{p}^\star - g_j) = 0$$
$$\sum_{j=1}^m \mu_j (h_j^T (AX^\star B + C)(AX^\star B + C)^T h_j - \nu_j^{\star 2}) = 0.$$

By decomposing the first condition into parts for the different optimization variables, we arrive at the vector/scalar valued gradients

$$\partial_{\mathbf{p}}\mathbf{L}(\mathbf{p}^\star, X^\star, \nu^\star, \Lambda, \lambda, \eta) = \partial_{\mathbf{p}} f(\mathbf{p}^\star) + \sum_{j=1}^{m} \lambda_j h_j^T = 0 \tag{5.17}$$

$$\partial_{\nu_j}\mathbf{L}(\mathbf{p}^\star, X^\star, \nu^\star, \Lambda, \lambda, \eta) = \lambda_j - 2\eta_j\nu_j = 0.$$

The first of these two, (5.17), is a typical optimality condition that appears in any linear programming problem if we use a linear objective function

$$c^T = \partial_{\mathbf{p}} f(\mathbf{p}^\star)$$

and shows how perturbations in the constraints locally change the objective value. For the matrix variable X, the gradient matrix of the Lagrangian is again derived using gradient techniques as in Section 5.3. Because

$$\langle \Lambda, E_i X_{ij} E_j^T \rangle = \operatorname{tr} \Lambda^T E_i X_{ij} E_j^T = \operatorname{tr}(E_i^T \Lambda E_j)^T X_{ij} = \langle E_i^T \Lambda E_j, X_{ij} \rangle$$

it follows that the partial derivative of the Lagrangian with respect to X_{ij} is given by

$$\partial_{X_{ij}}\mathbf{L}(\mathbf{p}^\star, X^\star, \nu^\star, \Lambda, \lambda, \eta) = E_i^T \Lambda E_j.$$

Hence, the Lagrangian is stationary with respect to the blocks X_{ij} if

$$E_i^T \Lambda E_j = 0, \quad 1 \le i \le j \le n$$

which implies that Λ must be upper block triangular with zero blocks on the diagonal. This is compactly represented by the requirement that

$$\Lambda = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E_i \Lambda^{ij} E_j^T$$

This is the same condition as was derived in the previous section. The gradient with respect to X gives

$$\partial_X \mathbf{L}(\mathbf{p}^\star, X^\star, \nu^\star, \Lambda, \lambda, \eta) = A^T \Big(\sum_{j=1}^{m} \eta_j h_j h_j^T\Big)(AX^\star B + C)B^T - \Lambda = 0. \tag{5.18}$$

This can easily be seen by the following equalities

$$h_j^T(AXB+C)(AXB+C)^T h_j = \operatorname{tr} h_j^T(AXB+C)(AXB+C)^T h_j = \|h_j^T AXB + h_j^T C\|_F^2$$

such that the derivative of this term is given by

$$(h_j^T A)^T (h_j^T AXB + h_j^T C)B^T = A^T h_j h_j^T (AXB + C)B^T$$

the rest follows from linearity.

Note that (5.18) equals the first-order optimality condition of the previous section. Since the Lagrange multipliers for the linear inequality constraints are positive or zero, that is $\eta_j \geq 0$ for all j, we can factor the positive semi-definite weighting matrix obtained by the positive sum of the rows of the constraint matrix

$$R = R^T = \sum_{j=1}^{m} \eta_j h_j h_j^T = F_R F_R^T.$$

Notice that as far as variance is concerned, it *appears* that we have minimized the objective function

$$\min_{X \in \mathbf{K}} \quad \mathrm{tr}(AXB + C)^T R (AXB + C).$$

which was shown in Section 5.3.1 (where one absorbs the factor of the weight R in A). This implies that the optimal solution X of the closed-loop MPC problem is also optimal for *some* finite horizon LQG problem. The actual objective function that corresponds to this LQG problem is unknown because the Lagrange multipliers η_j are unknown (if this was not the case, the closed-loop MPC problem could be solved directly). Nevertheless, this result is very important and will be formalized in the receding horizon implementation discussed in Chapter 7.

5.5 A heuristic algorithm

The second idea that was mentioned in the introduction of this chapter is to solve the optimality conditions of the CLMPC problem using a sequence of minimum variance problems that can each be solved very efficiently although there is no guarantee of convergence. Besides the argument that a matrix factorization approach is computationally much less demanding than a Kronecker algebraic approach, there is another application of this algorithm. A receding horizon implementation could be considered online, in which the objective value is iteratively improved at low computational cost, hence one iterates in time. This approach will not be pursued in depth, but will be illustrated by means of an example. Consider the following basic algorithm

1) Set a counter $k = 0$, set the initial optimal cost $\gamma_0 = \infty$, fix any forgetting factor $\alpha \in (0,1)$ and fix the weighting matrix of the finite horizon LQG problem to

$$R_z^0 = \sum_{j=1}^{m} \eta_j^0 h_j h_j^T$$

where $\eta_j^0 = 1/\sqrt{m}$. Define $\eta_j^\star = \eta_j^0$.

2) Increase the counter $k = k + 1$.

3) Solve the corresponding minimal variance problem

$$\begin{array}{ll} \min\limits_{X^k \in \mathbf{K}} & \operatorname{tr} F_R F_Z F_Z^T F_R^T \\ F_Z = \big((G_{zx} + G_{zu} X^k G_{vx}) F_P & (G_{zw} + G_{zu} X^k G_{vw}) F_W\big) \end{array}$$

using the innovations approach. Suppose this problem has been solved for the optimal X^k, then the optimal variance matrix Z^k is also known, and the back-off terms using the ellipsoidal relaxation are readily computed by

$$\nu_j^k = r\sqrt{h_j^T Z^k h_j}$$

4) Recall the discussion of Constrained Finite Horizon LQG in which the second step was to compute the optimal feedforward signal for a given feedback controller and hence given back-off's to the constraints. Given this back-off vector ν^k, compute the optimal solution of the dual linear program

$$\begin{array}{cc} \min\limits_{\lambda^k} & \sum_{j=1}^m (\nu_j^k - g_j)\lambda_j^k \\ \lambda_j^k \geq 0 & \\ \sum_{j=1}^m h_j \lambda_j^k + c = 0 & \end{array}$$

and primal linear program related to the original problem

$$\begin{array}{rcc} \gamma_k = & \min\limits_{\mathbf{z}} & c^T \mathbf{z}^k \\ & \nu_j^k + h_j^T \mathbf{z}^k \leq g_j & \end{array}$$

5) If $\gamma_k \leq \gamma_{k-1}$, scale the Lagrange multipliers for practical reasons

$$\tilde{\eta}_j^k = \begin{cases} \frac{\lambda_j^k}{2\nu_j^k} & \text{if} \quad \nu_j^k > 0 \\ 0 & \text{if} \quad \nu_j^k = 0 \end{cases}$$

and normalize the coefficients

$$\tilde{\eta}^k = \tilde{\eta}^k / \|\tilde{\eta}^k\|$$

to avoid radical changes (ratios can be quite large for some indices). Then, update the coefficients to the weighting matrix via the rule

$$\eta_j^k = \alpha \eta_j^{k-1} + (1-\alpha)\tilde{\eta}_j^k$$

and set the best value so far

$$\eta_j^\star = \eta_j^k, \quad \text{and} \quad \gamma^\star = \gamma_k$$

If $\gamma_k > \gamma_{k-1}$ set

$$\eta_j^k = \alpha \eta_j^\star + (1-\alpha)\eta_j^{k-1}$$

such that the weighting parameters converge to the last good value $\eta^\star$.

6) Compute the new weighting matrix as

$$R^k = \sum_{j=1}^{m} \eta_j^k h_j h_j^T$$

Note again that the coefficients can also be normalized in this step since scaling does not change the optimal LQG controller.

7) Stop if little progress is made, otherwise continue to step 2.

To illustrate the recursive method, the algorithm has been implemented on the column example of section 4.7 to solve the optimization problem **CO**. The result is plotted in figure 5.1, where the solid line represents the iterative improvement of $\eta^\star$. The optimum is found after 24 improving steps. The power of such a recursion could however be increased considerably by utilizing this approach in a receding horizon fashion.

5.6 Chapter summary

Current state of the art software for solving the closed-loop MPC problem requires that the optimization problem is vectorized. This blows up the size of the problem considerably and can prohibit on-line application for systems with many many inputs and outputs and long prediction horizons. In this chapter an alternative closed-loop prediction control strategy was developed called constrained finite horizon LQG control. In the first stage of this approach a sub-optimal feedback controller is designed that determines the amount of back-off in a subsequent feedforward optimization problem. The optimality conditions of CLMPC provides the relation between the two approaches and also reveals how iterative use of CFHLQG can be used to solve the CLMPC problem.

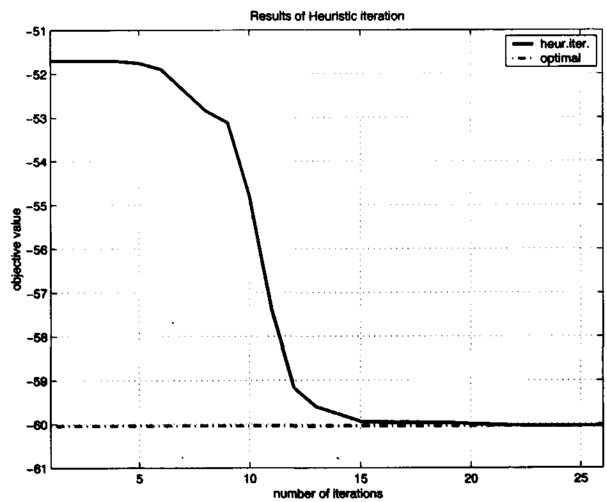


Figure 5.1: Optimal result for the distillation column. Dashed-dotted line: optimal value via conic solver, solid line: solution of heuristic iteration.

6 The Stationary Solution

In this chapter the stationary solution to the closed-loop MPC problem is presented that solves the problem of finding a linear time invariant controller and an optimal steady-state operating condition. This controller is chosen to minimize the amount of back-off to the constraints in directions that are economically attractive.

6.1 Introduction and problem formulation

The time-varying solution presented in the previous chapters allows for transitions in the state-space for both batch and continuous processes. In the case of continuous processes, the endpoint of such a transition usually is an optimal steady-state operating point. In this chapter, a method is presented to find these optimal steady-steady states as well as the feedback controller needed to keep the process safe within its constraints. In this chapter we will discuss the linear time-invariant only, but in case we want to find optimal steady states $(\bar{u}_0^r, \bar{v}_0^r, \bar{w}_0^r, \bar{x}_0^r, \bar{y}_0^r, \bar{z}_0^r)$ for the original nonlinear system in which the time derivative is zero $\dot{\bar{x}}_0^r = 0$

$$
\begin{aligned}
0 &= f(0, \bar{x}_0^r, \bar{v}_0^r, \bar{u}_0^r, \bar{w}_0^r) \\
\bar{y}_0^r &= C_y^x \bar{x}_0^r + C_y^v \bar{v}_0^r + D_y^u \bar{u}_0^r + D_y^w \bar{w}_0^r \\
\bar{z}_0^r &= C_z^x \bar{x}_0^r + C_z^v \bar{v}_0^r + D_z^u \bar{u}_0^r + D_z^w \bar{w}_0^r
\end{aligned}
$$

the techniques in this chapter must be used iteratively in precisely the same way as for the time-varying case as was discussed in Section 4.6 where the optimal solution of the convex LTI problem must be used as an update on the steady state of the nonlinear system.

$$
\begin{aligned}
&\bar{u}^r \simeq \bar{u}_0^r + u^r, \quad \bar{v}^r \simeq \bar{v}_0^r + v^r, \quad \bar{w}^r \simeq \bar{w}_0^r + w^r \\
&\bar{x}^r \simeq \bar{x}_0^r + x^r \quad \bar{y}^r \simeq \bar{y}_0^r + y^r, \quad \bar{z}^r \simeq \bar{z}_0^r + z^r.
\end{aligned}
$$

Hence, this update is computed on the basis of the linearization of the nonlinear equations above, that is

$$\begin{aligned} 0 &= \partial_{\bar{x}} f|_0\, x^r + \partial_{\bar{v}} f|_0\, v^r + \partial_{\bar{u}} f|_0\, u^r + \partial_{\bar{w}} f|_0\, w^r \\ y^r &= C_y^x x^r + C_y^v v^r + D_y^u u^r + D_y^w w^r \\ z^r &= C_z^x x^r + C_z^v v^r + D_z^u u^r + D_z^w w^r \end{aligned}$$

where

$$\partial_\star f|_0 = \partial_\star f(\bar{x}_0^r, \bar{v}_0^r, \bar{u}_0^r, \bar{w}_0^r).$$

Upon discretization on the sample times of these dynamics we arrive at the dynamics given by

$$\begin{pmatrix} x_{k+1}(\xi) \\ z_k(\xi) \\ y_k(\xi) \end{pmatrix} = \left(\begin{array}{c|cc} A & B^w & B \\ \hline C_z & 0 & D_z \\ C & D^w & 0 \end{array}\right) \begin{pmatrix} x_k(\xi) \\ w_k(\xi) \\ u_k(\xi) \end{pmatrix}$$

We suppose that (A, B) is stabilizable and (A, C) is detectable. As usual, any dynamic disturbance model is assumed to be incorporated in the plant model. Let $u_k(\xi)$ be a Gaussian stationary reference process with mean $\hat{u}$. The system's steady-state response processes $x_k(\xi)$, $z_k(\xi)$ have expectations $\hat{x}$, $\hat{z}$ which are determined by

$$\hat{x} = A\hat{x} + B\hat{u}, \quad \hat{z} = C_z\hat{x} + D_z\hat{u}.$$

In the stationary case, we require the reference to be in rest, and therefore all reference signals are assumed to be constant in time. For given constant reference trajectories

$$x_k^r \equiv x^r, \quad u_k^r \equiv u^r, \quad w_k^r \equiv w^r, \quad y_k^r \equiv y^r, \quad z_k^r \equiv z^r$$

assume that the system is controlled as

$$\begin{pmatrix} s_{k+1}(\xi) \\ u_k(\xi) \end{pmatrix} = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \begin{pmatrix} s_k(\xi) \\ y_k(\xi) - y^r \end{pmatrix} + \begin{pmatrix} 0 \\ u^r \end{pmatrix}.$$

In a standard fashion by setting

$$y_k^c(\xi) = y_k(\xi) - y^r \text{ and } u_k(\xi) = u_k^c(\xi) + u^r$$

we can clearly describe the controlled system as the interconnection of

$$\left(\begin{array}{c} x_{k+1}^c(\xi) \\ \hline z_k(\xi) \\ y_k^c(\xi) \end{array}\right) = \left(\begin{array}{c|cc} A & B^w & B \\ \hline C_z & 0 & D_z \\ C & D^w & 0 \end{array}\right) \left(\begin{array}{c} x_k^c(\xi) \\ \hline w_k(\xi) \\ u_k^c(\xi) \end{array}\right)$$

with the LTI controller

$$K: \begin{pmatrix} s_{k+1}(\xi) \\ u_k^c(\xi) \end{pmatrix} = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \begin{pmatrix} s_k(\xi) \\ y_k^c(\xi) \end{pmatrix}$$

resulting in the closed-loop system

$$\left(\begin{array}{c} x^c_{k+1}(\xi) \\ s_{k+1}(\xi) \\ \hline z_k(\xi) \end{array}\right) = \left(\begin{array}{cc} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{array}\right) \left(\begin{array}{c} x^c_k(\xi) \\ s_k(\xi) \\ \hline w_k(\xi) \end{array}\right)$$

where the closed-loop system matrices are given by

$$\left(\begin{array}{cc} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{array}\right) = \left(\begin{array}{cc|c} A + BD_cC & BC_c & B^w + BD_cD^w \\ B_cC & A_c & B_cD^w \\ \hline C_z + D_zD_cC & D_zC_c & D_zD_cD^w \end{array}\right). \tag{6.1}$$

If the reference inputs y^r, u^r are chosen to satisfy, for some not necessarily unique x^r, the open-loop equilibrium relations

$$x^r = Ax^r + Bu^r, \quad y^r = Cx^r + Fw^r, \quad z^r = C_zx^r + D_zu^r,$$

it is easy to see with these formulas that the expectation of the closed-loop state-process is given by

$$(\hat{x}, \hat{s}) = (x^r, 0).$$

Hence, all possible output expectations $\hat{z}$ equal the reference values z^r of the *controlled* system which are actually parameterized as

$$S_r := \{z^r : \exists (u^r, x^r),\ z^r = C_zx^r + D_zu^r,\ x^r = Ax^r + Bu^r\} \tag{6.2}$$

For given reference inputs y^r, u^r we assume (as in the LTV case) that the cost of the closed-loop steady-state response is measured by some smooth function f of z^r, and for *illustrational* reasons we restrict f to be linear

$$f(z^r) = c^T z^r$$

motivated by the typical objective functions in a chemical plant economy (maximal feed, minimal utility costs etc.). The reference inputs are chosen to minimize the cost by off-line optimization. It is emphasized that our controller K and our reference signal u^r are not re-optimized with every new measurement sample. As before, the output process $z(\xi)$ should be contained in some polytope with probability larger than some user-chosen level α

$$P(z \in \mathcal{P}) \geq \alpha \ \text{ where } \ \mathcal{P} = \{\zeta : \ H^T\zeta \leq g\}.$$

Satisfaction of this constraint is certainly influenced by suitable controller choices. This leads us to the following problem formulation:

Find a stabilizing LTI controller and reference inputs y^r, u^r such that the stationary output process of the controlled system satisfies $P(z \in \mathcal{P}) \geq \alpha$ and $c^T z^r$ is minimized.

Let us recall that the Gaussian processes $x(\xi)$ and $z(\xi)$ are fully described by their means $\hat{x}, \hat{z}$ and auto-covariance matrices X, Z. Indeed, let

$$X_k = E(x_k(\xi) - \hat{x}_k)(x_k(\xi) - \hat{x}_k)^T$$

then one immediately obtains the matrix valued dynamical system to propagate the variance matrices

$$\begin{aligned} X_{k+1} &= A_{cl} X_k A_{cl}^T + B_{cl} B_{cl}^T \\ Z_k &= C_{cl} X_k C_{cl}^T + D_{cl} D_{cl}^T \end{aligned}$$

If the closed-loop is striclty stable, $\rho(A_{cl}) < 1$, then the matrix state X_k of this system converges asymptotically to its equilibrium value X satisfying the Lyapunov type of equations

$$\begin{aligned} X &= A_{cl} X A_{cl}^T + B_{cl} B_{cl}^T \\ Z &= C_{cl} X C_{cl}^T + D_{cl} D_{cl}^T. \end{aligned} \tag{6.3}$$

Then, the probability constraint $P(z \in \mathcal{P}) \geq \alpha$ is equivalent to the integral constraint

$$\frac{1}{\sqrt{(2\pi)^n \det(Z)}} \int_{\mathcal{P}} e^{-\frac{1}{2}(\zeta - \bar{z})^T Z^{-1} (\zeta - \bar{z})} d\zeta \geq \alpha.$$

As for the non-stationary case, the goal is thus to shape the variance matrix Z by control in order to satisfy this bound and to reduce the cost $c^T z^r$ as far as possible. It follows that the back-off to the constraints is given by

$$\nu_j = r\sqrt{h_j^T Z h_j}.$$

With this preparation the following Steady-State Problem, in short **SSP** can be formulated

$$\textbf{(SSP)}: \quad \begin{array}{ll} \inf\limits_{K, X, Z, z^r \in S_r, \nu_j} & c^T z^r \\ X \succ A_{cl} X A_{cl}^T + B_{cl} B_{cl}^T & \\ Z \succ C_{cl} X C_{cl}^T + D_{cl} D_{cl}^T & \\ \nu_j + h_j^T z^r < g_j & \\ r\sqrt{h_j^T Z h_j} < \nu_j \quad \forall j & \end{array}$$

Recall that K is any stabilizing controller and that the equilibrium set S^r was defined in (6.2). Standard matrix perturbation arguments reveal that the matrix equations (6.3) can be replaced by the matrix inequalities in **SSP**. Due to the nonlinearities (in particular the square-root) it is neither clear whether this optimization problem is convex, nor is it easily seen whether it can be solved by efficient techniques. As one of the main results in this chapter we reveal how to solve (**SSP**) by Linear Matrix Inequality (LMI) techniques, and how to design controllers whose McMillan degree is at most as large as that of the system description. Contrary to the LTV solution, a two-step procedure is needed to find the global optimum. The solution of the stationary closed-loop MPC problem via the computation of upper and lower bounds presented in this chapter were obtained in fruitful collaboration with C.W. Scherer and are in abridged form available in Van Hessem *et al.* (2001).

6.2 A heuristic iteration

To explain why we need to follow two steps, we first examine a fast heuristic iteration based on a successive linearization of the constraints. Two constraints of (**SSP**) are the matrix inequalities corresponding to the standard $\mathbf{H}_2$-problem. Although these constraints are nonlinear in the closed-loop Lyapunov matrix X and the controller parameters, it is known how to linearize them by a suitable transformation (Scherer *et al.*, 1997). The technical obstruction to apply these techniques directly follows from the constraints

$$r\sqrt{h_j^T Z h_j} < \nu_j \quad \text{or equivalently} \quad r^2 h_j^T Z h_j < \nu_j^2 \tag{6.4}$$

since both ν_j and ν_j^2 appear as variables. Let us introduce a heuristic iteration procedure which shows remarkably good convergence properties at low computational cost (compared to the full solution that is discussed hereafter). All variables that are involved in the iteration are labelled with the iteration step k.

6.2.1 An iterative algorithm

Consider the following basic search algorithm.

1) Fix an initial guess $\nu_{0,j}$ as initial back-offs to the constraints. Such values can be computed by following the subsequent two step procedure: Design an optimal $\mathbf{H}_2$ controller

$$(\mathbf{H_2P}): \quad \inf_{\substack{K, X, Z, \mu \\ X \succ A_{cl} X A_{cl}^T + B_{cl} B_{cl}^T \\ Z \succ C_{cl} X C_{cl}^T + D_{cl} D_{cl}^T \\ \sum_j h_j^T Z h_j < \mu}} \mu$$

and solve, for the resulting output covariance matrix Z_0, the optimization problem

$$(\mathbf{LP}): \quad \gamma_k^{\text{iter}} = \inf_{\substack{z^r \in S_r, \nu_j \\ \nu_j + h_j^T z^r < g_j \quad j = 1, \ldots, m \\ r\sqrt{h_j^T Z_0 h_j} < \nu_j \quad j = 1, \ldots, m}} c^T z^r.$$

Both problems are solvable without any trouble, and the resulting optimal parameters $\nu_{0,j}$ can serve as initial guesses for the iterative procedure.

2) Given $\nu_{k,j}$, replace ν_j^2 in (6.4) with its linearization at $\nu_{k,j}$ to obtain

$$r^2 h_j^T Z h_j < 2\nu_{k,j}\nu_j - \nu_{k,j}^2 \quad j = 1, \ldots, m.$$

The linearization of (**SSP**) is then given by

$$
\begin{array}{lll}
\textbf{(L-SSP)}_k: & \inf & c^T z^r \\
& K, X, Z, z^r \in S_r, \nu_j & \\
& X \succ A_{cl} X A_{cl}^T + B_{cl} B_{cl}^T & \\
& Z \succ C_{cl} X C_{cl}^T + D_{cl} D_{cl}^T & \\
& \nu_j + h_j^T z^r < g_j \quad j = 1, \ldots, m & \\
& \nu_{k,j}^2 + r^2 h_j^T Z h_j < 2\nu_{k,j}\nu_j \quad j = 1, \ldots, m &
\end{array}
$$

and can be globally solved after nonlinear controller parameter transformation technique. The solution of $\textbf{(L-SSP)}_k$ leads to the updates $\nu_{k+1,j}$.

Step 2 is iterated until there is no significant improvement of the cost. More concretely, for a user-defined absolute error $\epsilon > 0$ the algorithm is stopped if two consecutive outcomes satisfy

$$|c^T z_{k+1}^r - c^T z_k^r| < \epsilon.$$

Note that by convexity, ν_j^2 majorizes its linearization at $\nu_{k,j}$. Since $\nu_{k+1,j}$ is feasible for $\textbf{L-SSP}_k$, this fact implies that it is also feasible for **SSP**, and thus also for $\textbf{(L-SSP)}_{k+1}$. We conclude that the iteration is well-defined, and that the iterates satisfy

$$\text{Optimal value of } \textbf{(SSP)} \le c^T z_{k+1}^r \le c^T z_k^r$$

which guarantees the convergence of the sequence $c^T z_k^r$ as $k \to \infty$ if the original problem has a bounded solution. Note that this algorithm does not necessarily converge to the global optimum but it does show good performance in application.

6.2.2 Convex constraints for dynamic output feedback

Output feedback LMI synthesis problems are generally are non-convex problems but there exist technical procedures by Scherer *et al.* (1997) and Masubuchi (1998) to render a large collection of bilinear matrix inequalities (BMI's) into LMI's. The **L-SSP** problem falls in this class, which allows us to globally solve it using these procedures. The important technicalities involved in the solution are stated below for direct reference. The standard constraints in $\mathbf{H}_2$ control can be written as

$$
\begin{aligned}
X \succ 0, \qquad X - \begin{pmatrix} A_{cl} & B_{cl} \end{pmatrix} \begin{pmatrix} X & O \\ O & I \end{pmatrix} \begin{pmatrix} A_{cl}^T \\ B_{cl}^T \end{pmatrix} \succ 0 \\
Z_j - \begin{pmatrix} C_{cl,j} & D_{cl,j} \end{pmatrix} \begin{pmatrix} X & O \\ O & I \end{pmatrix} \begin{pmatrix} C_{cl,j}^T \\ D_{cl,j}^T \end{pmatrix} \succ 0.
\end{aligned} \tag{6.5}
$$

Note that we are considering a slightly more general setup in the sense that multiple performance outputs z_k^j are handled. Then we can switch between one multidimensional output $z_k(\xi)$ or m scalar performance variables

$$z_k^j(\xi) := h_j^T z_k(\xi) = h_j^T C_z x_k + h_j^T D_z u_k = C_z^j x_k + D_z^j u_k$$

to compute the variance in the direction of the jth constraint. The tools presented here handle such cases without any problem. These constraints are the Schur complements (Zhou *et al.*, 1996; Horn and Johnson, 1999) of the block structured matrices below, such that the constraints (6.5) are equivalent to the constraints

$$\begin{pmatrix} X & A_{cl} & B_{cl} \\ A_{cl}^T & X^{-1} & O \\ B_{cl}^T & O & I \end{pmatrix} \succ 0, \quad \begin{pmatrix} Z & C_{cl,j} & D_{cl,j} \\ C_{cl,j}^T & X^{-1} & O \\ D_{cl,j}^T & O & I \end{pmatrix} \succ 0.$$

Then, by introducing of $P = X^{-1}$ the constraints can be reformulated to arrive at

$$\begin{pmatrix} P & PA_{cl} & PB_{cl} \\ A_{cl}^T P & P & O \\ B_{cl,j}^T P & O & I \end{pmatrix} \succ 0, \quad \begin{pmatrix} Z & C_{cl,j} & D_{cl,j} \\ C_{cl,j}^T & P & O \\ D_{cl,j}^T & O & I \end{pmatrix} \succ 0, \tag{6.6}$$

These are the analysis LMI's for fixed problem data $(A_{cl}, B_{cl}, C_{cl}, D_{cl})$ (thus a fixed controller) that are solved for the variables P, Z. These matrix inequalities are nonlinear and non-convex in the case of controller *synthesis* as then the controller parameters $(A_{cl}, B_{cl}, C_{cl}, D_{cl})$ are still to be chosen and products of these parameters with P are bilinear and therefore non-convex. These inequalities can be rendered convex via a suitable optimization variable transformation. Partition P and P^{-1} as

$$P = \begin{pmatrix} \mathbf{Y} & N \\ N^T & \star \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} \mathbf{X} & M \\ M^T & \star \end{pmatrix}$$

in the same block partitioning as the closed-loop system, (6.1), where both $\mathbf{X}, \mathbf{Y} \in S^n$ (where n is the dimension of the state-space). Next, define the matrices

$$\Pi_1 = \begin{pmatrix} \mathbf{X} & I \\ M^T & O \end{pmatrix}, \quad \Pi_2 = \begin{pmatrix} I & \mathbf{Y} \\ O & N^T \end{pmatrix}.$$

Then, $PP^{-1} = I$ implies $P\Pi_1 = \Pi_2$ and hence

$$\mathcal{P} := \Pi_1^T P \Pi_1 = \begin{pmatrix} \mathbf{X} & I \\ I & \mathbf{Y} \end{pmatrix}.$$

Define the transformed system as

$$\begin{pmatrix} \hat{\mathbf{A}} & \hat{\mathbf{B}} \\ \hat{\mathbf{C}} & \hat{\mathbf{D}} \end{pmatrix} = \begin{pmatrix} \mathbf{Y}A\mathbf{X} & O \\ O & O \end{pmatrix} + \begin{pmatrix} N & \mathbf{Y}B \\ O & I \end{pmatrix} \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \begin{pmatrix} M^T & O \\ C\mathbf{X} & I \end{pmatrix} \tag{6.7}$$

which results in the following transformation closed-loop system representation

$$\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C}_j & \mathcal{D}_j \end{array}\right) = \left(\begin{array}{c|c} \Pi_1^T P A_{cl} \Pi_1 & \Pi_1^T P B_{cl} \\ \hline C_{cl,j}\Pi_1 & \mathcal{D}_j \end{array}\right)$$
$$= \left(\begin{array}{cc|c} A\mathbf{X} + B\hat{\mathbf{C}} & A + B\hat{\mathbf{D}}C & B^w + B\hat{\mathbf{D}}D^w \\ \hat{\mathbf{A}} & \mathbf{Y}A + \hat{\mathbf{B}}C & \mathbf{Y}B^w + \hat{\mathbf{B}}D^w \\ \hline C_z^j\mathbf{X} + D_z^j\hat{\mathbf{C}} & C_z^j + D_z^j\hat{\mathbf{D}}C & D_z^j\hat{\mathbf{D}}D^w \end{array}\right). \tag{6.8}$$

The matrix inequality constraints can then be *linearized* using the congruence transformations

$$\begin{pmatrix} \Pi_1 & O & O \\ O & \Pi_1 & O \\ O & O & I \end{pmatrix}^T \begin{pmatrix} P & PA_{cl} & PB_{cl} \\ A_{cl}^T P & P & O \\ B_{cl}^T P & O & I \end{pmatrix} \begin{pmatrix} \Pi_1 & O & O \\ O & \Pi_1 & O \\ O & O & I \end{pmatrix} \succ 0,$$
$$\begin{pmatrix} I & O & O \\ O & \Pi_1 & O \\ O & O & I \end{pmatrix}^T \begin{pmatrix} Z & C_{cl,j} & D_{cl,j} \\ C_{cl,j}^T & P & O \\ D_{cl,j}^T & O & I \end{pmatrix} \begin{pmatrix} I & O & O \\ O & \Pi_1 & O \\ O & O & I \end{pmatrix} \succ 0 \tag{6.9}$$

leading to the so-called *synthesis* LMI's

$$\begin{pmatrix} \mathcal{P} & \mathcal{A} & \mathcal{B} \\ \mathcal{A}^T & \mathcal{P} & O \\ \mathcal{B}^T & O & I \end{pmatrix} \succ 0, \quad \begin{pmatrix} Z_j & \mathcal{C}_j & \mathcal{D}_j \\ \mathcal{C}_j^T & \mathcal{P} & O \\ \mathcal{D}_j^T & O & I \end{pmatrix} \succ 0$$

which are the original *analysis* LMI's after performing the transformations

$$P \to \mathcal{P}, \quad PA_{cl} \to \mathcal{A}, \quad PB_{cl} \to \mathcal{B}, \quad C_{cl,j} \to \mathcal{C}_j, \quad D_{cl,j} \to \mathcal{D}_j \tag{6.10}$$

Finally, upon substitution of the calligraphic blocks (6.8), the LMI's are obtained

$$\begin{pmatrix} \mathbf{X} & I & A\mathbf{X} + B\hat{\mathbf{C}} & A + B\hat{\mathbf{D}}C & B^w + B\hat{\mathbf{D}}D^w \\ \star & \mathbf{Y} & \hat{\mathbf{A}} & \mathbf{Y}A + \hat{\mathbf{B}}C & \mathbf{Y}B^w + \hat{\mathbf{B}}D^w \\ \star & \star & \mathbf{X} & I & O \\ \star & \star & \star & \mathbf{Y} & O \\ \star & \star & \star & \star & I \end{pmatrix} \succ 0 \tag{6.11}$$

and

$$\begin{pmatrix} Z_j & C_z^j\mathbf{X} + D_z^j\hat{\mathbf{C}} & C_z^j + D_z^j\hat{\mathbf{D}}C & D_z^j\hat{\mathbf{D}}D^w \\ \star & X & I & O \\ \star & \star & Y & O \\ \star & \star & \star & I \end{pmatrix} \succ 0 \tag{6.12}$$

which are indeed affine in the bold-faced parameters $(\mathbf{X}, \mathbf{Y}, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{D}})$. Once the solution for the variables $(\mathbf{X}, \mathbf{Y}, \hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}, \hat{\mathbf{D}})$ has been obtained, a back transformation must be performed to obtain the actual controller parameters. Recall that by definition we have

$$NM^T = I - \mathbf{Y}\mathbf{X}, \tag{6.13}$$

and since $I - \mathbf{YX} > 0$, we can find nonsingular factors M, N satisfying (6.13). Inverse of the transformation (6.7) is simple and the controller parameters are directly obtained from

$$\begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} = \begin{pmatrix} N & \mathbf{Y}B \\ O & I \end{pmatrix}^{-1} \begin{pmatrix} \hat{\mathbf{A}} - \mathbf{Y}A\mathbf{X} & \hat{\mathbf{B}} \\ \hat{\mathbf{C}} & \hat{\mathbf{D}} \end{pmatrix} \begin{pmatrix} M^T & O \\ C\mathbf{X} & I \end{pmatrix}^{-1} \tag{6.14}$$

The important issue here is that the constraints in the $\textbf{(L-SSP)}_k$ are convex in their representations (6.11) and (6.12). This allows to solve $\textbf{(L-SSP)}_k$ globally for its optimum, where the optimal controller is then retrieved via (6.14).

6.2.3 Example of heuristic algorithm

To illustrate the algorithm, it is applied to the mechanical system of section 4.4, (figure 4.8). In the previous example, the transition from the origin towards the optimal steady-state was computed, but no explicit characterization of the steady-state itself was given, see figure 4.9. The LTI dynamics of the system are given by

$$\left(\frac{x_{t+1}}{y_t}\right) = \left(\begin{array}{cc|ccc|c} .9756 & .0965 & .0316 & 0 & 0 & .0489 \\ -.4825 & .9225 & 0 & .0316 & 0 & .9649 \\ \hline 1 & 0 & 0 & 0 & .01 & 0 \end{array}\right) \left(\frac{x_t}{\frac{w_t}{u_t}}\right).$$

This system is stable with eigenvalues $0.9490 \pm 0.2141i$, observable and controllable. The performance output contains all inputs and states (as before)

$$z(\xi) = \begin{pmatrix} x(\xi) \\ u(\xi) \end{pmatrix}, \; C_z = \begin{pmatrix} I \\ 0 \end{pmatrix} \text{ and } D_z = \begin{pmatrix} 0 \\ I \end{pmatrix} \tag{6.15}$$

Any vertical position where the mass has zero velocity is a possible steady state, hence the set S of admissible steady-state outputs equals

$$S := \text{null}\begin{pmatrix} I - A & -B \end{pmatrix} = \{\lambda \begin{pmatrix} 2 & 0 & 1 \end{pmatrix}^T | \lambda \in \mathbb{R}\}.$$

The objective is to maximize the vertical deflection, for which the following linear objective is chosen

$$c^T = \begin{pmatrix} -1 & 0 & 0 \end{pmatrix}.$$

Further, the certainty level is $\alpha = .97$ corresponding to a radius of the confidence ellipsoid of $r = 3$. Two controllers with increasingly stringent constraints on the velocity and on the control input will be computed. For the initial design (subscript 1), it is assumed that the physical limitations on the motion of the mass and the input signal are given by the vector

$$g^T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1e3 & 1e3 \end{pmatrix}$$

(thus imposing no active constraint on the applied force). The optimal steady-state is found at $u_1 = 0.4436$, $x_1 = 0.8872$, $v_1 = \dot{x}_1 = 0$. The algorithm converges in four iterations starting from an optimal $\mathbf{H}_2$ controller. The resulting covariance matrix

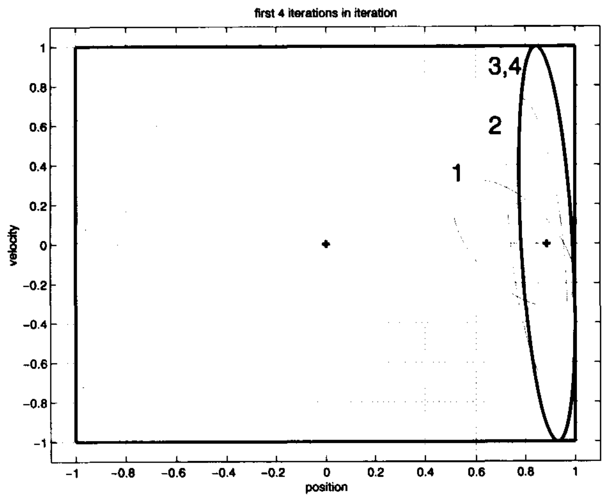


Figure 6.1: Result of the iterative scheme. Convergence after 4 iterations.

in the state-space is depicted by its 97% confidence ellipsoid in figure 6.1. Due to the inactive constraints on the inputs, the optimization pushes the ellipsoid against the position limit at the cost of increased variance in the velocity until ultimately the velocity constraints saturate. This is the interplay between control design and optimal set-point. To visualize the design, a simulation has been made showing the evolution in the time domain. The results are plotted in figure 6.2. In the time domain, the ellipsoid is indeed filled with dots, each dot represents one time instant. The position of the mass is located very close to its limiting constraint during the whole of the operational time. This small design example also reveals how one can influence the design of the optimal controller and how this can be traded off against economic benefit. Suppose the engineer is not satisfied with this design because of possible equipment wear or plant-model mismatch in the high frequency domain, which would not allow the high frequency content of the actuator force and the velocity signals and the low signal to noise ratio

$$\frac{3\sigma(u)}{\bar{u}} \approx 3.1, \quad 3\sigma(v) \approx 1.0.$$

Consequently, the controller is redesigned by replacing the constraints on the actuator input and velocity by more stringent ones, where the velocity is now bound by the constraints

$$0.3 \leq v \leq 0.3, \quad 0.5 \leq u \leq 0.5$$

which corresponds to choosing

$$g^T = \left(\begin{array}{cccccc} 1 & 1 & .3 & .3 & .5 & .5 \end{array} \right).$$

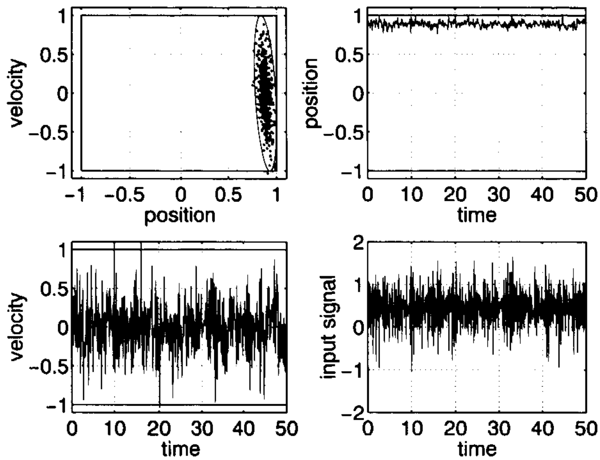


Figure 6.2: Result in the time domain for design 1.

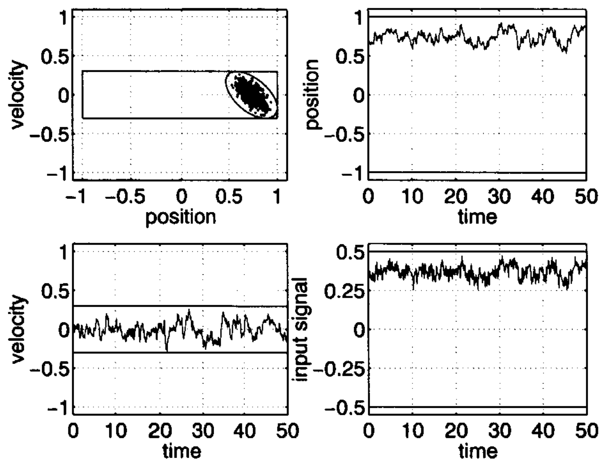


Figure 6.3: Results in the time domain for design 2.

The optimal values for the new design are given by

$$u_2 = 0.3674,\ x_2 = 0.7348,\ v_2 = \dot{x}_2 = 0.$$

for which the variance levels have reduced to

$$\frac{3\sigma(u)}{\bar{u}} \approx 0.36, \quad 3\sigma(v) \approx 0.3.$$

In this new design, one can directly compute the cost of these control requirements, i.e. the relative cost of design 2. compared to design 1. is

$$\frac{x_2 - x_1}{x_1} \times 100\% \approx -17\% \tag{6.16}$$

hence a 17% drop in profit rate. This provides a direct link between the economics of the process and the control design in which control motives may play a part. In figure 6.3, the time domain responses of the system are plotted. What immediately strikes the eye is the decrease in the higher frequency content of the signals, which shows that the engineering effort has been successful.

6.3 A full solution to the stationary problem

In this section, the global optimum is computed and for technical reasons that will soon be clear, two consecutive steps are needed to globally solve **SSP**. The procedure is to first determine the optimal values of the reference input u^r and the optimal back-off terms ν_j by computing upper and lower bounds on the optimal solution of **SSP**. Then, in a second step one solves a multi-objective $\mathbf{H}_2$ (LQG) problem using LMI techniques in which the optimal controller is determined. The key is to perform the following variable transformation in **SSP**

$$X_j := \frac{1}{\nu_j}X \text{ and } Z_j := \frac{1}{\nu_j}Z$$

thus, one divides the variance matrix by the back-off to each constraint. Then, by direct transformation, it follows that **SSP** is equivalent to

$$\begin{array}{lll}
(\mathbf{SSP\text{-}2}): & \displaystyle\inf_{K, X_j, Z_j, z^r \in S_r, \nu_j} & c^T z^r \\
& X_j \succ A_{cl} X_j A_{cl}^T + \frac{1}{\nu_j} B_{cl} B_{cl}^T \ \ \forall j & \\
& Z_j \succ C_{cl} X_j C_{cl}^T + \frac{1}{\nu_j} D_{cl} D_{cl}^T \ \ \forall j & \\
& \nu_j + h_j^T z^r < g_j \ \ \forall j & \\
& r h_j^T Z_j h_j < \nu_j \ \ \forall j &
\end{array}$$

which simplifies to

$$(\mathbf{SSP\text{-}3}): \quad \inf_{\substack{K, X_j, z^r \in S^r, \nu_j \\ X_j \succ A_{cl} X_j A_{cl}^T + \frac{1}{\nu_j} B_{cl} B_{cl}^T \ \forall j \\ \nu_j > r h_j^T C_{cl} X_j C_{cl}^T h_j + \frac{r}{\nu_j} h_j^T D_{cl} D_{cl}^T h_j \ \forall j \\ \nu_j + h_j^T z^r < g_j \ \forall j}} c^T z^r.$$

This new formulation reveals that controller analysis (fixed K) indeed amounts to solving a genuine LMI problem because there are no bilinear terms (the term $\frac{1}{\nu_j}$ disappears by rewriting the matrix inequality using a Schur complements). However, to optimize the controller as well, one observes the non-convex products of optimization variables

$$A_{cl}(K) X_j, \quad C_{cl}(K) X_j.$$

For the simple case that the constraint set $\mathcal{P}$ is just one half-space it is possible to apply again the general transformation as in subsection 6.2.2 to directly compute an optimal controller. A necessary condition of optimality is that there exists a $\lambda \geq 0$ such that

$$c + \lambda h_1 = 0.$$

The objective must be aligned with the normal to this single constraint and then the optimal back-off is precisely determined by the minimal variance in this direction and the optimal controller will be the corresponding LQG controller. Unfortunately, this technique fails if $\mathcal{P}$ is the intersection of at least two half-spaces since then technically more than one matrix X_j is involved in the problem formulation such that the nonlinear parameter transformation fails. The reason is that the problem can be transformed only once, while to remove the nonlinearities one transformation for each constraint is needed. Similarly as it has been suggested for multi-objective control problems, see (Scherer, 2000) and references therein, the following remedy based on the Youla-Kučera parameterization is used.

6.3.1 The Youla-Kučera parameterization of the closed-loop

In the stationary case we can apply the similar techniques as in the time varying case to obtain a solution because the algebraic structure of the problem formulation has not changed significantly. On the other hand, there is a significant change in the structure of the solution because in the stationary case we are interested in the behavior of the system at time infinity. If we would pursue the analysis in a lifted system representation, an infinite number of controller parameters would be needed, while in state space the controller is parameterized with a finite number of parameters.

It is well-known that the set of closed-loop transfer matrices $w \to z$ can be parameterized as

$$\mathbf{T} = T_1 + T_2 Q T_3$$

where T_1, T_2, T_3 are proper and stable transfer matrices and $Q \in R\mathbf{H}_2$ is a free parameter, see also the discussion in subsection 4.5. Consider again the plant dynamics

$$\begin{pmatrix} x_{k+1}(\xi) \\ z_k(\xi) \\ y_k(\xi) \end{pmatrix} = \left(\begin{array}{c|cc} A & B^w & B \\ \hline C_z & 0 & D_z \\ C & D^w & 0 \end{array}\right) \begin{pmatrix} x_k(\xi) \\ w_k(\xi) \\ u_k(\xi) \end{pmatrix}.$$

For this dynamic system, one first designs a feedback controller, usually chosen as a LQG controller, although any stabilizing controller will do. Suppose we solve the Riccati equations for X, Y (assuming there exist solutions)

$$X = AXA^T - AXC^T(CXC^T + D^w D^{wT})^{-1}CXA^T + B^w B^{wT}$$
$$Y = A^TYA - A^TYB(B^TYB + D_z^T D_z)^{-1}B^TYA + C_z^T C_z.$$

and let the Kalman gain N and the state-feedback gain L be given by

$$N = -APC^T(CXC^T + D^w D^{wT})^{-1}, \quad L = -(B^TYB + D_z^T D_z)^{-1}B^TYA$$

Then, the Youla-Kučera parameterization for the error dynamics form of the observer is given by the formulas (Maciejowski, 1994)

$$T_1 = \left(\begin{array}{cc|c} A+BL & -BL & B^w \\ O & A+NC & B^w + ND^w \\ \hline C_z + D_z L & -D_z L & O \end{array}\right) \tag{6.17}$$

$$T_2 = \left(\begin{array}{c|c} A+BL & B \\ \hline C_z + D_z L & D_z \end{array}\right), \quad T_3 = \left(\begin{array}{c|c} A+NC & B^w + ND^w \\ \hline C & D^w \end{array}\right). \tag{6.18}$$

In this case, the Youla parameter maps the innovations signal (for this particular choice of N) to an additive control signal. The transfer matrices $\mathbf{T}^j$ from w to $z_j = h_j^T z$ admit a similar description as

$$\mathbf{T}^j = h_j^T T_1 + h_j^T T_2 Q T_3 = T_1^j + T_2^j Q T_3$$

where

$$T_1^j = \left(\begin{array}{cc|c} A+BL & -BL & B^w \\ O & A+NC & B^w + ND^w \\ \hline h_j^T C_z + h_j^T D_z L & -h_j^T D_z L & O \end{array}\right)$$

$$T_2^j = \left(\begin{array}{c|c} A+BL & B \\ \hline h_j^T C_z + h_j^T D_z L & h_j^T D_z \end{array}\right)$$

with the corresponding realizations $(A_{cl}, B_{cl}, h_j^T C_{cl}, h_j^T D_{cl})$ (with some abuse of notation we still denote the corresponding closed-loop state-space realization with $(A_{cl}, B_{cl}, C_{cl}, D_{cl})$). By referring to the standard LMI-description of the discrete-time $\mathbf{H}_2$-norm inequality

$$r\sqrt{h_j^T Z h_j} = r\|\mathbf{T}^j(Q)\|_2 < \nu_j$$

we observe that (**SSP**-3) is nothing but

$$(\mathbf{SSP\text{-}4}): \quad \gamma^\star = \inf_{\substack{Q \in RH_2,\, z^r \in S_r,\, \nu_j \\ r\|\mathbf{T}^j(Q)\|_2 < \nu_j \;\; j=1,\ldots,m \\ \nu_j + h_j^T z^r < g_j \;\; j=1,\ldots,m}} c^T z^r.$$

Remark 23 $\mathbf{H}_2$. The Hardy space $\mathbf{H}_2$ consists of all complex-valued functions F which are analytic outside the open unit disc $D : \{z \in \mathbf{C} : \; |z| < 1\}$ that satisfy the condition

$$\|F\|_2 := \left(\sup_{r>1} \frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr} F(re^{j\omega})^* F(re^{j\omega}) d\omega\right)^{\frac{1}{2}} < \infty.$$

The subset of real rational function of $\mathbf{H}_2$ is denoted by $R\mathbf{H}_2$ and it can be shown that any such function is proper and stable, i.e. has all its poles in D. For such a function $F \in \mathbf{RH}_2$, it can be shown that its $\mathbf{H}_2$ norm is found by integration over the unit circle

$$\|F\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr} F(e^{j\omega})^* F(e^{j\omega}) d\omega\right)^{\frac{1}{2}} < \infty.$$

□

6.3.2 Computation of upper bounds

If we investigate (**SSP-4**) it appears as if the problem can be solved directly since it is a convex problem. However, because of the infinite dimensional constraint

$$Q \in R\mathbf{H}_2$$

we cannot solve it directly. A problem is that Q cannot be parameterized directly in terms of a state space or transfer function matrix as no convex techniques are then available to globally solve (**SSP-4**). Therefore, the search space will be restricted to finite dimensions and as a result it is not trivial to see how good or bad the approximation is even after optimization. Let us make a specific choice for the finite dimensional parameterization and see how good or bad it is afterwards by computing lower bounds in the next section. We shall follow the path set out in (Scherer, 1995; Hindi *et al.*, 1998b), where the Q-parameter is defined as a Finite Impulse Response (FIR) system

$$Q_v(z) = \sum_{k=0}^{v} Q_k \frac{1}{z^k}$$

with matrix valued coefficients $Q_k \in \mathbf{R}^{n_u \times n_y}$. It is customary to use the variable $z \in \mathbf{C}$ in $\mathcal{Z}$-transform $Q(z)$ of Q; it should not be confused with the performance output $z(\xi)$. Consider the following optimization problem

$$\inf_{Q \in R\mathbf{H}_2} \|T_1 + T_2 Q T_3\|_2$$

and suppose we have a found an almost optimal discrete time controller

$$Q^\star(z) = C(zI - A)^{-1}B + D$$

minimizing the norm of the closed-loop transfer function $\mathbf{T}$, then this function has a unique series expansion

$$\bar{Q}^\star(z) = \sum_{k=0}^{\infty} M_k z^{-k}, \tag{6.19}$$

where the M_k are the Markov parameters

$$M_k = \begin{cases} D & k = 0 \\ CA^{k-1}B & k \geq 1. \end{cases}$$

The region of convergence of the infinite series (6.19) contains the smaller disc $D_\delta := \{z \in \mathbf{C} \ : \ |z| > \rho(Q^\star)\}$. In turn, D_δ contains the unit circle, hence we can use the FIR expansion of $Q^\star(z)$ to evaluate the 2-norm of the optimal solution, (see remark 23)

$$\|Q^\star\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} \operatorname{tr} \bar{Q}^\star(e^{j\omega})^* \bar{Q}^\star(e^{j\omega}) d\omega\right)^{\frac{1}{2}}$$

Stated otherwise, the subspace

$$S = \operatorname{span}\{1, \frac{1}{z}, \frac{1}{z^2}, \dots\}.$$

lies dense in $R\mathbf{H}_2$ and this motivates to solve the finite dimensional approximate problem

$$\inf_{Q_v \in S_v} \|T_1 + T_2 Q_v T_3\|_2, \quad \text{where} \quad S_v := \operatorname{span}\{1, \frac{1}{z}, \dots, \frac{1}{z^v}\}$$

By using the denseness of the orthogonal basis, we arrive at the result that

$$\|T_1 + T_2 Q_v T_3\|_2 \to \|T_1 + T_2 Q^\star T_3\|_2 \quad \text{as } n \to \infty$$

and consequently we can numerically find a ϵ-suboptimal controller $Q_n^\star$ by choosing n sufficiently large. A typical state-space representation of the FIR system is given by

$$\left(\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right) = \left(\begin{array}{cccccc|c} O & I & O & \cdots & O & O \\ O & O & I & \cdots & O & O \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ O & O & \cdots & O & I & O \\ \hline Q_v & Q_{v-1} & \cdots & Q_2 & Q_1 & Q_0 \end{array}\right).$$

Depending on the specific structure of a problem, one might consider to use alternative FIR parameterizations. Due to the specific dependence of the resulting realization $(A_{cl}, B_{cl}, h_j^T C_{cl}, h_j^T D_{cl})$ of $\mathbf{T}^j$ on $(Q_0, Q_1, \ldots, Q_n)$, it turns out possible to apply the general procedure as suggested by Scherer (2000) in order to transform (SSP-3) with K replaced by Q_v into a genuine LMI problem with optimal value u_v. The starting point for this procedure is again the pre-stabilized system, (6.18). The FIR system Q is added this pre-stabilized system to arrive at

$$\left(\begin{array}{c} x^c_{k+1}(\xi) \\ s_{k+1}(\xi) \\ q_{k+1}(\xi) \\ \hline z^j_k(\xi) \\ \tilde{y}_k(\xi) \\ y_k(\xi) \end{array}\right) = \left(\begin{array}{ccc|cc} A+BL & -BL & O & B^w & B \\ O & A+NC & O & B^w + ND^w & O \\ O & B_Q C & A_Q & B_Q D^w & O \\ \hline C^j_z + D^j_z L & -D^j_z L & O & O & D^j_z \\ O & O & I & O & O \\ O & C & O & D^w & O \end{array}\right) \left(\begin{array}{c} x^c_k(\xi) \\ s_k(\xi) \\ q_k(\xi) \\ \hline w_k \\ u_k \end{array}\right)$$

where $C_j = h_j^T C_z$, $D_j = h_j^T D_z$ and $z^j_k = h_j^T z_k$. The closed-loop system is obtained via the (now) static output feedback

$$u_k(\xi) = \underbrace{\begin{pmatrix} C_Q & D_Q \end{pmatrix}}_{=:N_Q} \begin{pmatrix} y_k(\xi) \\ \tilde{y}_k(\xi) \end{pmatrix}.$$

The crucial property is the upper block-triangular structure of the system matrix such that after static output feedback $u = N_Q y$

$$\left(\begin{array}{cc|cc} A_1 & \hat{A} & B_1 & \hat{B} \\ O & A_2 & B_2 & O \\ \hline C^{1,j}_z & C^{2,j}_z & O & D^j_z \\ O & \hat{C} & D^w & O \end{array}\right) \xrightarrow{u = N_Q y} \left(\begin{array}{cc|c} A_1 & \hat{A} + \hat{B} N_Q \hat{C} & B_1 + \hat{B} N_Q D^w \\ O & A_2 & B_2 \\ \hline C^{1,j}_z & C^{2,j}_z + D^j_z N_Q \hat{C} & D^j_z N_Q D^w \end{array}\right)$$

the closed-loop transfer matrix

$$\mathbf{T}^j = C_{cl,j}(zI - A_{cl})^{-1} B_{cl,j} + D_{cl,j}$$

is an affine function of the controller parameters. In the static output feedback case, the trick is now to exploit the block triangular structure of the closed-loop system. Introduce the partitioned matrices

$$P = \begin{pmatrix} X & U \\ U^T & Y \end{pmatrix}$$

and introduce the matrices

$$\begin{pmatrix} \Psi_1 & \Psi_2 \\ \Psi_2^T & \Psi_3 \end{pmatrix} = \begin{pmatrix} X^{-1} & -X^{-1}U \\ U^T X^{-1} & Y - U^T X^{-1} U \end{pmatrix}$$

Then, the definitions

$$\Pi_1 := \begin{pmatrix} \Psi_1 & O \\ \Psi_2^T & I \end{pmatrix}, \; \Pi_2 := \begin{pmatrix} I & -\Psi_2^T \\ O & \Psi_3 \end{pmatrix}, \text{ imply } \Pi_1^T P \Pi_1 = \begin{pmatrix} \Psi_1 & O \\ O & \Psi_3 \end{pmatrix}$$

Moreover, define

$$\left(\begin{array}{c|c}\mathcal{A} & \mathcal{B}_j \\ \hline \mathcal{C}_j & \mathcal{D}_j\end{array}\right) = \left(\begin{array}{c|c}\Pi_1^T P A_{cl}\Pi_1 & \Pi_1^T P B_{cl,j} \\ \hline C_{cl,j}\Pi_1 & D_{cl,j}\end{array}\right)$$
$$= \left(\begin{array}{cc|c} A_1\Psi_1 & A_1\Psi_2 - \Psi_2 A_2 + \hat{A} + \hat{B}N_Q\hat{C} & B_1 + \hat{B}N_Q D^w - \Psi_2 B_2 \\ O & \Psi_3 A_2 & \Psi_3 B_2 \\ \hline C_z^{1,j}\Psi_1 & C_z^{2,j} + D_z^j N_Q \hat{C} + C_z^{1,j}\Psi_2 & D_z^j N_Q D^w \end{array}\right).$$

Then, with the same congruence transformation, as in (6.9), and the same substitutions as in (6.10) we arrive at the linear matrix inequalities

$$\begin{pmatrix} \Psi_1 & O & A_1\Psi_1 & A_1\Psi_2 - \Psi_2 A_2 + \hat{A} + \hat{B}N_Q\hat{C} & B_1 + \hat{B}N_Q D^w - \Psi_2 B_2 \\ \star & \Psi_3 & O & \Psi_3 A_2 & \Psi_3 B_2 \\ \star & \star & \Psi_1 & O & O \\ \star & \star & \star & \Psi_3 & O \\ \star & \star & \star & \star & \nu_j I \end{pmatrix} \succ 0$$

and

$$\begin{pmatrix} \mathbf{Z}_j & C_z^{1,j}\Psi_1 & C_z^{2,j} + D_z^j N_Q\hat{C} + C_z^{1,j}\Psi_2 & D_z^j N_Q D^w \\ \star & \Psi_1 & O & O \\ \star & \star & \Psi_3 & O \\ \star & \star & \star & \nu_j I \end{pmatrix} \succ 0$$

which are indeed affine in the optimization variables $(\Psi_1, \Psi_3, \Psi_2, \mathbf{Z}_j, N_Q)$. Finally, this allows us to compute the approximate solution via

$$(\mathbf{SSP\text{-}5}): \quad \gamma^v = \inf_{\substack{Q_v \in S_v,\, z^r \in S_r,\, \nu_j \\ r\|\mathbf{T}^j(Q_v)\|_2 < \nu_j \ \forall j \\ \nu_j + h_j^T z^r < g_j \ \forall j}} c^T z^r.$$

Due to the ever increasing length of the FIR expansion v of the Youla parameter, it is easy to see that we have

$$\gamma^\star \leq \gamma^{v+1} \leq \gamma^v,$$

(at every increase in length, one can set the last parameter $Q_{Nv+1} = 0$ to observe that the previous solution is also feasible). This shows that one can compute a series of non-increasing upper bounds γ^v of the optimal value. A density argument reveals that γ^v actually converge to $\gamma^\star$ for $v \to \infty$, see (Scherer, 1995).

6.3.3 Computation of lower bounds

In the previous subsection it was shown that increasingly long FIR systems asymptotically converges to the optimal controller. However, there is no guaranteed rate of convergence and therefore the level of sub-optimality is unknown. Suppose that it is possible to compute a lower bound γ_v on the optimal value of (**SSP-5**) for each length of FIR system, then at any instance, the distance from the upper bound γ^v to the optimum $\gamma^\star$ is bounded by

$$\gamma^v - \gamma^\star \leq \gamma^v - \gamma_v \tag{6.20}$$

which allows us to make a sensible choice on the acceptable length of the FIR system. To develop the machinery, define the operator $\mathcal{T}_v$ that maps the transfer matrix $T(z) = C(zI - A)^{-1}B + D$ (with stable A) into the truncated Toeplitz matrix as

$$\mathcal{T}_v(T) = \begin{pmatrix} D & 0 & 0 \\ CB & D & 0 \\ \vdots & \vdots & \vdots \\ CA^{v-2}B & \cdots & D \end{pmatrix},$$

and $E_0 := (I \; 0 \; \cdots \; 0)^T$ such that $\mathcal{T}_v(T)E_0$ is just the first block column of $\mathcal{T}_v(T)$. Since the two norm of a stable system can be evaluated using the Markov parameters is follows that

$$\|T\|_2^2 = \|D\|_2^2 + \sum_{k=0}^{\infty} \|CA^kB\|_2^2 = \|\mathcal{T}_v(T)E_0\|_2^2$$

and it is not difficult to see that

$$\|\mathcal{T}_v(T)E_v\|_2 \leq \|\mathcal{T}_{v+1}(T)E_v\|_2 \leq \|T\|_2 \tag{6.21}$$

and in the limit one obtains

$$\lim_{v \to \infty} \|\mathcal{T}_v(T)E_v\|_2 = \|T\|_2.$$

Moreover, one verifies for any $Q \in \mathcal{RH}_2$ that

$$\mathcal{T}_v(\mathbf{T}^j)E_0 = \mathcal{T}_v(h_j^T T_1)E_0 + \mathcal{T}_v(h_j^T T_2)\mathcal{T}_v(Q)\mathcal{T}_v(T_3)E_0$$

and one observes that only the first v Markov parameters of Q enter this expression. This motivates to define the optimization problem

$$\gamma_v = \inf_{\substack{Q \in \boldsymbol{RH}_2,\, z^r \in S_r,\, \nu_j \\ r\|\mathcal{T}_v(h_j^T T_1)E_0 + \mathcal{T}_v(h_j^T T_2)\mathcal{T}_v(Q)\mathcal{T}_v(T_3)E_0\|_2 < \nu_j \;\; \forall j \\ \nu_j + h_j^T z^r < g_j \;\; \forall j}} c^T z^r.$$

In view of (6.21) we conclude that $\gamma_v \leq \gamma_{v+1} \leq \gamma^\star$. Hence, l_v defines a nondecreasing sequence of lower bounds of the optimal value of **(SSP)**. One can show that γ_v actually converges to $\gamma^\star$ (Scherer, 1999b). Note that this is nothing but a finite dimensional convex quadratic program such that l_v is easy to compute, no parameter transformation is needed to render the constraints convex. Squaring both sides of the inequality and introducing the slack variable P_j, transforms the nonlinear inequality

$$P_j \succ \frac{1}{\nu_j}\left(\mathcal{T}_v(\mathbf{T}^j)E_0\right)^T\left(\mathcal{T}_v(\mathbf{T}^j)E_0\right)$$

into the equivalent to the LMI

$$\begin{pmatrix} P_j & \left(\mathcal{T}_v(\mathbf{T}^j)E_0\right)^T \\ \mathcal{T}_v(\mathbf{T}^j)E_0 & \nu_j I \end{pmatrix} \succ 0$$

that is readily implemented due to the affine parameterization of $\mathbf{T}_j$. The original constraint is then enforced via the LMI

$$\operatorname{tr} P_j < \nu_j.$$

6.3.4 Construction of Optimal Low Order Controllers

The computation of the upper and lower bounds γ^v and γ_v on the optimal value $\gamma^\star$ in the last two subsections is based on the determination of a corresponding Youla parameter whose McMillan degree increases with v. This means that the total degree of the whole system is twice that of the plant (due to pre-stabilization) plus the McMillan degree of the FIR system. We intend to demonstrate how to construct a close-to-optimal controller with the same McMillan degree as that of the plant.

Theorem 24 Given any $\epsilon > 0$ one can determine a feedback controller K with the same McMillan degree as the plant that achieves a performance γ^K such that $\gamma^K < \gamma_\star + \epsilon$.

Proof. By the FIR techniques discussed in this chapter we know that for any $\epsilon > 0$ we can find a Q with FIR length v that achieves a γ^v such that $\gamma^v < \gamma_\star + \epsilon$. Let us denote the corresponding back-off values by ν_j^ϵ and the steady-state output by z_ϵ^r. Moreover, denote the resulting FIR Youla parameter as Q^ϵ and let it correspond to the stabilizing controller K^ϵ for the original system. Since Q^ϵ, ν_j^ϵ, z_ϵ^r are feasible for **(SSP-4)** with

$$c^T z_\epsilon^r < \gamma_\star + \epsilon$$

we can conclude that K^ϵ, ν_j^ϵ, z_ϵ^r are feasible for **(SSP)**. Let us now fix the back-offs in **(SSP)** to ν_j^ϵ. We infer that there exists a high-order controller K^ϵ for which the value of **(SSP)** is smaller than $\gamma^\star + \epsilon$. This allows to constructively find z^r and a controller with the same McMillan degree as the underlying plant which achieves $c^T z^r < \gamma^\star + \epsilon$. □

We have shown how to reduce, in a systematic and computationally efficient fashion, the controller order without performance degradation. Hence, the computation of the close-to-optimal back-off values ν_j^ϵ requires the solution of potentially large-scale optimization problems, but that this does not come at the expense of inflation of the controllers' McMillan degrees. This interesting feature should be contrasted with other multi-objective control problems for which such a reduction is in generally not possible without conservatism.

6.3.5 Example of full solution

Let us continue to example of the mechanical system. For this example case, the computations of the upper and lower bounds have been carried out for various lengths of the FIR parameter. For illustrational purposes the constraints were defined as before with $g^T = \begin{pmatrix} 1 & 1 & .3 & .3 & .5 & .5 \end{pmatrix}$. In figure 6.4, the solutions are plotted along side the solution of the iterative algorithm. For a FIR system with 20 Markov parameters, the following upper and lower bounds are found

$$\gamma_{20} = 0.4042, \quad \gamma^{20} = 0.4044, \quad \frac{\gamma^{20} - \gamma_{20}}{\gamma^{20}} \times 100\% \approx 0.06\%$$

The iterative algorithm converged in four iterations to an optimal value of

$$\gamma^{\text{iter}_4} = 0.4042$$

with optimal parameters $u_3 = 0.4042$, $x_3 = 0.8083$, $\dot{x}_3 = 0$. This solution lies close to the true optimal value, since it is between the best upper and lower bounds. Although there is no guarantee that the iterative algorithm converges to the actual optimal value, it does have the advantage that the total computation time lies 3 orders of magnitude below that of the upper and lower bound computation.

6.4 Chapter summary

In this chapter, a full solution to the stationary closed-loop MPC problem was given. The solution to this problem consists of a LTI controller and an optimal steady-state. In this stationary case, the optimal controller is found by dynamic feedback of the innovations sequence as in the non-stationary case, but the techniques employed to find the optimal solution are more involved. In a first step, a sequence of upper and lower bounds on the optimal objective value is computed until a satisfactory accuracy is obtained. This fixes the necessary given back-off such that the optimal LTI controller can be computed in the second step. This technique is computationally too demanding to solve for large process systems and therefore a heuristic algorithm has been developed that is comparable to the non-stationary case.

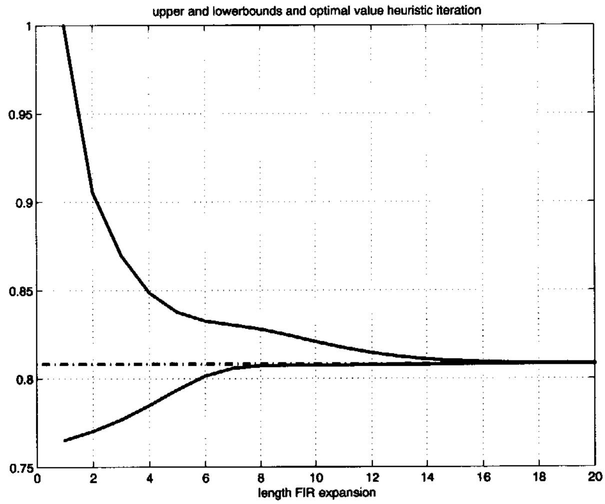


Figure 6.4: Result of upper and lower bound computation *(solid)* and the heuristic algorithm after 4 iterations, *(dash-dotted)*.

7 A Recursive Solution for the Receding Horizon Implementation

In this chapter a recursive solution to the closed-loop MPC problem is presented. A recursive solution is a crucial requirement to set up a receding horizon implementation of the control law. The technical contribution of this chapter is to construct an optimal sequence of solutions that is independent of the specific sample path of the system.

7.1 Introduction

Up to this point, we have only considered snapshot solutions to the closed-loop MPC problem. At time zero, the problem is solved for a finite time horizon for both the feedforward trajectory and the feedback controller and as long as the actual time is in this time window we have a full solution to the control problem (contrary to open-loop MPC of course). However, in the case of continuous processing, the actual time will eventually exceed the time window over which the control law is defined and a new optimization problem needs to be defined and solved. Although this is one possible (though extreme) solution, a better and more robust solution is to continuously re-optimize the feedforward and feedback control action in a receding horizon fashion. This re-optimization requires that the sequence of solutions thereby constructed is continuous in its behavior. We focus on the case that this re-optimization is done every time sample, but in applications one can take any integer multiple of the sample time within the control horizon.

Let us first emphasize some differences to open-loop model predictive control. In *open-loop* MPC, only a feedforward is calculated, and this feedforward is used to control the system in a receding horizon fashion. In this case, the receding horizon control is a strict necessity to stabilize the system. In *closed-loop* MPC, the feedforward does not fulfill the regulation or stabilization task and the receding horizon implementation does not provide feedback in regulatory sense. It is the feedback controller that stabilizes the system, while the receding horizon *implementation* is used only to provide continuity in time (therefore we cannot speak of receding horizon control). The feedforward is used for tracking purposes and contrary to open-loop MPC, it should not even change every time sample due to arbitrary high frequent disturbances.

In many control design methods, a feedback controller is computed once in transfer function matrix or state-space form and then left to its online control task. Discrete time state-space controllers are very efficient for computing control moves given the output measurements due to the recursive nature of their representation, no more than simple matrix multiplication is needed. The great advantage of such a recursive description that cannot be overemphasized, is that measured data is processed recursively by updating the controller *state* and it is completely clear how to shift time by one sample. This reveals the strongest drawback of lifted or algebraic system descriptions; these representations lack an internal state and must therefore grow unbounded to keep track of past measurement data.

In a receding horizon approach to finite horizon control to continuous processes, a continuum of solutions must be created. In each iteration, a prediction problem is considered on the time window $\{k, \dots, k+n\}$ and a single iteration later a new prediction problem must be considered on the time window $\{k+1, \dots, k+n+1\}$. Hence in our context we will continuously be looking for a feedforward input and a feedback controller

$$\mathbf{u}_k^r = \begin{pmatrix} u_k^r \\ u_{k+1}^r \\ \vdots \\ u_{k+n}^r \end{pmatrix}, \quad K_k = \begin{pmatrix} K_{k,k} & O & \cdots & O \\ K_{k+1,k} & K_{k+1,k+1} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ K_{k+n,k} & K_{k+n,k+1} & \cdots & K_{k+n,k+n} \end{pmatrix}$$

defined over a future horizon of n control samples. In chemical process applications, the processes are usually slow and the prediction horizons quite long compared to the dominant time constant of the process under consideration. In practise, the actual extension of the horizon with a single time sample has therefore a relatively low impact on the actual controlled behavior of the system. On the other hand, the feedback of the current measurement to an additive future control sequence determines the closed-loop behavior. Therefore, the central problem in deriving the receding horizon implementation for closed-loop model predictive control lies at the beginning of the horizon, as the horizon shifts from $t = k$ to $t = k+1$. In light of this discussion we will derive the receding horizon implementation for a batch problem defined over n samples only, by considering a so-called shrinking horizon scenario which allows us to study the main feedback mechanism.

Consider the feedback law that was established in Chapter 4

$$\begin{pmatrix} u_0^c(\xi) \\ u_1^c(\xi) \\ \vdots \\ u_{k+n}^c(\xi) \end{pmatrix} = \begin{pmatrix} K_{00} & O & \cdots & O \\ K_{10} & K_{11} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ K_{n0} & K_{n1} & \cdots & K_{nn} \end{pmatrix} \begin{pmatrix} v_0^c(\xi) \\ v_1^c(\xi) \\ \vdots \\ v_n^c(\xi) \end{pmatrix}$$

and after the first measurements become available, we compute the first innovations v_k^c. At each time instant k, the control law for the remainder of the horizon is then given as

$$\begin{pmatrix} u_k^c(\xi) \\ u_{k+1}^c(\xi) \\ \vdots \\ u_{k+n}^c(\xi) \end{pmatrix} = \sum_{j=1}^{k-1} \begin{pmatrix} K_{k,j} \\ K_{k+1,j} \\ \vdots \\ K_{n,j} \end{pmatrix} v_j^c + \begin{pmatrix} K_{k,k} & O & \cdots & O \\ K_{k+1,k} & K_{k+1,k+1} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ K_{n,k} & K_{n,k+1} & \cdots & K_{nn} \end{pmatrix} \begin{pmatrix} v_k^c(\xi) \\ v_{k+1}^c(\xi) \\ \vdots \\ v_n^c(\xi) \end{pmatrix}$$

This implies that even for the computation of the future control moves, we need to keep track of all past measurement data. This is eventually unacceptable and therefore we aim at replacing this control law with a combination of innovations feedback and state feedback using the control law

$$\begin{pmatrix} u_k^c(\xi) \\ u_{k+1}^c(\xi) \\ \vdots \\ u_{k+n}^c(\xi) \end{pmatrix} = \begin{pmatrix} L_k \\ L_{k+1} \\ \vdots \\ L_n \end{pmatrix} \hat{x}_k^c + \begin{pmatrix} K_{k,k} & O & \cdots & O \\ K_{k+1,k} & K_{k+1,k+1} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ K_{n,k} & K_{n,k+1} & \cdots & K_{nn} \end{pmatrix} \begin{pmatrix} v_k^c(\xi) \\ v_{k+1}^c(\xi) \\ \vdots \\ v_n^c(\xi) \end{pmatrix}$$

where $\hat{x}_k^c$ is the state estimate. Hence, instead of keeping track of v_j^c, we use this sequence of innovations recursively to update the state estimate. The main problem is to show that there is no loss of performance in this substitution.

7.2 Problem formulation

Consider again a discrete time-varying stochastic system

$$\begin{pmatrix} x_{k+1}(\xi) \\ z_k(\xi) \\ y_k(\xi) \end{pmatrix} = \begin{pmatrix} A_k & B_k^w & B_k \\ C_k^z & E_k^z & D_k^z \\ C_k & D_k^w & O \end{pmatrix} \begin{pmatrix} x_k(\xi) \\ w_k(\xi) \\ u_k(\xi) \end{pmatrix} \tag{7.1}$$

where $w_k(\xi)$ is a resulting white noise sequence with variance matrix W_k with the property $B_k^w W_k D^{w}{}_k^T = 0$ (process and measurement noise are independent). Let us lift this system over a time horizon of n samples as before. As argued in the introduction, the difficulty of the receding horizon implementation lies at the beginning of the horizon. To derive a consistent receding horizon law, we must look at the finite horizon batch problem in which the horizon gets shorter with each cycle (extending the horizon with a time sample is then straightforward). At each cycle, the optimal solution should coincide with the previous one if there are no changes in the optimization problem formulation, that is, the objective function or inequality constraints. In order to represent this batch iteration process compactly, define the following stochastic processes

$$\mathbf{u}_k(\xi) = \begin{pmatrix} u_k(\xi) \\ \vdots \\ u_n(\xi) \end{pmatrix}, \mathbf{w}_k(\xi) = \begin{pmatrix} w_k(\xi) \\ \vdots \\ w_n(\xi) \end{pmatrix}, \mathbf{y}_k(\xi) = \begin{pmatrix} y_k(\xi) \\ \vdots \\ y_n(\xi) \end{pmatrix}, \mathbf{z}_k(\xi) = \begin{pmatrix} z_k(\xi) \\ \vdots \\ z_n(\xi) \end{pmatrix}$$

representing the part from each signal $\mathbf{u}, \mathbf{w}, \mathbf{y}, \mathbf{z}$ from sample k to sample n corresponding to the remainder of the batch process time after time t_k. For ease of presentation and without loss of generality we will set the reference trajectories to zero in the next few sections[1], that is

$$\mathbf{u}_k^r \equiv 0, \ \mathbf{w}_k^r \equiv 0, \ \mathbf{x}_k^r \equiv 0, \ \mathbf{y}_k^r \equiv 0, \ \mathbf{z}_k^r \equiv 0.$$

These sections are concerned with state estimation and the results are not influenced by the deterministic reference signals. As a consequence we can use the signals

$$\mathbf{u}_k(\xi) = \mathbf{u}_k^c(\xi), \ \mathbf{w}_k(\xi) = \mathbf{w}_k^c(\xi), \ \mathbf{x}_k(\xi) = \mathbf{x}_k^c(\xi), \ \mathbf{y}_k(\xi) = \mathbf{y}_k^c(\xi), \ \mathbf{z}_k(\xi) = \mathbf{z}_k^c(\xi)$$

which facilitates easier comparison of the results below to existing literature on state estimation. One of the problems in building a receding horizon implementation is to keep track of the past measurements. Recall the algebraic expression for the observer error dynamics from Chapter 4, equation (4.39)

$$\left(\begin{array}{c} e_0(\xi) \\ \hline e_1(\xi) \\ \vdots \\ e_n(\xi) \end{array}\right) = \left(\begin{array}{c} I \\ \hline \Phi_{1,0}^e \\ \vdots \\ \Phi_{n,0}^e \end{array}\right) e_0(\xi) + \left(\begin{array}{c|ccc} O & O & \cdots & O \\ \hline B_0^e & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n,1}^e B_0^e & \Phi_{n,2}^e B_1^e & \cdots & O \end{array}\right) \begin{pmatrix} w_0(\xi) \\ w_1(\xi) \\ \vdots \\ w_n(\xi) \end{pmatrix}.$$

where the transition matrix is given by

$$\Phi_{k,j}^e = A_{k-1}^e A_{k-2}^e \cdots A_j^e, \quad \Phi_{j,j}^e = I, \ A_k^e = A_k - N_k C_k \text{ and } B_k^e = B_k^w - N_k D_k^w.$$

The corresponding innovations sequence, (Kailath, 1968), is given by

$$\begin{aligned} v_k(\xi) :&= y_k(\xi) - \hat{y}_k(\xi) \\ &= C_k(x_k(\xi) - \hat{x}_k(\xi)) + D_k^w w_k(\xi) \\ &= C_k e_k(\xi) + D_k^w w_k(\xi) \end{aligned}$$

and when the innovations sequence is put in its lifted form we obtain

$$\mathbf{v}(\xi) = G_{ve}^0 e_0(\xi) + G_{vw}^0 \mathbf{w}(\xi) \tag{7.2}$$

[1]Sections 7.2, 7.3, 7.4, 7.5 to be precise.

or in terms of the matrices $(\Phi^e_{k,j}, B^e_k, C_k, D^w_k)$

$$\begin{pmatrix} v_0(\xi) \\ \hline v_1(\xi) \\ \vdots \\ v_n(\xi) \end{pmatrix} = \begin{pmatrix} C_0 \\ \hline C_1\Phi^e_{1,0} \\ \vdots \\ C_n\Phi^e_{n,0} \end{pmatrix} e_0(\xi) + \left(\begin{array}{c|ccc} D^w_0 & O & \cdots & O \\ \hline C_1B^e_0 & D^w_1 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ C_n\Phi^e_{n,1}B^e_0 & C_n\Phi^e_{n,2}B^e_1 & \cdots & D^w_n \end{array}\right) \begin{pmatrix} w_0(\xi) \\ \hline w_1(\xi) \\ \vdots \\ w_n(\xi) \end{pmatrix} \tag{7.3}$$

It follows immediately that at the next time sample, the remainder of the innovations sequence is given by

$$\begin{pmatrix} v_1(\xi) \\ v_2(\xi) \\ \vdots \\ v_n(\xi) \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2\Phi^e_{2,1} \\ \vdots \\ C_n\Phi^e_{n,1} \end{pmatrix} e_1(\xi) + \begin{pmatrix} D^w_1 & O & \cdots & O \\ C_2B^e_1 & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ C_n\Phi^e_{n,2}B^e_1 & C_n\Phi^e_{n,3}B^e_2 & \cdots & D^w_n \end{pmatrix} \begin{pmatrix} w_1(\xi) \\ w_2(\xi) \\ \vdots \\ w_n(\xi) \end{pmatrix}$$

In the same fashion, the system transfer matrices $G^k_{\cdot,\cdot}$ are reduced to represent mappings between these partial signals. The system matrices considered in consecutive iterations have the following structure, see also (Furuta and Wongsaisuwan, 1993),

$$\begin{aligned} G^k_{ee} &= \begin{pmatrix} I \\ \hline G^{k+1}_{ee}A^e_k \end{pmatrix}, \quad & G^k_{ew} &= \left(\begin{array}{c|c} O & O \\ \hline G^{k+1}_{ee}B^e_k & G^{k+1}_{ew} \end{array}\right), \\ G^k_{ve} &= \begin{pmatrix} C_k \\ \hline G^{k+1}_{ve}A^e_k \end{pmatrix}, \quad & G^k_{vw} &= \left(\begin{array}{c|c} D^w_k & O \\ \hline G^{k+1}_{ve}B^e_k & G^{k+1}_{vw} \end{array}\right), \\ G^k_{zx} &= \begin{pmatrix} C^z_k \\ \hline G^{k+1}_{zx}A_k \end{pmatrix}, \quad & G^k_{zu} &= \left(\begin{array}{c|c} D^z_k & O \\ \hline G^{k+1}_{zx}B_k & G^{k+1}_{zu} \end{array}\right) \end{aligned} \tag{7.4}$$

where the decomposition matches the decomposition above, hence one slices off one time sample at the time from the system matrices. Then, (7.2) is compactly written as

$$\mathbf{v}_k(\xi) = G^k_{ve}e_k(\xi) + G^k_{vw}\mathbf{w}_k(\xi) \tag{7.5}$$

and it is immediate that the performance output for the remainder of the batch is given by

$$\mathbf{z}_k(\xi) = G^k_{zx}x_k(\xi) + G^k_{zu}\mathbf{u}_k(\xi) + G^k_{zw}\mathbf{w}_k(\xi). \tag{7.6}$$

Thus many ingredients for a recursive implementation are there, however, to guarantee optimality, the variance matrices as well as the control law must also be computed recursively. This brings us to the problem formulation of this chapter that is to give the recursive implementation of the closed-loop MPC problem.

Remark 25 *A quasi-stationary angle.* A few remarks are in order. A key issue is that we do *not* want the actual controller to depend on the specific realization of the measured output process $\mathbf{y}(\xi)$. Recall the stationary solution in Chapter 6, where an optimal set point and an optimal controller

$$\mathbf{z}^r, K$$

were computed to maximize the profit by using the controller K to minimize the back-off to the constraints. In this case, one does not update the controller at every single measurement sample. Even in the case that an estimate of the initial condition is available, there is no need to update the controller parameters, but only the controller state. On the other hand, the controller *is* updated if the constraints or the objective function change because these relocate the economic optimum. This view point corresponds to the following property of the optimization problem. Suppose there are no changes in the system dynamics, constraints, objective or properties of the disturbances. Then all future solutions should coincide with the solution obtained in the first iteration. □

7.3 Recursive construction of lifted FHLQG controllers

In the main result on the receding horizon implementation, we will rely on several properties of finite horizon LQG controllers. In Chapter 5, the relations between FHLQG control and CLMPC were already touched upon, but in this chapter we will formalize these results. The reason for our interest in the FHLQG problem, is that it is relatively easy to build a receding horizon implementation due to its internal observer and state feedback structure. Let us return to our time-varying stochastic system and let the control objective function be given as

$$E\sum_{k=0}^{N} z_k(\xi)^T Q_k z_k(\xi), \quad Q_k \succeq 0. \tag{7.7}$$

Theorem 26 recaps the standard result in Finite Horizon LQG theory that we will need to construct recursive controllers for the CLMPC problem later on.

Theorem 26 Suppose we are given the time-varying system equation (7.1) and the finite time objective equation (7.7). Let the control law be given as

$$\mathbf{u}_0(\xi) = K_0\mathbf{v}_0(\xi) \tag{7.8}$$

where $K_0 \in \mathbf{K}_0$ is a non-anticipative controller

$$K_0 = \begin{pmatrix} K^{11} & O & \cdots & O \\ K^{21} & K^{22} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ K^{n1} & K^{n2} & \cdots & K^{nn} \end{pmatrix}$$

and $\mathbf{v}_0$ is the innovations sequence. Then, at any time instant $0 \leq k \leq n-1$ in the future, there exist matrices

$$L_k \in \mathbf{R}^{(n-k)n_u \times n_x} \text{ and } K_k \in \mathbf{R}^{(n-k)n_u \times (n-k)n_y}$$

such that the control law

$$\mathbf{u}_k(\xi) = L_k \hat{x}_k(\xi) + K_k \mathbf{v}_k(\xi)$$

generates the optimal control sequence for the remainder of the horizon.

Proof. It is well known that the optimal control sequence to the FHLQG problem given by a Kalman filter and a state feedback. In discrete time, the optimal Kalman filter dynamics are given by (Lewis, 1986)

$$\begin{aligned} \hat{x}_{k+1}(\xi) &= A_k \hat{x}_k(\xi) + B_k u_k(\xi) + N_k v_k(\xi) \\ v_k(\xi) &= y_k(\xi) - \hat{y}_k(\xi) \end{aligned} \tag{7.9}$$

where N_k is again the Kalman gain matrix. Then, the optimal LQG control sequence can be generated by from this estimate via

$$u_k(\xi) = F_k \hat{x}_k(\xi) + M_k v_k(\xi) \tag{7.10}$$

where the innovations feedthrough matrix is given by

$$M_k = F_k P_k^e C_k^T (C_k P_k^e C_k^T + F_k W_k F_k^T)^{-1}$$

The covariance matrix P_k^e will be computed in Section 7.5. Then, application of the control input (7.10) to the observer dynamics (7.9) gives in closed-loop

$$\hat{x}_{k+1}(\xi) = (A_k + B_k F_k)\hat{x}_k(\xi) + (N_k + B_k M_k) v_k(\xi)$$

Represented as a lifted system one obtains

$$\begin{pmatrix} \hat{x}_0(\xi) \\ \hat{x}_1(\xi) \\ \vdots \\ \hat{x}_n(\xi) \end{pmatrix} = \begin{pmatrix} I \\ \Phi_{1,0}^c \\ \vdots \\ \Phi_{n,0}^c \end{pmatrix} \hat{x}_0(\xi) + \begin{pmatrix} O & O & \cdots & O \\ N_0^c & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n,1}^c N_0^c & \Phi_{n,2}^c N_1^c & \cdots & O \end{pmatrix} \begin{pmatrix} v_0(\xi) \\ v_1(\xi) \\ \vdots \\ v_n(\xi) \end{pmatrix} \tag{7.11}$$

where the transition matrices are given by

$$\Phi_{k,j}^c = A_{k-1}^c A_{k-2}^c \cdots A_j^c,\ \Phi_{j,j}^e = I,\ A_k^c = A_k - B_k F_k, \text{ and } N_k^c = N_k + B_k M_k$$

Using this expression, the input sequence is given by

$$\left(\begin{array}{c} u_0(\xi) \\ \hline u_1(\xi) \\ \vdots \\ u_n(\xi) \end{array}\right) = \left(\begin{array}{c|ccc} F_0 & O & \cdots & O \\ \hline O & F_1 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & F_n \end{array}\right) \left(\begin{array}{c} \hat{x}_0(\xi) \\ \hline \hat{x}_1(\xi) \\ \vdots \\ \hat{x}_n(\xi) \end{array}\right) + \left(\begin{array}{c|ccc} M_0 & O & \cdots & O \\ \hline O & M_1 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & M_n \end{array}\right) \left(\begin{array}{c} v_0(\xi) \\ \hline v_1(\xi) \\ \vdots \\ v_n(\xi) \end{array}\right)$$

Carrying out the multiplication shows how the input depends on the innovation sequence, and thereby reveals the structure of the innovations feedback law

$$\begin{pmatrix} u_0(\xi) \\ \hline u_1(\xi) \\ \vdots \\ u_n(\xi) \end{pmatrix} = \begin{pmatrix} F_0 \\ \hline F_1\Phi^c_{1,0} \\ \vdots \\ F_n\Phi^c_{n,0} \end{pmatrix} \hat{x}_0(\xi) + \left(\begin{array}{c|ccc} M_0 & O & \cdots & O \\ \hline F_1 N^c_0 & M_1 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ F_n\Phi^c_{n,1}N^c_0 & F_n\Phi^c_{n,2}N^c_1 & \cdots & M_n \end{array}\right) \begin{pmatrix} v_0(\xi) \\ \hline v_1(\xi) \\ \vdots \\ v_n(\xi) \end{pmatrix}$$

One time sample later one arrives at

$$\begin{pmatrix} u_1(\xi) \\ u_2(\xi) \\ \vdots \\ u_n(\xi) \end{pmatrix} = \overbrace{\begin{pmatrix} F_1 \\ F_2\Phi^c_{2,1} \\ \vdots \\ F_n\Phi^c_{n,1} \end{pmatrix}}^{L_1} \hat{x}_1(\xi) + \overbrace{\begin{pmatrix} M_1 & O & \cdots & O \\ F_2 N^c_1 & M_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ F_n\Phi^c_{n,1}N^c_1 & L^n\Phi^c_{n,2}N^c_2 & \cdots & M_n \end{pmatrix}}^{K_1} \begin{pmatrix} v_1(\xi) \\ v_2(\xi) \\ \vdots \\ v_n(\xi) \end{pmatrix}$$

by using the recursive representation of the dynamics (7.4). The detailed derivation of this step is shown in Remark 27 below. This can be represented compactly as

$$\mathbf{u}_1(\xi) = L_1\hat{x}_1(\xi) + K_1\mathbf{v}_1(\xi)$$

Because this *cut* can be made at any location in the horizon, a recursive solution and therefore a receding horizon implementation for this control law

$$\mathbf{u}_k(\xi) = L_k\hat{x}_k(\xi) + K_k\mathbf{v}_k(\xi)$$

is easily derived, where the state and innovations feedback matrices are given by

$$L_k = \begin{pmatrix} F_k \\ F_{k+1}\Phi^c_{k+1,k} \\ \vdots \\ F_n\Phi^c_{n,k} \end{pmatrix}, \quad K_k = \begin{pmatrix} M_k & O & \cdots & O \\ F_{k+1}N^c_k & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ F_n\Phi^c_{n,1}N^c_1 & F_n\Phi^c_{n,2}N^c_2 & \cdots & M_n \end{pmatrix}$$

□

This recursive construction of the controller is the motivation for the definition of a receding horizon implementation. The next step is to exploit this procedure to get a recursive solution to the CLMPC problem.

Remark 27 From the top block row in the matrices we see that

$$u_0(\xi) = F_0\hat{x}_0(\xi) + M_0 v_0(\xi) \tag{7.12}$$

and by definition of the Kalman filter we have that

$$\hat{x}_1(\xi) = A_0\hat{x}_1(\xi) + B_0 u_0(\xi) + N_0 v_0(\xi). \tag{7.13}$$

Upon substitution of (7.12) in (7.13) one obtains

$$\begin{aligned} \hat{x}_1(\xi) &= (A_0 + B_0 F_0)\hat{x}_1(\xi) + (N_0 + B_0 M_0)v_0(\xi) \\ &= A_0^c \hat{x}_1(\xi) + N_0^c v_0(\xi) \end{aligned}$$

Then, exploiting the recursive representation of the dynamics (7.4)

$$\begin{aligned} &\Phi_{1,0}^c = A_0^c \\ &\Phi_{2,0}^c = \Phi_{2,1}^c A_0^c \\ &\vdots \\ &\Phi_{n,0}^c = \Phi_{n,1}^c A_0^c \end{aligned}$$

we observe that

$$\begin{pmatrix} u_1(\xi) \\ u_2(\xi) \\ \vdots \\ u_n(\xi) \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2\Phi_{2,1}^c \\ \vdots \\ F_n\Phi_{n,1}^c \end{pmatrix} A_0^c \hat{x}_0(\xi) + \begin{pmatrix} F_1 \\ F_2\Phi_{2,1}^c \\ \vdots \\ F_n\Phi_{n,1}^c \end{pmatrix} N_0^c v_0(\xi) +$$
$$\begin{pmatrix} M_1 & O & \cdots & O \\ F_2 N_1^c & M_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ F_n\Phi_{n,1}^c N_1^c & L^n \Phi_{n,2}^c N_2^c & \cdots & M_n \end{pmatrix} \begin{pmatrix} v_1(\xi) \\ v_2(\xi) \\ \vdots \\ v_n(\xi) \end{pmatrix}$$

and when collecting the first common terms

$$\begin{aligned} \begin{pmatrix} u_1(\xi) \\ u_2(\xi) \\ \vdots \\ u_n(\xi) \end{pmatrix} &= \begin{pmatrix} F_1 \\ F_2\Phi_{2,1}^c \\ \vdots \\ F_n\Phi_{n,1}^c \end{pmatrix} (A_0^c \hat{x}_0(\xi) + N_0^c v_0(\xi)) \\ &+ \begin{pmatrix} M_1 & O & \cdots & O \\ F_2 N_1^c & M_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ F_n\Phi_{n,1}^c N_1^c & L^n \Phi_{n,2}^c N_2^c & \cdots & M_n \end{pmatrix} \begin{pmatrix} v_1(\xi) \\ v_2(\xi) \\ \vdots \\ v_n(\xi) \end{pmatrix} \end{aligned}$$

and hence

$$\begin{pmatrix} u_1(\xi) \\ u_2(\xi) \\ \vdots \\ u_n(\xi) \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2\Phi_{2,1}^c \\ \vdots \\ F_n\Phi_{n,1}^c \end{pmatrix} \hat{x}_1(\xi) + \begin{pmatrix} M_1 & O & \cdots & O \\ F_2 N_1^c & M_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ F_n\Phi_{n,1}^c N_1^c & L^n \Phi_{n,2}^c N_2^c & \cdots & M_n \end{pmatrix} \begin{pmatrix} v_1(\xi) \\ v_2(\xi) \\ \vdots \\ v_n(\xi) \end{pmatrix}$$

Repetition of this procedure gives the same result for arbitrary time instances k. $\square$

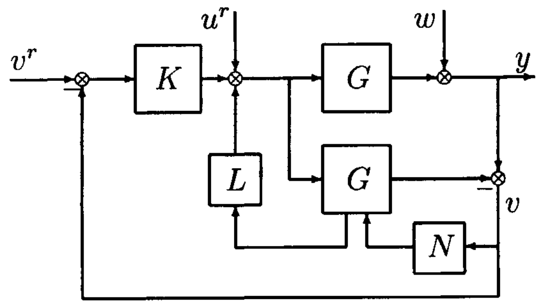


Figure 7.1: Innovations feedback with state-feedback; a closed-loop predictive alternative to open-loop internal model control.

7.4 Definition of a receding horizon implementation

Theorem 26 shows that if a controller admits an observer/state feedback structure, then restricted control laws of the form

$$\mathbf{u}_k(\xi) = L_k \hat{x}_k(\xi) + K_k \mathbf{v}_k(\xi) \tag{7.14}$$

can replace the original control law

$$\mathbf{u}_0(\xi) = K_0 \mathbf{v}_0(\xi) \tag{7.15}$$

on the remainder of the horizon without any loss of performance. The resulting control structure is visualized in Figure 7.1. If the covariance matrix

$$Z_0 = E\mathbf{z}_0(\xi)\mathbf{z}_0(\xi)^T$$

is realized by the original control law (7.15), then the lower right block of Z_0, given by the covariance matrix

$$Z_k = E\mathbf{z}_k(\xi)\mathbf{z}_k(\xi)^T$$

is realized by the control law (7.14). Once this consistency in optimality has been established on the finite horizon, one simply discards the measurements after they have been used to update the state estimate. Then the new optimization problem amounts to finding the receding horizon control law in terms of L_k and K_k. In practical implementation, a time sample is then added at the end of the horizon to construct a continuous implementation as in standard receding horizon implementations for open-loop MPC.

Definition 28 *A receding horizon implementation.* Suppose a control law

$$\mathbf{u}_0(\xi) = K_0 \mathbf{v}_0(\xi)$$

is given on a finite horizon, mapping a finite number of innovations into a finite number of inputs. The receding horizon implementation of this control law is defined as

$$\mathbf{u}_k(\xi) = L_k \hat{x}_k(\xi) + K_k \mathbf{v}_k(\xi)$$

for some matrices

$$L_k \in \mathbf{R}^{(n-k)n_u \times n_x} \text{ and } K_k \in \mathbf{R}^{(n-k)n_u \times (n-k)n_y}$$

and state estimate $\hat{x}_k$ generated by the Kalman filter from any sample k onwards on the remainder of the horizon. □

Let us also introduce a way to measure whether a receding horizon implementation is good or not. To this end, suppose we have solved the CLMPC problem for the controller K_0 and reference input signal $\mathbf{u}_0$. Define the performance over the *tail* of the horizon (from sample k onwards) as

$$J_k(K_0, \mathbf{u}_0^r) = \sum_{j=k}^{n} f_j\left(z_j^r(K_0, \mathbf{u}_0^r)\right)$$

where we implicitly assume the objective to be of this structural form. Then define the receding horizon objective value as

$$J_{RH}(L_k, K_k, \mathbf{u}_k^r) = \sum_{j=k}^{n} f_j\left(z_j^r(L_k, K_k, \mathbf{u}_k^r)\right).$$

The receding horizon implementation will be called optimal if the objective value of the receding horizon implementation is equal to the original objective value restricted to the remainder of the horizon $\{k, k+1 \ldots, n\}$

$$J_{RH}^{\star}(L_k, K_k, u^r) = J_k^{\star}(K_0, u^r)$$

where the superscript $\star$ denotes the optimal values.

Remark 29 In the CLMPC optimization problem the optimal controller is chosen from the set of all possible linear transformations of the output measurements. Because the use of an observer together with state and innovations feedback is a specific choice of a linear transformation, no improvement of the performance can occur using the receding horizon implementation, hence

$$J_{RH}^{\star}(L_k, K_k, \mathbf{u}_k^r) \geq J_k^{\star}(K_0, \mathbf{u}_0^r)$$

Hence, to prove optimality it suffices to prove that

$$J_{RH}^{\star}(L_k, K_k, \mathbf{u}_k^r) \leq J_k^{\star}(K_0, \mathbf{u}_0^r)$$

It will be proved that under certain conditions, the CLMPC controller is optimal for some FHLQG objective and then use of theorem 26 will give us the optimal receding horizon implementation. □

Remark 30 *Stability and innovations feedback.* The innovations approach to closed-loop MPC turns out to be the right choice of controller structure when CLMPC is put in a receding horizon implementation as we will soon show. However, one of the problems associated with the feedback of the innovation sequence is that one easily runs into trouble with stability. Let us analyze the situation. At time t_0, we are solving the finite horizon control problem, using n future measurements

$$y_0, y_1, \ldots, y_n$$

for some finite but possibly large $n \in \mathbb{N}$, (it makes sense to choose an horizon length covering a few times the dominant time constant in view of the simultaneous feedforward design). From Theorem 18 it follows that any output feedback controller can be written as an innovations feedback controller, and consequently the same problem can be written as a feedback of the innovation sequence

$$v_0, v_1, \ldots, v_n.$$

Both sequences have an equivalent information content and it is not difficult to compute one control sequence from the other. However, at a certain moment in time we are forced by memory and computational limitations to discard the first samples

$$y_0 \quad \text{and} \quad v_0$$

and at that time, both sequences are no longer equivalent with regard to their possibilities in feedback. Suppose only the last measurement is used for static output feedback (as a possible extreme case)

$$y_k(\xi) = Cx_k(\xi), \quad u_k(\xi) = K_y y_k(\xi).$$

Then, we are still able to do (partial) pole-placement via

$$x_{k+1} = Ax_k(\xi) + Bu_k(\xi) = (A + BKC)x_k(\xi)$$

such that it is not structurally impossible to stabilize the system, while it is structurally impossible in the innovations feedback case. Once only a finite number of samples of the innovation sequence $\mathbf{v}$ are used in linear feedback, the poles of the corresponding recursive description cannot be placed as will be shown subsequently. Recall that an open-loop system including the error dynamics observer is given by

$$\begin{pmatrix} x_{k+1}(\xi) \\ e_{k+1}(\xi) \end{pmatrix} = \begin{pmatrix} A & O \\ O & A - NC \end{pmatrix} \begin{pmatrix} x_k(\xi) \\ e_k(\xi) \end{pmatrix} + \begin{pmatrix} B \\ O \end{pmatrix} u_k(\xi) + \begin{pmatrix} B^w \\ B^w - ND^w \end{pmatrix} w_k(\xi)$$

where N is the (stationary) Kalman predictor gain. Then, introduce a feedback of the innovation and the state estimate

$$\begin{aligned} u_k(\xi) &= F\hat{x}_k(\xi) + K_v v_k(\xi) \\ &= Fx_k(\xi) + (K_v C - F)e_k(\xi) + K_v D^w w_k(\xi) \end{aligned} \tag{7.16}$$

leading to the closed-loop system

$$\begin{pmatrix} x_{k+1}(\xi) \\ e_{k+1}(\xi) \end{pmatrix} = \begin{pmatrix} A+BF & B(K_vC-F) \\ O & A-NC \end{pmatrix} \begin{pmatrix} x_k(\xi) \\ e_k(\xi) \end{pmatrix} + \begin{pmatrix} B^w + BK_vD^w \\ B^w - ND^w \end{pmatrix} w_k(\xi)$$

If the state-feedback is set to zero $F \equiv 0$, then there is no way of stabilizing open-loop unstable systems as then the poles of the closed-loop system contain the open-loop poles of A. The intuitive reason is illustrated by the following. Consider the semi-deterministic case in which an autonomous system with uncertain initial conditions

$$x_{k+1} = Ax_k, \quad y_k = Cx_k, \quad x_0 = x^0$$

is observed by the Luenberger observer

$$\hat{x}_{k+1} = A\hat{x}_k + Nv_k, \quad v_k = Ce_k, \quad e_k = x_k - \hat{x}_k$$

and suppose we use the feedback law

$$u_k = K_v v_k.$$

It follows immediately that the observer in error dynamics form is given by

$$e_{k+1} = (A-NC)e_k, \quad v_k = Ce_k, \quad e_0 = x^0$$

and assuming that the observer gain N is chosen stabilizing this implies

$$e_k \to 0 \quad \Rightarrow \quad v_k \to 0 \quad \text{as } k \to \infty.$$

Hence, the innovations sequence converges to zero if our observer is exponentially stable. But this means that the control input converges to zero even if the state estimate $\hat{x}$ is still non-zero! With a properly chosen state feedback gain, the control action can only be zero if both the innovation sequence v_k and the state-estimate $\hat{x}_k$ are zero. □

7.5 Recursive computation of the variance matrix

With the formal definition above, we can set up the receding horizon implementation of the closed-loop MPC problem. To do so, we must keep track of the uncertainty in the initial condition and the estimate thereof in order to compute the effect of variance of the future controlled variables. Let the control law

$$\mathbf{u}_0(\xi) = K_0 \mathbf{v}_0(\xi) \tag{7.17}$$

be put into a receding horizon implementation. Then, the control law above is replaced by

$$\mathbf{u}_k(\xi) = L_k \hat{x}_k(\xi) + K_k \mathbf{v}_k(\xi) \tag{7.18}$$

from sample k onwards. By substitution of the expression for the innovation sequence (7.5) it follows that

$$\mathbf{u}_k(\xi) = L_k \hat{x}_k(\xi) + K_k G^k_{ve} e_k(\xi) + K_k G^k_{vw} \mathbf{w}_k(\xi) \tag{7.19}$$

where the plant dynamics are given by equations (7.4). In the new optimization problem a single time step later, the matrix

$$Z_k = E\mathbf{z}_k(\xi)\mathbf{z}_k(\xi)^T$$

determines the future back-off and to compute the performance output $\mathbf{z}_k$, (7.19) is substituted in (7.6) to obtain

$$\begin{aligned}
\mathbf{z}_k(\xi) &= G^k_{zx} x_k(\xi) + G^k_{zu}\mathbf{u}_k(\xi) + G^k_{zw}\mathbf{w}_k(\xi) \\
&= G^k_{zx}\hat{x}_k(\xi) + G^k_{zx} e_k(\xi) + G^k_{zu}\mathbf{u}_k(\xi) + G^k_{zw}\mathbf{w}_k(\xi) \\
&= (G^k_{zx} + G^k_{zu} L_k)\hat{x}_k(\xi) + (G^k_{zx} + G^k_{zu} K_k G^k_{ve}) e_k(\xi) + \\
&\quad + (G^k_{zw} + G^k_{zu} K_k G^k_{vw})\mathbf{w}_k(\xi).
\end{aligned} \tag{7.20}$$

To compute Z_k efficiently, a recursion is needed to keep track of the joint variance matrix of the estimation error and the state estimate at the beginning of the control horizon (time sample k). From equation (7.20) we know that $\mathbf{z}_k(\xi)$ in closed-loop is some function of

$$\hat{x}_k(\xi),\ e_k(\xi),\ \text{and } \mathbf{w}_k(\xi).$$

Since $\hat{x}_k(\xi), e_k(\xi)$ are independent of $\mathbf{w}_k(\xi)$ it follows that we only need to keep track of the joint variance matrix

$$V_k = E\begin{pmatrix} \hat{x}_k(\xi) \\ e_k(\xi) \end{pmatrix}\begin{pmatrix} \hat{x}_k(\xi) \\ e_k(\xi) \end{pmatrix}^T$$

of $\hat{x}_k(\xi)$ and $e_k(\xi)$. Observe that the actual control move applied at each instant is given by L^1_k and K^{11}_k which are the first n_u rows of the control law L_k and K_k respectively, see (7.18),

$$u_k = L^1_k \hat{x}_k + K^{11}_k v_k.$$

Using Kwakernaak and Sivan (1972) as reference, we can directly write

$$\begin{aligned}
\begin{pmatrix} \hat{x}_{k+1}(\xi) \\ e_{k+1}(\xi) \end{pmatrix} &= \begin{pmatrix} A_k + B_k L^1_k & (N_k + B_k K^{11}_k) C_k \\ O & A_k - N_k C_k \end{pmatrix}\begin{pmatrix} \hat{x}_k(\xi) \\ e_k(\xi) \end{pmatrix} \\
&\quad + \begin{pmatrix} (N_k + B_k K^{11}_k) D^w_k \\ B^w_k - N_k D^w_k \end{pmatrix} w_k(\xi)
\end{aligned}$$

such that the joint variance matrix is recursively given by

$$\begin{aligned}
V_{k+1} &= \begin{pmatrix} A_k + B_k L^1_k & (N_k + B_k K^{11}_k) C_k \\ O & A_k - N_k C_k \end{pmatrix} V_k \begin{pmatrix} \star & \star \\ \star & \star \end{pmatrix}^T \\
&\quad + \begin{pmatrix} (N_k + B_k K^{11}_k) D^w_k \\ B^w_k - N_k D^w_k \end{pmatrix} W_k \begin{pmatrix} \star \\ \star \end{pmatrix}^T
\end{aligned}$$

with initial condition

$$V_0 = \begin{pmatrix} O & O \\ O & P_0 \end{pmatrix}.$$

From the projection theorem (Kailath, 1968) it is immediate that the estimation error $e_k(\xi)$ is orthogonal to estimate $\hat{x}_k(\xi)$ for any k. Hence, the joint variance matrix V_k is block-diagonal for each k by construction, therefore the variance matrices can be constructed efficiently by the following Riccati recursions for the estimation error

$$P^e_{k+1} = A_k P^e_k A_k^T - N_k(C_k P^e_k C_k^T + D^w_k W_k D^{wT}_k)N_k^T + B^w_k W_k (B^w_k)^T \tag{7.21}$$

with boundary condition

$$P^e_0 = P_0.$$

Then, given the recursion for the estimation error, the recursion for the variance matrix of the state-estimate follows from

$$\begin{aligned} P^{\hat{x}}_{k+1} &= (A_k + B_k L^1_k) P^{\hat{x}}_k (A_k + B_k L^1_k)^T \\ &\quad + (N_k + B_k K^{11}_k)(C_k P^e_k C_k^T + D^w_k W_k D^{wT}_k)(N_k + B_k K^{11}_k)^T \end{aligned}$$

with boundary condition

$$P^{\hat{x}}_0 = O.$$

L^1_k and K^{11}_k are given externally in every cycle by the solution of the closed-loop MPC problem. In both the CLMPC and the FHLQG problem, the Kalman predictor gain is given by

$$N_k = A_k P^e_k C_k^T (C_k P^e_k C_k^T + D^w_k W_k D^{wT}_k)^{-1}. \tag{7.22}$$

In the specific case of FHLQG control, the direct feedthrough term is determined from the Kalman and state feedback gain via

$$K^{11}_k = M_k = F_k P^e_k C_k^T (C_k P^e_k C_k^T + D^w_k W_k D^{wT}_k)^{-1}.$$

Then upon substitution one finds the expression

$$N_k + B_k K^{11}_k = (A_k + B_k L^1_k) P^e_k C_k^T (C_k P^e_k C_k^T + D^w_k W_k D^{wT}_k)^{-1}$$

which leads to the Riccati recursion

$$\begin{aligned} P^{\hat{x}}_{k+1} &= (A_k + B_k L^1_k) P^{\hat{x}}_k (A_k + B_k L^1_k)^T \\ &\quad + (A_k + B_k L^1_k) P^e_k C_k^T (C_k P^e_k C_k^T + D^w_k W_k D^{wT}_k)^{-1} C_k P^e_k (A_k + B_k L^1_k)^T \end{aligned}$$

This corresponds to the one step ahead prediction using the closed-loop system matrix, (compare to the Kalman predictor gain (7.22) which is prediction using the open-loop system matrix). The factored variance matrix of the initial condition and disturbances are

$$P^x_k = E x_k(\xi) x_k(\xi)^T = P^{\hat{x}}_k + P^e_k = F_P F_P^T, \quad W_k = F_W F_W^T.$$

By orthogonality we can directly sum the error and estimate variance matrices obtained with the Riccati recursions.

7.6 Recursive construction of the CLMPC controller

In the previous sections we have established recursive relations for the systems and signals, for the control law via the receding horizon implementation and for the variance matrices. It remains to show that we can recursively solve the optimization problem. In this section we formalize the previous investigations between LQG control and closed-loop MPC. The main idea is to show that the optimal solution to the closed-loop MPC problem is also optimal for *some* finite horizon LQG problem for which we have an optimal receding horizon implementation. Recall the CLMPC problem was defined as

$$\begin{array}{lcr}\textbf{(CLMPC)} & \displaystyle\min_{\mathbf{u}^r\in\mathbf{R}^{nn_u},\nu\in\mathbf{R}^m,K\in\mathbf{K}_0} & f(\mathbf{z}^r) \\ & \mathbf{z}^r = G_{zx}x_0^r + G_{zu}\mathbf{u}^r + G_{zw}\mathbf{w}^r & \\ & \nu_j + h_j^T\mathbf{z}^r \leq g_j,\ j=1,\dots,m & \\ & Z(K) = G_{zx}^K P G_{zx}^{K\,T} + G_{zw}^K W G_{zw}^{K\,T} & \\ & r\sqrt{h_j^T Z(K) h_j} \leq \nu_j,\ j=1,\dots,m & \end{array} \tag{7.23}$$

Suppose we square the last constraint and assume for ease of presentation that $r=1$ and define the data matrices

$$\begin{array}{ll} A = G_{zu} & B = \begin{pmatrix} G_{ve}F_P & G_{vw}F_W \end{pmatrix} \\ C = \begin{pmatrix} G_{zx}F_P & G_{zw}F_W \end{pmatrix} & X = K. \end{array}$$

Then Theorem 31 below summarizes the result of Section 5.4.

Theorem 31 Consider the following optimization problem (NOP)

$$\begin{array}{lcr} & \displaystyle\min_{\mathbf{p}\in\mathbf{R}^{nn_p},\nu\in\mathbb{R}^m,X\in\mathbf{R}^{nn_u\times nn_y}} & f(\mathbf{p}) \\ \text{(NOP)} & \nu_j + h_j^T\mathbf{p} \leq g_j, \quad j=1,\dots,m & \\ & X = \sum_{i,j} E_i X_{ij} E_j^T & \\ & h_j^T(AXB+C)(AXB+C)^T h_j \leq \nu_j^2, \quad j=1,\dots,m & \end{array} \tag{7.24}$$

where $\mathbf{X}$ is the set lower block triangular matrices, f is some smooth function. Suppose the solution $(\mathbf{p}^\star, X^\star, \nu^\star)$ is a local minimizer of (NOP) and at this feasible point a constraint qualification is satisfied. Then $X^\star$ is also optimal for the optimization problem

$$\min_{X = \sum_{i,j} E_i X_{ij} E_j^T} \quad \mathrm{tr}(AXB+C)^T R (AXB+C) \tag{7.25}$$

for some $R = R^T \succeq 0$.
Proof. See section 5.4. Define the corresponding Lagrangian of the (non-convex)

optimization problem to be

$$\mathbf{L}(\mathbf{p}, X, \nu, \Lambda, \lambda, \eta) = f(\mathbf{p}) + \sum_{j=1}^{m} \lambda_j h_j^T \mathbf{p} + \sum_{j=1}^{m} (\lambda_j \nu_j - \eta_j \nu_j^2) - \sum_{j=1}^{m} \lambda_j g_j$$
$$+ \sum_{j=1}^{m} \eta_j h_j^T (AXB + C)(AXB + C)^T h_j + \operatorname{tr} \Lambda^T (\sum_{i,j} E_i X_{ij} E_j^T - X).$$

Then, because the constraint qualification holds at $(\mathbf{p}^\star, X^\star, \nu^\star)$, it follows that there exist Lagrange multipliers

$$\lambda \in \mathbf{R}_+^m, \quad \mu \in \mathbf{R}_+^m, \quad \Lambda \in \mathbf{R}^{nn_u \times nn_y}$$

such that

$$\partial_{\mathbf{p}, X, \nu} L(\mathbf{p}^\star, X^\star, \nu^\star, \Lambda, \lambda, \eta) = 0$$
$$\sum_{j=1}^{m} \lambda_j (\nu^\star + h_j^T \mathbf{p}^\star - g_j) = 0$$
$$\sum_{j=1}^{m} \mu_j (h_j^T (AX^\star B + C)(AX^\star B + C)^T h_j - \nu_j^{\star 2}) = 0.$$

By application of the matrix gradient techniques as discussed in Section 5.4 one arrives at the result that the following equations are satisfied

$$A^T \left(\sum_{j=1}^{m} \eta_j h_j h_j^T \right) (AX^\star B + C) B^T = \Lambda,$$
$$\Lambda = \sum_{j > i \geq 1} E_i \Lambda_{ij} E_j^T,$$
$$X^\star = \sum_{i \geq j \geq 1} E_i X_{ij}^\star E_j^T \tag{7.26}$$

for some Lagrange multipliers $\eta \in \mathbf{R}_m^+$ and $\Lambda \in \mathbf{R}^{nn_u \times nn_y}$. Then, if we define

$$R = \sum_{j=1}^{m} \eta_j h_j h_j^T$$

it follows that R, being the positive sum of positive semi-definite symmetric rank-one matrices, is positive semi-definite itself. Therefore, the optimal controller satisfies the necessary optimality condition (7.26) that coincides with the necessary optimality condition of the optimization problem (7.25). Since $R \succeq 0$, this latter problem is convex and hence the necessary optimality condition (7.26) is also sufficient for $X^\star$ to be optimal for (7.25). □

Now we are able to state the main result.

Theorem 32 Consider the closed-loop MPC problem

$$\begin{array}{lr} \min\limits_{\mathbf{u}_0^r \in \mathbf{R}^{Nn_u}, \nu \in \mathbf{R}^m, K \in \mathbf{K}} & f(\mathbf{z}_0^r) \\ \nu_j + h_j^T \mathbf{z}_0^r \leq g_j, \quad j = 1, \ldots, m & \\ \mathbf{z}_0^r = G_{zx} x_0^r + G_{zu} \mathbf{u}_0^r + G_{zw} \mathbf{w}_0^r & \\ r h_j^T (AKB + C)(AKB + C)^T h_j \leq \nu_j^2, \quad j = 1, \ldots, m & \end{array} \tag{7.27}$$

where $A = G_{zu}$, $B = \begin{pmatrix} G_{ve}F_P & G_{vw}F_W \end{pmatrix}$ and $C = \begin{pmatrix} G_{zx}F_P & G_{zw}F_W \end{pmatrix}$. Let the rows h_j of the constraint matrix be such that each vector h_j constrains the performance output $\mathbf{z}^r$ on *single* time instances only and suppose there exists a solution $(\mathbf{u}^\star, \nu^\star, K^\star)$ at which a constraint qualification holds. Then, the receding horizon implementation is optimal for the closed-loop MPC problem.

Proof. By Theorem 31 it follows that the solution to the closed-loop MPC problem is optimal for some finite horizon LQG controller. If the constraints cover single time instances only, then for each constraint j there exist an index k_j and a vector $l_j^T \in \mathbf{R}^{n_z}$ such that

$$h_j^T \mathbf{z}_0 = l_j^T z_{k_j}$$

for all stacked vectors $\mathbf{z}_0 \in \mathbf{R}^{nn_z}$. It follows that the matrix

$$R = \sum_{j=1}^{m} \eta_j h_j h_j^T$$

is block diagonal, where each block R_k on the diagonal of R is a weighted sum of those l_j vectors corresponding to that index k in the time horizon. Therefore the corresponding finite horizon LQG objective function is of the form

$$E \sum_{k=0}^{N} z_k(\xi)^T R_k z_k(\xi), \quad R_k \succeq 0.$$

Then, the optimal solution consists of a Kalman predictor combined with a state feedback and innovations feedthrough where infinite gains are excluded by the existence of a solution to the closed-loop MPC problem. By Theorem 26 it follows that the receding horizon strategy is optimal for this finite horizon LQG problem and consequently also for the closed-loop MPC problem. □

7.7 Penalties on the rate of change of the input

Let us briefly illustrate why it is not restrictive to require that the performance variables are constrained on single time instances only. The reason is that one can usually reformulate the problem by extending the state space to include past values of the performance variables. In advanced process control this is important because it is common to quadratically penalize the changes in the control inputs in the objective function

$$E\sum_{k=1}^{n}(z_k(\xi)-z_k^r)^T Q_k(z_k(\xi)-z_k^r)+\Delta u_k(\xi)^T R_k \Delta u_k(\xi).$$

where the changes in inputs is given by

$$\Delta u_k(\xi)=u_k(\xi)-u_{k-1}(\xi).$$

and evenly common are inequality constraints on the rate of change of the inputs. In this section we are interested in the receding horizon implementation of the closed-loop model predictive control law in the case that there are rate constraints on the control input sequence. Again the relation between the two is given by Lagrangian duality, but it is clear that if we want to make use of Theorem 32 to reveal this, the problem must be reformulated such that only constraints defined on single time instances are left. Since the quadratic penalty on the change of the input covers cross-products of inputs on different time instances, the standard LQG result applies only after adding the previous input to the state vector of the LTV system

$$x_k^+(\xi)=\begin{pmatrix} x_k(\xi)\\ u_{k-1}(\xi)\end{pmatrix}.$$

The variational system is a recursive linear discrete time dynamical system rewritten with relative inputs $\Delta u_k(\xi)$ as

$$x_{k+1}^+(\xi)=\left(\begin{array}{cc|cc} A_k & B_k & B_k & B_k^w\\ O & I & I & O\end{array}\right)\left(\begin{array}{c} x_k^+(\xi)\\ \hline \Delta u_k(\xi)\\ w_k(\xi)\end{array}\right).$$

The optimal LQG solution will then be a state feedback of (the estimate of) the extended state including the estimate of the control input. An observer for this system is obtained as

$$\hat{x}_{k+1}^+(\xi)=\left(\begin{array}{cc|cc} A_k & B_k & B_k & N_k\\ O & I & I & O\end{array}\right)\left(\begin{array}{c} \hat{x}_k^+(\xi)\\ \hline \Delta \hat{u}_k(\xi)\\ v_k(\xi)\end{array}\right).$$

But, since the control input itself is available information for state estimation one will in general have

$$\hat{u}_k(\xi)=u_k(\xi)$$

such that N_k is the usual Kalman predictor gain. The input to the LTV system is given by the state feedback on the extended system and the innovation feedthrough

$$\begin{aligned}\Delta u_k(\xi)&=L_k^1\hat{x}_k^+(\xi)+K_k^{11}v_k(\xi)\\ &=L_k^{1,x}\hat{x}_k(\xi)+L_k^{1,u}u_k(\xi)+K_k^{11}v_k(\xi).\end{aligned}$$

Applying this feedback to the stochastic system leads to the closed-loop system

$$\left(\begin{array}{c}\hat{x}_{k+1}^{+}(\xi)\\ \hline e_{k+1}(\xi)\end{array}\right)=\left(\begin{array}{cc|c}A_k+B_kL_k^{1,x} & B_k(I+L_k^{1,u}) & (N_k+B_kK_k^{11})C_k\\ L_k^{1,x} & I+L_k^{1,u} & K_k^{11}C_k\\ \hline O & O & A_k-N_kC_k\end{array}\right)\left(\begin{array}{c}\hat{x}_k^{+}(\xi)\\ \hline e_k(\xi)\end{array}\right)$$

$$+\left(\begin{array}{c}(N_k+B_kK_k^{11})D_k^w\\ K_k^{11}D_k^w\\ \hline G_k-N_kD_k^w\end{array}\right)w_k(\xi)$$

for which the (block-diagonal) joint variance matrix is determined as

$$V_k=E\begin{pmatrix}\hat{x}_k^{+}(\xi)\\ e_k(\xi)\end{pmatrix}\begin{pmatrix}\hat{x}_k^{+}(\xi)\\ e_k(\xi)\end{pmatrix}^T,\quad V_0=\begin{pmatrix}O & O\\ O & P_0\end{pmatrix},\quad Ew_kw_k^T=W_k.$$

The blocks are determined via the estimation Riccati recursion (7.21), while for the estimate one finds

$$P_{k+1}^{\hat{x}}=\begin{pmatrix}A_k+B_kL_k^{1,x} & B_k(I+L_k^{1,u})\\ L_k^{1,x} & I+L_k^{1,u}\end{pmatrix}P_k^{\hat{x}}\begin{pmatrix}\star & \star\\ \star & \star\end{pmatrix}^T$$

$$+\begin{pmatrix}N_k+B_kK_k^{11}\\ K_k^{11}\end{pmatrix}(C_kP_k^eC_k^T+D_k^wW_kD_k^{wT})\begin{pmatrix}\star\\ \star\end{pmatrix}^T.$$

This analysis explains how the estimation problem is set-up and how one computes the necessary initial variance matrices, the next step is to construct the actual optimization problem. To this end consider the following extended performance output

$$\mathbf{z}_k^{+}(\xi)=\begin{pmatrix}\mathbf{z}_k(\xi)\\ \boldsymbol{\Delta}\mathbf{u}_k(\xi)\end{pmatrix}$$

where the performance outputs are as usual given by

$$\mathbf{z}_k(\xi)=G_{zx}^kx_k(\xi)+G_{zu}^k\mathbf{u}_k(\xi)+G_{zw}^k\mathbf{w}_k(\xi).$$

To go from absolute inputs to relative inputs we have the following relations

$$\mathbf{u}_k(\xi)=Ju_{k-1}(\xi)+T\boldsymbol{\Delta}\mathbf{u}_k(\xi)$$

where

$$T=\begin{pmatrix}I & O & O & \cdots & O\\ I & I & O & \cdots & O\\ I & I & I & \cdots & O\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ I & \cdots & \cdots & I & I\end{pmatrix},\quad J=\begin{pmatrix}I\\ I\\ I\\ \vdots\\ I\end{pmatrix}$$

hence, the performance output is represented in the change in inputs

$$\mathbf{z}_k^{+}(\xi)=\left(\begin{array}{cc|ccc}G_{zx}^k & G_{zu}^kJ & G_{zx}^k & G_{zu}^kT & G_{zw}^k\\ O & O & O & I & O\end{array}\right)\left(\begin{array}{c}x_k^{+}(\xi)\\ \hline \mathbf{e}_k(\xi)\\ \boldsymbol{\Delta}\mathbf{u}_k(\xi)\\ \mathbf{w}_k(\xi)\end{array}\right)$$

The feedback in lifted form is given by the following relative control sequence

$$\begin{aligned}\boldsymbol{\Delta}\mathbf{u}_k(\xi) &= L_k\hat{x}_k^+(\xi) + K_k\mathbf{v}_k(\xi)\\ &= L_k^x\hat{x}_k(\xi) + L_k^u u_{k-1}(\xi) + K_k\mathbf{v}_k(\xi)\end{aligned}$$

or corresponding absolute control sequence

$$\mathbf{u}_k(\xi) = TL_k^x\hat{x}_k(\xi) + (J + TL_k^u)u_{k-1}(\xi) + TK_k\mathbf{v}_k(\xi).$$

The closed-loop performance outputs are compactly represented by

$$\mathbf{z}_k^+(\xi) = (A^kM_kB^k + C^k)\begin{pmatrix}\hat{x}_k^+\\ e_k\\ \mathbf{w}_k\end{pmatrix}$$

where the matrices A^k, B^k, M_k, C^k are given in terms of the system dynamics by

$$A^k = \begin{pmatrix} G_{zu}^kT \\ I \end{pmatrix}, \qquad B^k = \left(\begin{array}{cc|cc} I & O & O & O\\ O & I & O & O\\ \hline O & O & G_{ve}^k & G_{vw}^k\end{array}\right).$$

$$C^k = \left(\begin{array}{cc|cc} G_{zx}^k & G_{zu}^kJ & G_{zx}^k & G_{zw}^k\\ \hline O & O & O & O\end{array}\right) \qquad M_k = \begin{pmatrix} L_k^x & L_k^u & K_k\end{pmatrix}$$

How to solve this problem is shown in the next section.

7.8 State feedback implementation issues

The final step is the actual calculation of this additional state feedback gain L_k. To construct L_k we apply the same techniques as before, that is either the Cholesky factorization in the case of the FHLQG problem or the Kronecker solution in the case of true CLMPC. The future input is in both cases and in any iteration given by the feedback control law consisting of a state feedback gain and innovations feedback gain

$$\mathbf{u}_k(\xi) = L_k\hat{x}_k(\xi) + K_k\mathbf{v}_k(\xi).$$

The performance output, expressed in terms of the state estimate $\hat{x}_k(\xi)$ and estimation error $e_k(\xi)$, is given by

$$\mathbf{z}_k(\xi) = (G_{zx}^k + G_{zu}^kL_k)\hat{x}_k(\xi) + (G_{zx}^k + G_{zu}^kK_kG_{ve}^k)e_k(\xi) + (G_{zw}^k + G_{zu}^kK_kG_{vw}^k)\mathbf{w}_k(\xi).$$

To simplify notation, the optimization parameters are collected in a single lower block triangular matrix (since L_k is a full block matrix)

$$M_k = \begin{pmatrix} L_k & K_k \end{pmatrix} \in \mathbf{M_k}$$

where $\mathbf{M_k}$ is the set of lower block triangular matrices

$$\mathbf{M}_k = \left\{ \begin{pmatrix} L & K \end{pmatrix} : \; L \in \mathbb{R}^{nn_u \times n_x}, \; K \in \mathbf{K}_k \right\}$$

and in analogy with the original problem

$$\mathbf{K}_k = \left\{ \sum_{i=1}^{n-k} \sum_{j=1}^{i} E_i K_{ij} E_j^T \; : \; K_{ij} \in \mathbb{R}^{n_u \times n_y} \right\}.$$

For notational convenience we further introduce the matrices

$$\begin{aligned} A^k &= G_{zu}^k, \\ B^k &= \begin{pmatrix} F_{P^{\hat{x}_k}} & O & O \\ O & G_{ve}^k F_{P^e} & G_{vw}^k F_W \end{pmatrix} \\ C^k &= \begin{pmatrix} G_{zx}^k F_{P^{\hat{x}_k}} & G_{zx}^k F_{P^e} & G_{zw}^k F_W \end{pmatrix} \end{aligned}$$

Then, it is immediate that the variance matrix of the performance output process is given by the product

$$Z_k = (A^k M_k B^k + C^k)(A^k M_k B^k + C^k)^T$$

which is factored by construction and therefore the back-off is determined as

$$\nu_j = r\sqrt{(h_j^k)^T Z_k h_j^k} = r\|(h_j^k)^T A^k M_k B^k + (h_j^k)^T C^k\|_2. \tag{7.28}$$

As before, the optimization parameters enter in an affine fashion and therefore, the CLMPC with the additional state feedback is again found to be a second order cone program. In the notation of subsection 4.4.5 we have the optimization parameter

$$\begin{aligned} \mathbf{m} &= \mathbf{vec}(M) \\ &= \mathbf{vec}\begin{pmatrix} L & \sum_{i \geq j \geq 1} E_i K_{ij} E_j^T \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{vec}(L) \\ \sum_{i \geq j \geq 1} E_j \otimes E_i \mathbf{vec}(K_{ij}) \end{pmatrix} \end{aligned}$$

Define $p_{ij} = \mathbf{vec}(K_{ij})$, $\mathbf{l} = \mathbf{vec}(L)$ and

$$U := \begin{pmatrix} E_1 \otimes E_1 & E_1 \otimes E_2 & E_2 \otimes E_2 & \cdots \end{pmatrix}, \quad \mathbf{p}^T = \begin{pmatrix} p_{11}^T & p_{12}^T & p_{22}^T & \cdots \end{pmatrix}$$

to arrive at the compact representation

$$\mathbf{m} = \begin{pmatrix} \mathbf{l} \\ U\mathbf{p} \end{pmatrix} = \begin{pmatrix} I & O \\ O & U \end{pmatrix} \begin{pmatrix} \mathbf{l} \\ \mathbf{p} \end{pmatrix}$$

Substitution into (7.28) gives the vectorized back-off formula

$$\nu_j(\mathbf{l}, \mathbf{p}) = r\|C^k h_j^k + B^{k^T} \otimes (h_j^{k^T} A^k) \begin{pmatrix} I & O \\ O & U \end{pmatrix} \begin{pmatrix} \mathbf{l} \\ \mathbf{p} \end{pmatrix}\|_2$$

With this transformation, all variables appear as vectors in the optimization problem, which shows that the problem is in the format

$$\begin{array}{cc} \min\limits_{\mathbf{p},\mathbf{l},\mathbf{u}^r} & f(\mathbf{z}^r) \\ \mathbf{z}^r = G_{zx}x_0^r + G_{zu}\mathbf{u}^r + G_{zw}\mathbf{w}^r & \\ \nu_j(\mathbf{l},\mathbf{p}) + h_j^T\mathbf{z}^r \leq g_j, \quad j = 1,\ldots,m. & \\ \nu_j(\mathbf{l},\mathbf{p}) = r\|C^k h_j^k + B^{k^T} \otimes (h_j^{k^T} A^k) \begin{pmatrix} I & O \\ O & U \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{p} \end{pmatrix} \|_2 & \end{array}$$

Note that in the same fashion as before one obtains finite horizon LQG solutions by solving the first order optimality condition

$$A^{k^T}(A^k M_k B^k + C^k)B^{k^T} = \Lambda_k, \quad \Lambda_k \in \mathbf{\Lambda}_k \tag{7.29}$$

where

$$\mathbf{\Lambda}_k = \left\{ \begin{pmatrix} O & \sum_{i=1}^{n-k}\sum_{j=i}^{n-k} E_i \Lambda^{ij} E_j^T \end{pmatrix} : \ \Lambda^{ij} \in \mathbb{R}^{n_u \times n_y} \right\} \tag{7.30}$$

where a little caution is needed when it comes to the full rank condition of B^k. In this new case, the existence of measurement noise is not sufficient to ensure that $B^k B^{k^T}$ has an inverse, see also the discussion in remark 22.

7.9 Chapter summary

In this chapter the receding horizon implementation of the closed-loop MPC problem has been developed. The key property of the optimal feedback controller of the closed-loop MPC problem is that it is also optimal for some finite horizon LQG controller. This optimal feedback controller is therefore constructed recursively using a combination of state and innovations feedback. Because such a controller is observer-based, measurement data is processed very efficiently by implementing a Kalman predictor to compute the state-estimate and the innovations sequence. Because of the state feedback structure, the closed-loop variance matrices are also constructed recursively by means of Riccati difference equations. This means that the whole optimization problem is constructed recursively with a *fixed* complexity in terms of the amount of optimization parameters bounding the on-line computation time.

8 An Industrial Polymerization Reactor

In this chapter the techniques discussed in this chapter are applied to a realistic simulator of a polymerization reactor process. The performance of constrained finite horizon LQG control is compared to an open-loop MPC approach.

8.1 Introduction

The theory that was developed in the previous chapters will be applied to a high density polyethylene (HDPE) continuous polymerization reactor. The main purpose of this implementation is twofold. First, it will be illustrated how the theory is applied to dynamic nonlinear chemical process systems using a nonlinear differential-algebraic model, the current standard in modern process modelling. Second, the performance of closed-loop predictive control will be compared to a set-up consisting of an extended Kalman filter and a linear time-varying MPC, which is taken as the industrial state-of-the-art in applied nonlinear model-based process control, see (Lee and Ricker, 1994).

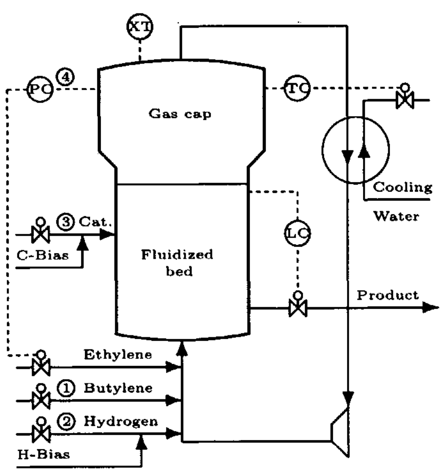
It cannot be overemphasized that the HDPE process is used for illustration purposes *only*. For a full discussion on the control and operation of the HDPE process we refer to the series of papers by McAuley and co-workers (1991,1992,1993) in the chemical engineering literature. The model used in this thesis is based on the model equations available in literature (Choi and Ray, 1992; McAuley *et al.*, 1990). The actual implementations was done by R.L. Tousain (2002) and extended by W. Van Brempt (2000). We greatly appreciate that they have made their modelling efforts available to us. The model is build in a generic process modelling language gPROMS, (PSE, 2003), that is suited for large scale nonlinear differential algebraic modelling.

8.2 The HDPE process

The HDPE process is a continuous fluidized bed polymerization reactor using solid (Ziegler-Natta) catalyst particles. The schematic process lay out is given in figure 8.1. The reactor consists of a gascap, a fluidized bed and a material recycle loop. The main feed components are Ethylene, Butylene and Hydrogen. Nitrogen does flow through the reactor, but it is an inert gas for the reactions and is used to maintain the fluidized bed and cooling power. In normal operation, the Nitrogen feed is zero.

The main reactions that take place in the fluidized bed are (McAuley *et al.*, 1990): 1) *Catalyst Activation*: potential sites on catalyst surface are activated at the start of polymerization process. 2) *Chain Initialization*: first (co)monomer occupies active sites to form living chains. 3) *Chain Propagation*: new (co)monomers are inserted between active sites and living chains. 4) *Chain Transfer*: (co)monomer terminates living chains and forms new living chains on catalyst. 5) *Catalyst Deactivation*: active sites on catalyst particle are irreversibly turned into a dead sites.

The process has a basic control system consisting of three main PI loops. The pressure in the gas cap is controlled by the main feed of Ethylene and the set point of the pressure controller u_4 therefore provides the main means to control production. The reactions are highly exothermic and the temperature PI controller adjusts the cooling water flow to a counter-current heat exchanger to remove this heat of reaction. The cooling water flow through the valve is constrained by pressure drop limitations. Violation of this constraint leads to reaction run-away by the inherent open-loop instability. The reactor product flow is regulated by level control.



Controlled variables			
z_1	ρ	[kg/m^3]	density
z_2	LNMI	[-]	melt-index
z_3	FWC	[kg/hr]	cool water flow
z_4	PF	[kg/hr]	product flow
Measured outputs			
y_1	$x(C_2H_4)$	[kg/kg]	Ethylene gasfrac.
y_2	$x(C_4H_8)$	[kg/kg]	Butylene gasfrac.
y_3	$x(H_2)$	[kg/kg]	Hydrogen gasfrac.
Manipulated variables			
$u_1(z_5)$	C_4H_8	[kg/hr]	Butylene
$u_2(z_6)$	H_2	[kg/hr]	Hydrogen
$u_3(z_7)$	$TiCl_4$	[kg/hr]	Catalyst
$u_4(z_8)$	SP_p	[bar]	pressure setpoint
Disturbances			
d_1	H-Bias	[kg/hr]	Hydrogen bias flow
d_2	C-Bias	[kg/hr]	Catalyst bias flow
w_i	–	[-]	measurement noise

Figure 8.1: Process schematic of the fluidized bed reactor

8.3 A software architecture for advanced process control

A basic requirement for advanced process control is that a dynamic model of the process is available. Linear model predictive control is well established in industry, (Qin and Badgewell, 2003), and standard software is sold off the shelf. As discussed in Chapter 1, the next level of automation used dynamic first principle models of processes and the current industrially available optimization and MPC solutions are unable to use these nonlinear models online. Within the EU-funded INCOOP[1], (**IN**tegration of **CO**ntrol and plant-wide **OP**timization) a generic interface for simulation, optimization and control was developed (Tousain, 2000; Van Hessem and Tousain, 2001; Kadam *et al.*, 2002). In this set-up, a central data server was used to exchange commands and data between the modelling platform gPROMS and the computational environment MATLAB (The Mathworks, 2002), in which all control algorithms were programmed. Although all connections are OPC compliant and can be connected to an actual plant, we have limited ourselves to a simulated process in this chapter.

8.4 Open-loop implementation of a grade change scenario

The continuous HDPE process is a multi-product plant able of producing polyethylene polymer in several grade specifications. In an industrial environment, these product specifications are traditionally defined in terms of the density and melt-index. The melt-index is related to the viscosity, elasticity, tensile and impact strength and stress crack resistance, while the density is related to stiffness, hardness, transparency, flexibility and heat resistance (McAuley and MacGregor, 1992). Without discussion of the details, we will consider a grade change from grade **A** to grade **B** defined by constraints on the density z_1 and melt-index z_2 of the polymer, see table 8.1.

Table 8.1: Grade definitions

Grade		**A**	$\rightarrow$	**B**	Δ
$\bar{z}_1^g$	[kg/m^3]	942.9		937.9	-5
$\bar{z}_2^g$	[-]	1.1		3.1	$+2$

In the first step the nominal transition is optimized, after which the performance is evaluated for the disturbance scenario. To find the optimal transition starting from grade **A**, a quadratic objective function as proposed by McAuley and MacGregor (1992) is considered that penalizes the quadratic norm of the difference between the desired end-specifications of grade **B**. An additional weight on the rate of change of

[1]GRD1-1999-10628 project in *Competitive and Sustainable Growth* Programme (1999-2003)

the control inputs

$$\Delta\bar{u}_k^r := \bar{u}_{k+1}^r - \bar{u}_k^r$$

is added to avoid excessive control moves from the outset. This gives the following objective function

$$J(\Delta\mathbf{u}_0^r) = \sum_{k=0}^{N}\sum_{j=1}^{3} r_j^k \Delta\bar{u}_j^{r2} + \int_0^T \sum_{j=1}^{3} q_j(t)(\bar{z}_j^r(t) - \bar{z}_j^g(t))^2 dt.$$

During the transition, we want the following inequality constraints to be satisfied

Table 8.2: Constraints on process variables

Variable		lo	up
z_3	[10^4kg/h]	2.5	5.2
z_5	[10^2kg/h]	0.0	1.5
z_6	[kg/h]	0.1	1.0
z_7	[kg/h]	0.2	2.0

This brings us to the deterministic reference optimization problem to be solved for the grade change scenario (GOP)

$$\begin{array}{ll} \min\limits_{\mathbf{u}_0^r \in \mathbf{R}^{Nn_u}} & \sum_{k=0}^{N}\sum_{j=1}^{3} r_j^k \Delta\bar{u}_j^{r2} + \int_0^T \sum_{j=1}^{3} q_j(\bar{z}_j^r - \bar{z}_j^g)^2 dt \\ & 0 = f(\dot{\bar{x}}^r, \bar{x}^r, \bar{v}^r, \bar{u}^r, \bar{w}^r) \\ & \bar{z}^r = C_z^x \bar{x}^r + C_z^v \bar{v}^r + D_z^u \bar{u}^r + D_z^w \bar{w}^r \\ & \bar{u}^r(t) = \bar{u}_k^r, \quad t \in [t_k, t_{k+1}) \\ & z_3^{\text{lo}} \le \bar{z}^r(t) \le z_3^{\text{up}} \\ & z_5^{\text{lo}} \le \bar{z}^r(t) \le z_5^{\text{up}} \\ & z_6^{\text{lo}} \le \bar{z}^r(t) \le z_6^{\text{up}} \\ & z_7^{\text{lo}} \le \bar{z}^r(t) \le z_7^{\text{up}} \end{array} \tag{8.1}$$

Notice that the controls are assumed to be constant over each time interval $[t_k, t_{k+1})$ and therefore appear as a double sum in the objective. To facilitate efficient implementation, we discretize the integral via a zero-order integration scheme (trapezoidal rule can alternatively be applied to give computationally cheap evaluations). Sampling the performance outputs on the control sample times

$$\bar{z}_{j,k}^r = \bar{z}_j^r(t_k)$$

then gives objective function

$$\begin{aligned}
J(\Delta \mathbf{u}_l) &= \sum_{k=0}^{N}\sum_{j=1}^{3} r_k^j (\Delta \bar{u}_{l,k}^{r,j})^2 + Ts\sum_{k=0}^{N}\sum_{j=1}^{3} q_j^k (\bar{z}_{l,j}^r(t_k) - \bar{z}_j^g(t_k))^2 \\
&= \sum_{k=0}^{N} \Delta \bar{u}_{k,l}^{r}{}^T R_k \Delta \bar{u}_{l,k}^r + (\bar{z}_{l,k}^r - \bar{z}_k^g)^T Q_k (\bar{z}_{l,k}^r - \bar{z}_k^g) \\
&= \mathbf{\Delta \bar{u}}_l^{r\,T} R \mathbf{\Delta \bar{u}}_l^r + (\bar{\mathbf{z}}_l^r - \bar{\mathbf{z}}^g)^T Q (\bar{\mathbf{z}}_l^r - \bar{\mathbf{z}}^g)
\end{aligned}$$

where $R = \mathrm{diag}(R_1, R_2, \dots, R_N)$ and $Q = \mathrm{diag}(Q_1, Q_2, \dots, Q_N)$. For clarity

$$\Delta \bar{u}_{l,k}^{r,j}$$

means the rate of change of the reference trajectory of the j^{th} input in iteration l, on the time instant k.

Because the summation over the control sample time T_s has a high resolution in the time window of length T, there is a negligible loss of dynamic performance in the on-line implementation due to this approximation. The path constraints in the optimization problem are given in table 8.2 and are enforced on the control sample times only. Then a sequential receding horizon approach to the dynamic optimization problem is used as was discussed in Section 4.6. Along an initial guess for the optimal trajectory

$$\bar{x}_l,\ \bar{v}_l,\ \bar{u}_l,\ \bar{w}_l, \bar{z}_l$$

that satisfies the nonlinear dynamics in iteration l, we linearize the dynamics

$$\begin{aligned}
0 &= \partial_{\dot{\bar{x}}} f|_l\, \dot{x}^r + \partial_{\bar{x}} f|_l\, x^r + \partial_{\bar{v}} f|_l\, v^r + \partial_{\bar{u}} f|_l\, u^r + \partial_{\bar{w}} f|_l\, w^r, \qquad x^r(0) = x_0^r \\
z^r &= C_z^x x^r + C_z^v v^r + D_z^u u^r + D_z^w w^r
\end{aligned}$$

to obtain the linearized optimization problem for the search directions d_l. The update on the control moves is given by

$$\bar{\mathbf{u}}_{l+1}^r = \bar{\mathbf{u}}_l^r + \mathbf{u}_l^r = \bar{\mathbf{u}}_l^r + \alpha d_l$$

because we apply the full Newton steps without line search we set $\alpha = 1$. Then, the approximate update $\mathbf{z}_0^r$ on the controlled variables is given by

$$\bar{\mathbf{z}}_{l+1} \simeq \bar{\mathbf{z}}_l + \mathbf{z}_l^r = \bar{\mathbf{z}}_l + G_{zu}^l d_l.$$

Hence, in terms of d_l, we obtain the linearized grade change optimization problem $(\mathrm{LGOP})_l$

$$\begin{aligned}
\min_{d_l \in \mathbf{R}^{Nn_u}} \quad & 2\mathbf{\Delta \bar{u}}_l^{r\,T} R \Delta d_l + \Delta d_l^T R \Delta d_l + 2(\bar{\mathbf{z}}_l^r - \bar{\mathbf{z}}^g)^T Q G_{zu}^l d_l + d_l^T G_{zu}^l{}^T R G_{zu}^l d_l \\
& \mathbf{z}_l^r = G_{zx} x_{l,0}^r + G_{zu}^l d_l \\
& h_j^T \mathbf{z}_l^r \leq g_j - h_j^T \bar{\mathbf{z}}_{l,k}^r
\end{aligned}$$

This is a standard QP that is easily and efficiently solved for d_l leading to the updated controls $\bar{\mathbf{u}}_{l+1}$. These controls are then used to integrate the model again leading to a new trajectory

$$\bar{x}_{l+1},\ \bar{v}_{l+1},\ \bar{u}_{l+1},\ \bar{w}_{l+},\bar{z}_{l+1}.$$

To illustrate this full Newton step approach the first four iterations have been plotted in figure 8.2.

8.5 Disturbance scenario for closed-loop simulations

In the next sections, the closed-loop performance will be analyzed. Two persistent disturbances are considered on the feed flows of hydrogen and the catalyst both strongly counteracting the desired transition. The local flow controllers compensate these measurement biases by introducing biases in the mass flows. The Hydrogen bias (H-Bias) strongly influences the melt-index and a negative value of this disturbance is chosen to counteract the transition from **A** to **B**. The advanced controller must increase the hydrogen feed to compensate for this bias. A positive bias in the catalyst feed (C-Bias) leads to a significant increase in energy hold-up and is sufficiently severe to let the temperature controller TC increase the cooling water flow up to saturation of the flow constraint. The steady state effects of the biases on the performance process variables is displayed in table 8.3. Measurement noise is active on all output measurements.

Table 8.3: Steady-state effects catalyst and hydrogen biases

[kg/h]	nominal	ΔH_2	$\Delta TiCl_4$	both	Δz	
3σ		-0.050	$+0.025$			[kg/h]
z_1	942.9	942.9	943.2	943.2	$+0.3$	[kg/m^3]
z_2	1.1	0.72	1.0	0.70	-0.3	[-]
z_3	5.0	5.0	5.4	5.4	$+0.4$	[10^4 kg/h]
z_4	3.65	3.65	3.76	3.76	$+0.11$	[10^3 kg/h]

In the simulations to come, there will be violations of the cooling water flow constraint. The inequality constraints are not taken into account during simulation but only during optimization, which is done to make the results of the open- and closed-loop predictive control techniques comparable.

8.6 Kalman filter based linear time-varying MPC

In this section we will show the results of the most commonly proposed nonlinear model predictive control strategy applied to the HPDE reactor. This strategy is

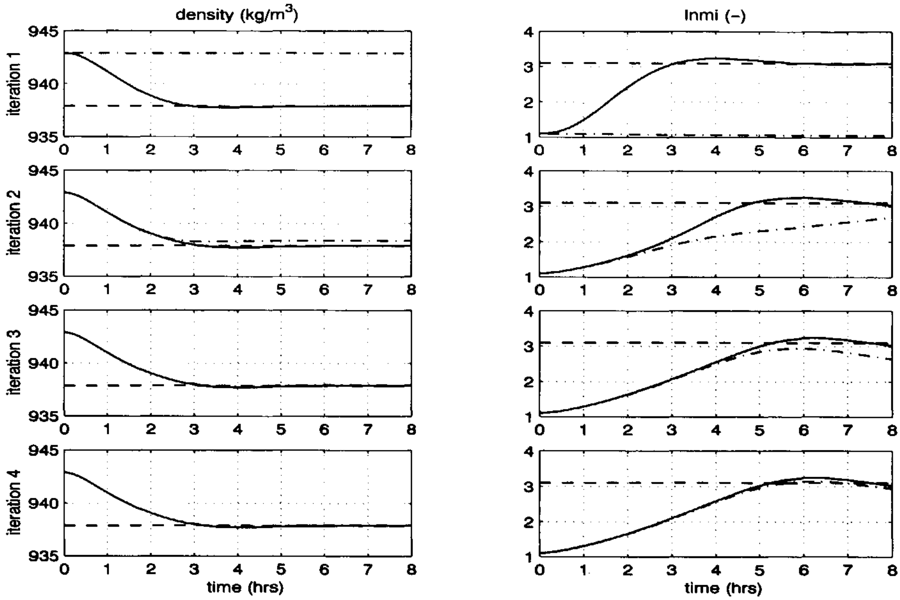


Figure 8.2: First four iterations in LTV method. Performance after optimization before simulation (solid), performance before optimization after simulation (dash-dotted), nominal target performance (solid).

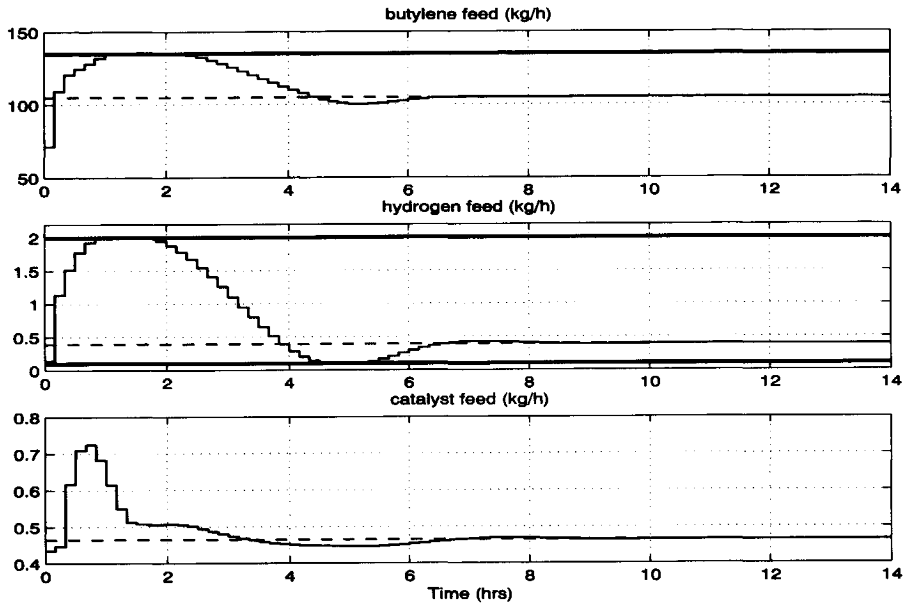


Figure 8.3: Open-loop trajectories of the control inputs. Applied inputs (solid) and nominal target inputs (dashed).

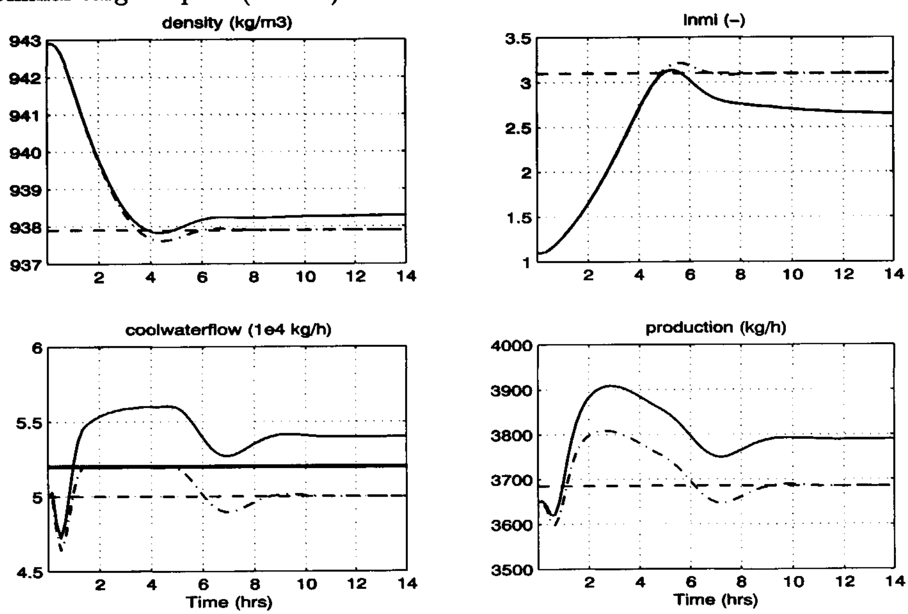


Figure 8.4: Open-loop trajectories of the controlled variables for the grade change. Real performance (solid), reference performance (dash-dotted), target performance grade **B**(dashed).

adequately discussed by Lee and Ricker (1994) and which has also been used within the INCOOP project in relation to rigorous dynamic optimization. In this approach, an extended Kalman filter is used to estimate the state of the process, while a linear time-varying MPC is used for *both* tracking performance and disturbance rejection. Detailed descriptions of the extended Kalman filter (EKF) are available in literature (Robertson *et al.*, 1996; Lewis, 1986; Jazwinsky, 1970) and therefore we will be brief in its discussion.

8.6.1 The extended Kalman filter

Let us use the following notation $\hat{x}_{t_1|t_2}$ denotes the best minimal variance estimate of $x(t_1)$ using all available measurement data y upto time t_2. Suppose we are given the covariance matrix P^e_{k-1} of the estimation error and an initial estimate $\hat{x}_{k-1|k-1}$. First, we integrate the model to find the estimate $\hat{x}_{k|k-1}$. That is, we seek solutions

$$\hat{\bar{x}},\ \hat{\bar{v}},\ \hat{\bar{y}}$$

that, for given the input trajectories

$$\bar{u},\ \bar{w},$$

satisfy the model equations

$$\begin{aligned} 0 &= f(\dot{\hat{\bar{x}}}, \hat{\bar{x}}, \hat{\bar{v}}, \bar{u}, \bar{w}), \quad \hat{\bar{x}}(t_{k-1}) = \hat{x}_{k-1|k-1} \\ \hat{\bar{y}} &= C^x_y \hat{\bar{x}} + C^v_y \hat{\bar{v}} + D^u_y \bar{u} + D^w_y \bar{w} \end{aligned}$$

over the time span $[t_{k-1}, t_k]$ and then set

$$\hat{\bar{x}}_{k|k-1} = \hat{\bar{x}}(t_k) \text{ and } \hat{\bar{y}}_{k|k-1} = \hat{\bar{y}}(t_k).$$

As soon as the measurement y_k arrives we compute the innovations

$$v_k = y_k - \hat{\bar{y}}_{k|k-1}$$

and we update the state estimate as

$$\hat{\bar{x}}_{k|k} = \hat{\bar{x}}_{k|k-1} + \bar{N}_k v_k$$

where the Kalman Gain is given by

$$\bar{N}_k = P^e_k C^T_k (C_k P^e_k C^T_k + D^w_k W_k D^{wT}_k)^{-1}.$$

The estimation error covariance matrix P^e_k is then updated via

$$P^e_{k+1} = A_k P^e_k A^T_k - A_k \bar{N}_k (C_k P^e_k C^T_k + D^w_k W_k D^{wT}_k) \bar{N}^T_k A^T_k + B^w_k W_k B^{wT}_k$$

where (A_k, B^w_k, C_k, D^w_k) are the local discrete time matrices derived from the nonlinear model f at $(\dot{\hat{\bar{x}}}(t_k), \hat{\bar{x}}(t_k), \hat{\bar{v}}(t_k), \bar{u}(t_k), \bar{w}(t_k))$. Several variations on this procedure are possible, such as making multiple iterations, or using a recursive prediction error method (Kozub and Macgregor, 1992) but this is not pursued here.

8.6.2 The open-loop MPC

The basic optimization problem in the MPC controller is the following. Suppose at time t_k we are given the estimates for the states $\hat{\bar{x}}_{k+1|k}$ and an initial guess for the nonlinear input sequence $\bar{u}_l^p$ and disturbances $\bar{w}_l^p$ over a future horizon $[t_{k+1}, t_{k+1} + nT_s]$ of n samples with sample time T_s. The superscript p is used to denote prediction and should be interpreted as in Section 5.2, while the subscript l denotes cycle or iteration in the optimization. Then, using this information we generate prediction trajectories trajectories satisfying the nonlinear model equations

$$\begin{aligned}0 &= f(\dot{\bar{x}}_l^p, \bar{x}_l^p, \bar{v}_l^p, \bar{u}_l^p, \bar{w}_l^p), \quad \bar{x}^p(t_{k+1}) = \hat{\bar{x}}_{k+1|k}\\ \bar{z}^p &= C_z^x \bar{x}_l^p + C_z^v \bar{v}_l^p + D_z^u \bar{u}_l^p + D_z^w \bar{w}_l^p.\end{aligned}$$

We seek to solve the following optimization problem

$$\begin{array}{ll}\min\limits_{\bar{u}_{l+1}^p} & \int_{t_k}^{t_k+nT_s} \Delta\bar{u}_{l+1}^{p\ T} R_k \Delta\bar{u}_{l+1}^p + (\bar{z}_{l+1}^p - \bar{z}^g)^T Q_k (\bar{z}_{l+1}^p - \bar{z}^g) dt\\ & 0 = f(\dot{\bar{x}}_{l+1}^p, \bar{x}_{l+1}^p, \bar{v}_{l+1}^p, \bar{u}_{l+1}^p, \bar{w}^p), \quad \bar{x}^p(t_{k+1}) = \hat{x}_{k+1|k}\\ & \bar{z}^p = C_y^x \bar{x}^p + C_z^v \bar{v}^p + D_z^u \bar{u}^p + D_z^w \bar{w}^p\\ & h_j^T z_{l+1}^p(t) \leq g_j, \ j = 1, \ldots, m\end{array}$$

in receding horizon. As usual, we seek the updates that satisfy the linearization of the system dynamics along the predicted trajectory

$$\begin{aligned}0 &= \partial_{\dot{\bar{x}}} f|_l \, \dot{x}^p + \partial_{\bar{x}} f|_l \, x^p + \partial_{\bar{v}} f|_l \, v^p + \partial_{\bar{u}} f|_l \, u^p + \partial_{\bar{w}} f|_l \, w^p, \quad x^p(t_{k+1}) = x_k^p\\ z^p &= C_z^x x^p + C_z^v v^p + D_z^u u^p + D_z^w w^p\end{aligned}$$

and for the optimization purposes we discretize and lift these dynamics by standard sampling and zero-order hold operations. Then by solving the following QP,

$$\begin{array}{ll}\min\limits_{\mathbf{u}_l^p} & (\Delta\bar{\mathbf{u}}_l^p + \Delta\mathbf{u}_l^p)^T R_k (\Delta\bar{\mathbf{u}}_l^p + \Delta\mathbf{u}_l^p) + (\bar{\mathbf{z}}_l^p - \bar{\mathbf{z}}^g + \mathbf{z}_l^p)^T Q_k (\bar{\mathbf{z}}_l^p - \bar{\mathbf{z}}^g + \mathbf{z}_l^p)\\ & \mathbf{z}^p = G_{zx} x_0^p + G_{zu} \mathbf{u}^p + G_{zw} \mathbf{w}^p.\\ & h_j^T \bar{\mathbf{z}}_l^p + h_j^T \mathbf{z}_l^p \leq g_j, \ j = 1, \ldots, m\end{array}$$

we arrive at the desired updates

$$\bar{\mathbf{u}}_{l+1}^p = \bar{\mathbf{u}}_l^p + \mathbf{u}_l^r, \quad \text{and} \quad \bar{\mathbf{z}}_{l+1}^p \simeq \bar{\mathbf{z}}_l^p + \mathbf{z}_l^p.$$

The control problem is then defined as the minimization of the objective function subject to the dynamics and the inequality constraints in table 8.2.

$$J_k(\bar{\mathbf{u}}_{l+1}) = \sum_{i=k}^{k+n} (\Delta\bar{u}_{i,l+1})^T R_k \Delta\bar{u}_{i,l+1} + (\bar{z}_{i,l+1} - \bar{z}^g)^T Q_k (\bar{z}_{i,l+1} - \bar{z}^g).$$

In this standard *open-loop* MPC problem we take the references to be constant and equal to the desired end-values of grade **B**.

$$\bar{z}_k^g \equiv \bar{z}^B \quad \text{for all } k$$

Alternatively (although not pursued here) one can pre-filter the reference trajectory in a two degrees-of-freedom design by means of an open-loop dynamic trajectory optimization. This pre-filtering of the reference control signal and feedforward is automatic in *closed-loop* MPC and constrained finite horizon LQG control due to the variational structure in the control architecture, see also the discussion on feedforward in section 4.2.3. In case the constraints

$$h_j^T \bar{\mathbf{z}}^p \leq g_j$$

cannot be satisfied online, a constraint relaxation scheme is used in which the constraints are replaced by

$$h_j^T \bar{\mathbf{z}}^p \leq g_j + \epsilon_j$$

and the objective function is given by ($\beta = 10^6$)

$$J_{l+1}^{\mathrm{RELAX}}(\bar{\mathbf{u}}^{l+1}) = \beta\|\epsilon\|^2 + J_{l+1}(\bar{\mathbf{u}}^{l+1})$$

8.6.3 Simulation results

The closed-loop results are plotted in figure 8.6. The transition for both the density and melt-index are comparable to the off-line determined optimum, and tracking is good except for the cool-water constraint violation of approximately 4 hours. It peaks at approximately 2.5% of the cooling capacity and leads to approximately 1% on average over the duration of the constraint violation. The violation is induced by the PI loop between the reactor temperature and cooling water flow. If the constraint would have been enforced on the actual plant and not just in relaxed fashion in the optimization problem, the reaction would have lead to reactor runaway and the reactor would have been shut-down. The feed flows corresponding to the closed-loop transition are plotted in figure 8.7. The constraint relaxation scheme that is used in the EKF+MPC implementation to keep the optimization problem feasible at all times results in the chaotic control moves, Figure 8.5.

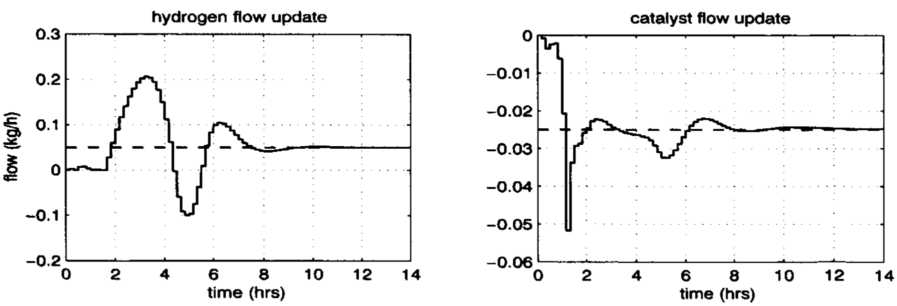


Figure 8.5: Bias compensation using EKF+LTV-MPC. Real values (solid), H-Bias (dashed), C-Bias (dashed).

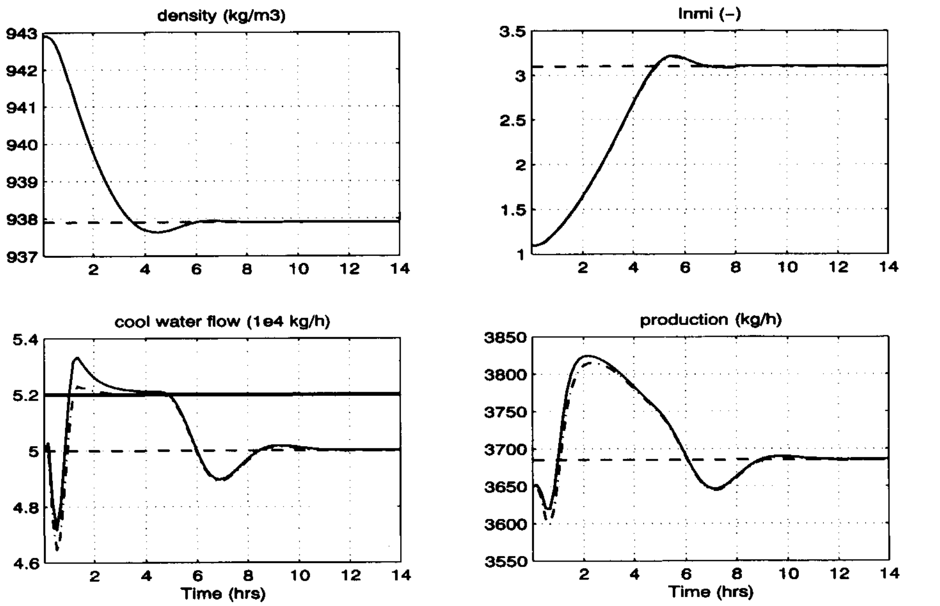


Figure 8.6: Closed-loop trajectories of controlled variables using EKF+LTV-MPC. Real performance (solid), estimated performance (dash-dotted), target performance (dashed).

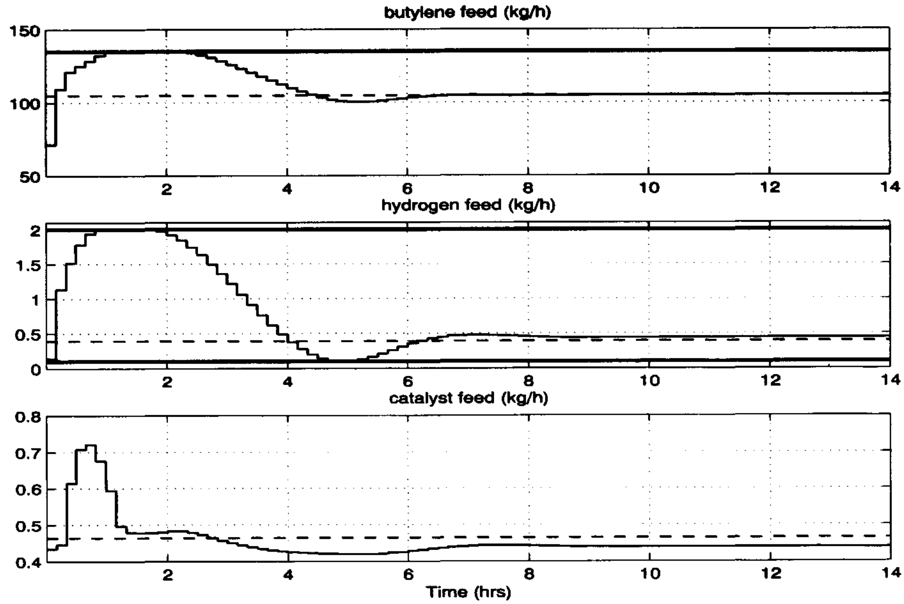


Figure 8.7: Closed-loop trajectories manipulated variables using EKF+LTV-MPC. Applied inputs (solid) and nominal target values (dashed).

8.7 Constrained finite horizon LQG control

In this section we will apply the closed-loop prediction techniques as introduced in this thesis on the HDPE process. Current solution methods for the true closed-loop MPC problem, in which one simultaneously optimizes the feedforward and feedback control, still requires vectorization of the problem which prohibits online application at this time. Although a formal complexity analysis falls outside our scope, it turned out that the column example discussed in Chapter 4 was at the limit of current computing power[2]. To cope with this reality, the (strongly related) constrained finite horizon LQG control strategy was developed in Chapter 5 as a cheap alternative. We will apply it here to compute the feedback controller (CFHLQG^A) and feedforward (CFHLQG^B) in consecutive steps. In application, this is a good starting point anyway because from a structural point of view, both techniques are identical up to the computation of $K, \mathbf{u}^r$ such that only minor changes in the control setup are needed. The computation of the controller K is not restrictive for online application because it can efficiently be computed using the control Riccati recursion or directly using the matrix valued least squares techniques discussed in Chapter 5. The feedforward computation is of precisely the same complexity as standard open-loop MPC; a single QP must be solved. Therefore, the computational effort of CFHLQG is the same as for open-loop MPC. As explained above we need to solve two problems, the feedback problem (CFHLQG^A) and the feedforward problem (CFHLQG^B). The details of each approach can be found in Chapter 5.

8.7.1 The CFHLQG^A problem

For the CFHLQG^A problem, we start with the same quadratic objective in the minimal variance problem as in the deterministic quadratic control law of the MPC, that is, the same weighting matrices Q, R and we use the same second-order information on the disturbances $w(\xi)$ as the Kalman filter. Because the weighting matrix R is defined as a penalty on the rate of change of the inputs, we must follows the procedure as explained in detail in Section 7.7. Hence, we define the extended state space

$$x_k^+(\xi) = \begin{pmatrix} x_k^c(\xi) \\ u_{k-1}^c(\xi) \end{pmatrix}$$

and the extended performance variables

$$z_k^+(\xi) = \begin{pmatrix} z_k^c(\xi) \\ \Delta u_k^c(\xi) \end{pmatrix}, \quad Q_k^+ = \begin{pmatrix} Q_k & O \\ O & R_k \end{pmatrix}$$

Then, let the weighting matrix Q^+ have the blocks Q_k^+ on the block diagonal for each time instant in the horizon and let us factor

$$Q^+ = F_{Q^+} F_{Q^+}^T$$

[2]Simulation was carried out on a Pentium III, 1GHz

then the minimal variance problem that must be solved is given by

$$\min_{M_k \in \mathbf{M}_n} \quad \operatorname{tr} F_{Q^+}^T (A^k M_k B^k + C^k) \begin{pmatrix} V & O \\ O & W \end{pmatrix} (A^k M_k B^k + C^k)^T F_{Q^+}$$

where A_k, B_k, C_k, V_k, W_k are defined in Section 7.7. From M_k we can extract the feedback matrices and define the feedback law

$$\mathbf{\Delta u}_k(\xi) = L_k \hat{x}_k^+(\xi) + K_k \mathbf{v}_k(\xi). \tag{8.2}$$

8.7.2 The CFHLQG^B problem

We are given an initial guess for the reference trajectories $\bar{u}_l^r$, $\bar{v}_l^r$, $\bar{v}_l^r, \bar{x}_l^r$, $\bar{y}_l^r$, $\bar{z}_l^r$ to be optimized that satisfy the nonlinear model equations

$$\begin{aligned} 0 &= f(\dot{\bar{x}}_l^r, \bar{x}_l^r, \bar{v}_l^r, \bar{u}_l^r, \bar{w}_l^r), \qquad \bar{x}_l^r(t_{k+1}) = \bar{x}_{l,k+1}^r \\ \bar{y}_l^r &= C_y^x \bar{x}_l^r + C_y^v \bar{v}_l^r + D_y^u \bar{u}_l^r + D_y^w \bar{w}_l^r \\ \bar{z}_l^r &= C_z^x \bar{x}_l^r + C_z^v \bar{v}_l^r + D_z^u \bar{u}_l^r + D_z^w \bar{w}_l^r \end{aligned} \tag{8.3}$$

and we seek updates based on the linearized dynamics along this trajectory

$$\begin{aligned} 0 = \partial_{\dot{\bar{x}}} f|_l \, \dot{x}_l^r + \partial_{\bar{x}} f|_l \, x_l^r + \partial_{\bar{v}} f|_l \, v_l^r + \partial_{\bar{u}} f|_l \, u_l^r + \partial_{\bar{w}} f|_l \, w_l^r, \quad x_l^r(t_{k+1}) = x_{l,k+1}^r \\ y_l^r = C_y^x x_l^r + C_y^v v_l^r + D_y^u u_l^r + D_y^w w_l^r \\ z_l^r = C_z^x x_l^r + C_z^v v_l^r + D_z^u u_l^r + D_z^w w_l^r \end{aligned}$$

The same objective function as in the open-loop MPC problem is used for (CFHLQG^B), with that difference that we substitute the predictions with the references

$$\mathbf{u}^p \mapsto \mathbf{u}^r, \ \mathbf{u}^p \mapsto \mathbf{u}^r, \ \mathbf{y}^p \mapsto \mathbf{y}^r, \ \mathbf{z}^p \mapsto \mathbf{z}^r$$

and the back-off ν is subtracted form the feasible set that is the inequality constraints are replaced by

$$\nu_j + h_j^T \bar{\mathbf{z}}^r + h_j^T \mathbf{z}^r \leq g_j$$

Then by solving the following QP,

$$\begin{aligned} \min_{\mathbf{u}_l^r} \quad & (\mathbf{\Delta \bar{u}}_l^r + \mathbf{\Delta u}_l^r)^T R_k (\mathbf{\Delta \bar{u}}_l^r + \mathbf{\Delta u}_l^r) + (\bar{\mathbf{z}}_l^r - \bar{\mathbf{z}}^g + \mathbf{z}_l^r)^T Q_k (\bar{\mathbf{z}}_l^r - \bar{\mathbf{z}}^g + \mathbf{z}_l^r) \\ & \mathbf{z}^r = G_{zx} x_0^r + G_{zu} \mathbf{u}^r + G_{zw} \mathbf{w}^r. \\ & \nu_j + h_j^T \bar{\mathbf{z}}^r + h_j^T \mathbf{z}^r \leq g_j, \ j = 1, \ldots, m \end{aligned}$$

we arrive at the desired updates

$$\bar{\mathbf{u}}_{l+1}^r = \bar{\mathbf{u}}_l^r + \mathbf{u}_l^r \text{ and } \bar{\mathbf{z}}_{l+1}^r \simeq \bar{\mathbf{z}}_l^r + \mathbf{z}_l^r \text{ and } \bar{\mathbf{y}}_{l+1}^r \simeq \bar{\mathbf{y}}_l^r + \mathbf{y}_l^r.$$

8.7.3 Simulations results

The above controller computation leads to actual controller action of

$$\bar{u}(\xi)_k = \bar{u}_k^r + u_k^c(\xi).$$

The actual realization of $u_k^c(\xi)$ is readily computed from (8.2) and the estimate and innovations signal generated by the Kalman filter that was also used for the open-loop MPC implementation.

The results for the density and melt-index are given in figure 8.9. Without inequality constraints, both control systems, that is the EKF+MPC and CFHLQG show comparable performance in tracking of the main quality variables density and melt-index and in disturbance rejection because the same quadratic weighting matrices have been used in both approaches. Note that this has a practical advantage that tuning matrices for existing MPC can be used as initial starting point for a CFH-LQG implementation. Because the CFHLQG controller consists of two optimization problems, reference tracking (CFHLQG^B) can be tuned separately from disturbance rejection (CFHLQG^A) such that improved performance can be obtained.

In the constraint case on the other hand, the EKF+MPC forces cannot maintain the cooling water constraint and there is a long unacceptable period (50% of the transition time) in which this main constraint is violated. The closed-loop predictive controller keeps back-off to this constraint and no violation occurs, compare Figures 8.6 and 8.9. Figures 8.11 and 8.12 shows close-ups of the reference trajectories of the cooling water flow and Hydrogen feed, where the back-off is visualized via the 1-dimensional confidence intervals (these intervals are the projection of the confidence ellipsoids on the corresponding controlled variable space). Due to the back-off taken in the feedforward solution, inequality constraints play no role in the feedback control moves induced by disturbances and no relaxtation scheme with some arbitrary high coefficient β is needed. Consequently, the control input moves are very smooth compared to the restless action in the EKF+MPC case, compare Figures 8.8 and 8.5.

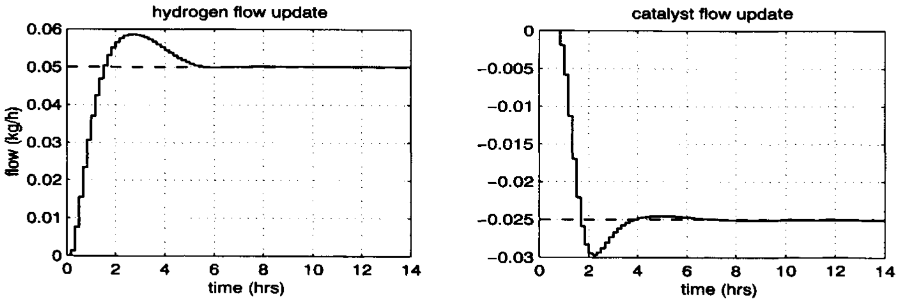


Figure 8.8: Bias compensation using the closed-loop predictive controller. Real values (solid), H-Bias (dashed), C-Bias (dashed).

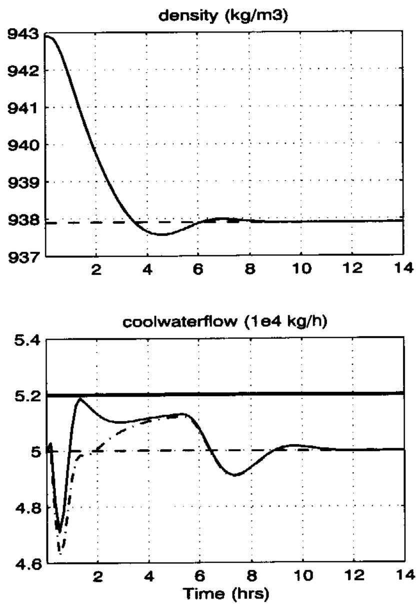


Figure 8.9: Closed-loop trajectories controlled variables with back-off. Real performance (solid), reference performance (dash-dotted), target performance (dashed).

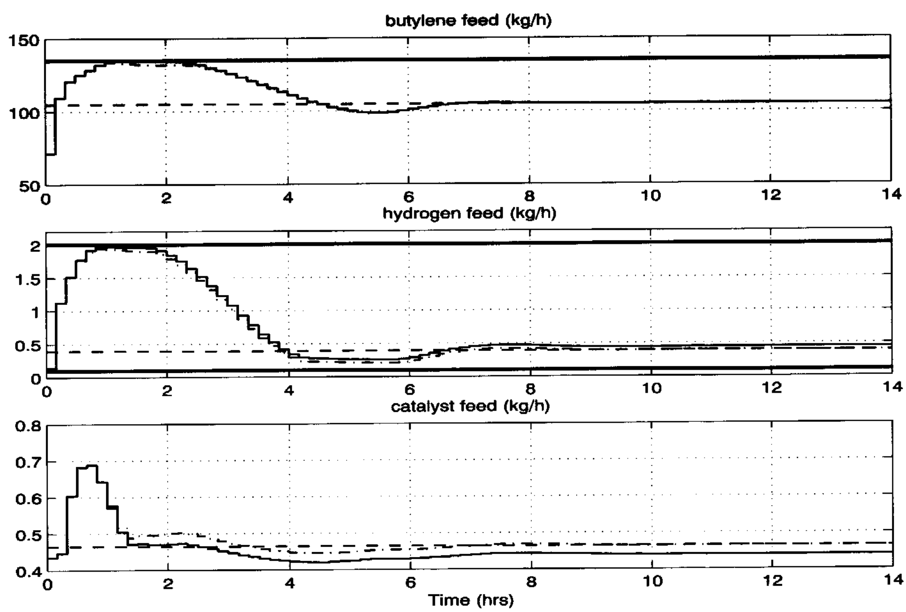


Figure 8.10: Closed-loop trajectories manipulated variables with back-off. Applied inputs (solid), reference inputs (dashed-dotted) and nominal target values (dashed).

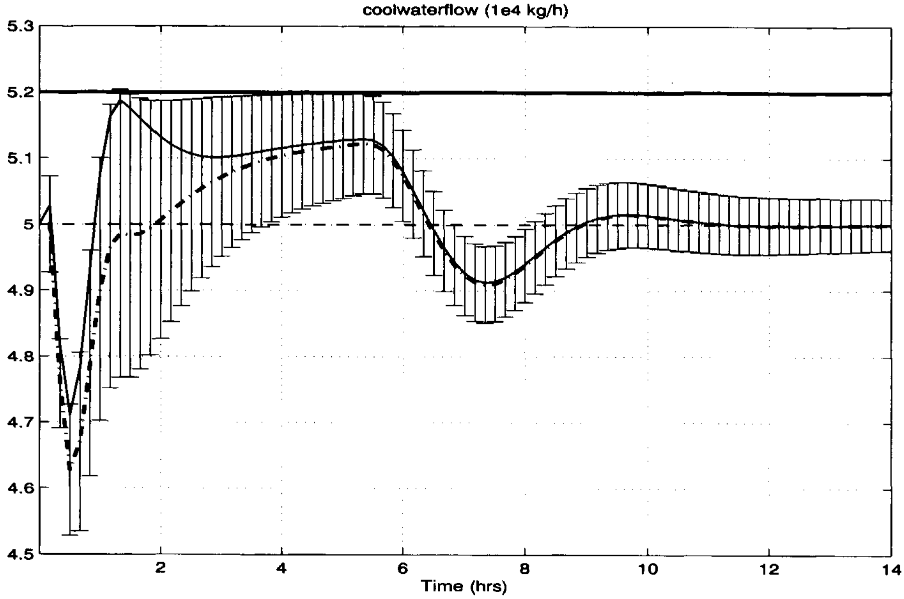


Figure 8.11: Close-up closed-loop cool water flow. Real (solid), reference (dash-dotted), target (dashed).

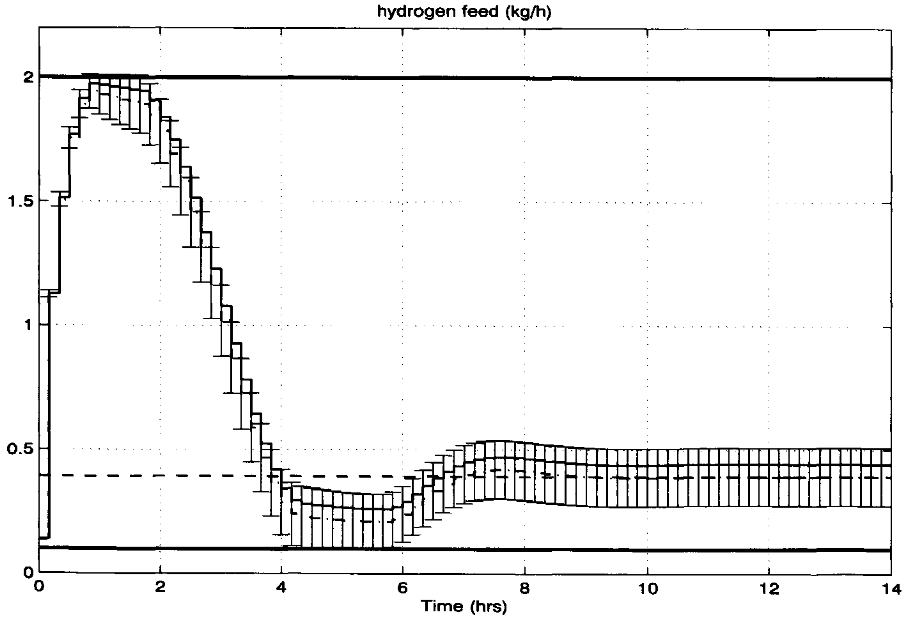


Figure 8.12: Close-up of closed-loop hydrogen feed flow. Applied (solid), reference (dashed-dotted) and nominal target value (dashed).

8.8 Chapter summary

In this section the newly developed techniques were applied to a continuous polymerization reactor. A grade change problem is discussed and control is used for both tracking (feedforward) and disturbance rejection (feedback). A persistent disturbance scenario on the feed flows of Hydrogen and catalyst were considered that strongly counteract the grade change. A constrained finite horizon LQG variant of closed-loop predictive control was implemented and compared to the industrial state of the art as discussed in (Lee and Ricker, 1994). The main conclusion is that the predictive controller is, in contrast to open-loop MPC, able to keep the trajectories feasible in the disturbance case using very smooth control action for disturbance rejection. This shows that closed-loop predictive control is applicable to realistic process systems and is able to outperform open-loop strategies.

9 Conclusions and recommendations

At the beginning of this thesis, an analysis of the historical development of advanced process control compared to the historical development of high performance control for linear unconstrained systems was made. The crucial aspects of the closed-loop control design methodology were identified (Chapter 1) which led to a subdivision of the research objective in seven research questions (Chapter 2). In this chapter, the research is concluded and a future outlook is given in the recommendations.

9.1 Results and conclusions

Formulating an advanced process control problem. The advanced process control community has over the last twenty five years focussed on open-loop model predictive control methods. Although open-loop MPC has become a mature technology and a standard commodity, its closed-loop behavior is unequal to its open-loop future prediction even for simplest of possible disturbance cases. This structural discrepancy of model predictive control is caused by its ignorance towards future disturbances and the implicit nature of its receding horizon feedback mechanism. *A necessary condition to remove these limitations is to introduce the standard or generalized plant in a predictive set-up.* This means that unknown future disturbances and measured outputs are explicitly introduced as signals besides the manipulated and controlled variables. This apparently simple step has far reaching consequences for the analysis and synthesis of inequality constrained control systems. It breaks completely with the traditional MPC solution that only considers manipulated and controlled variables in the future while past disturbances and measurements are only considered via state-estimation.

Integration of optimization and control. Economic benefit is the primary driver to implement advanced process control on an existing plant. Industrial RTO and APC have two structural objectives: 1) an economic objective that consists of finding economically optimal steady-states or in more complex cases dynamic transitions and

2) a control objective that consists of a dynamic open-loop regulatory optimization problem used to track these optimal set points or reference trajectories and reject process disturbances. The separation of the overall operational business objective of a plant in two optimization problems is inconsistent and suboptimal. It reflects the traditional approach in which a dynamic optimization is carried out with no consideration to future disturbances because it is implicitly assumed that a perfect control system will reject these disturbances not limited by the dynamics of the plant itself. To remove this inconsistency, the optimal performance of the plant-controller interconnection should be added as a constraint on the economic optimization problem. The suboptimality follows from the fact that the optimal controller depends on the economic objective function and cannot be found without extensive simulation studies. *A necessary condition to solve this problem is to quantify the closed-loop performance in terms of the controller parameters and to guarantee a priori that the computed optimal trajectories will be realized.* Closed-loop MPC solves the economic optimization and control problem simultaneously (sufficient condition), while constrained finite horizon LQG control provides a numerically efficient approximation. The key point in closed-loop predictive control methods is that the closed-loop predictions match the actual closed-loop behavior.

Feedback, sensitivity and constraint handling. The paradox of open-loop MPC is that it is applied to reject disturbances, but it does not consider disturbances in its problem formulation. The advantage is the simplicity of the implementation but it comes at a high cost of difficult analysis and almost impossible synthesis due to the implicit nature of the required receding horizon control action. Therefore, MPC cannot be considered a solution to advanced control problems in the long run. To build a systematic control synthesis procedure, *the* basic characteristic of feedback design methods namely the possibility to change the process sensitivity in closed-loop by means of control is introduced. Because the standard plant is used, in which both future disturbances and future measurements are available, this sensitivity is expressed in terms of direct feedback that maps the future measurements to future control moves. The calculation of the process sensitivity has traditionally been difficult because the inequality constraints introduce non-differentiable input-output behavior. This is not the case in the new framework, because the introduction of future disturbances requires that back-off to the inequality constraints is introduced. The process can then locally move freely without activating the constraints which recovers linearity (or differentiability) of the process behavior. The strength of this approach is the bootstrap effect that is introduced. The back-off is a function of the disturbance properties and the process sensitivity, which is a function of the controller to be chosen. For linear time-varying dynamics, linear inequality constraints and Gaussian disturbances the problem of finding the optimal controller can be rendered *convex* by suitable transformations. Two crucial observations play an important role in this transformation. First, by either using the Q-parameterization or the Youla-Kučera parameterization of the closed-loop system, the sensitivity function and (consequently the factor of the closed-loop variance) becomes an affine function of the controller parameters. Secondly, by introducing an ellipsoidal technique combined with modern results in conic programming, the back-off is computed as the norm of this parameterization giving so-called convex second-order cone constraints.

Feedforward trajectory design. Feedforward is a major factor in high performance control design. The two types of feedforward are *disturbance* feedforward and *control input* feedforward. Disturbance feedforward is determined by exogenous sources outside the system boundary and can therefore be added to the reference trajectory for the disturbance channel. Control input feedforward (from hereon referred to as *the* feedforward) is a different matter because it is optimized and therefore coupled to the choice of feedback controller. *In the presented framework, the feedforward defines the economic profit of the plant and serves as a dynamic reference trajectory for the control inputs.* It is determined by specific grade and load changes, changes in constraints, structural changes in properties of disturbances and changes in the economic objective function. The feedforward and thereby the planned online profit is not influenced by specific disturbance realizations since those are handled by the feedback discussed above. The coupling between feedforward and feedback and thus plant economics and disturbance rejection is the amount of back-off to the inequality constraints. The better the feedback controller is, measured in variance in dominant economic variables, the more closely the constraints can be approached. This enlarges the feasible set of feedforward trajectories and increases the profitability of the process by constrained pushing. Closed-loop MPC problem handles this coupling in full generality by simultaneously optimizing the feedforward trajectory and the feedback controller. Because this problem has a convex representation we are guaranteed to find the global optimum in a numerically efficient way.

Convergence and stationary behavior. The closed-loop MPC problem is a time-varying feedforward/feedback problem. This holds even in the case that the process has linear time-invariant dynamics because of the different active sets of inequality constraints at each different time sample in a transition. Nevertheless, for continuous processes the target of such transitions will in the end be an economically optimal steady-state. This steady-steady state will not lie on the constraints because of the necessary back-off to the constraints, where the minimal back-off is determined by the optimal controller. This optimal solution can usually not be computed directly because this requires an infinitely long prediction and control horizon. Instead, this optimal steady-state and LTI controller can be computed directly using LMI methods. In this case, a two-step procedure is presented again based on the Youla-Kučera parameterization of all stabilizing controllers. In the first step, the optimal back-off and steady-state are computed using a sequence of finite impulse response controllers. Then in a second step, for fixed back-off, the optimal state-space controller can be computed using standard output feedback techniques. The use of linear matrix inequalities in this approach is a limiting factor in large scale application, and therefore a fast alternative heuristic algorithm as in the linear time-varying case has been developed using stationary LQG results in combination with linear or quadratic programming (depending on the objective function).

The receding horizon implementation. In standard open-loop MPC, feedback is obtained by computing a new open-loop control sequence at each time instant over a shifted time window and for this reason open-loop MPC is also referred to as receding horizon control. The reason for the development of closed-loop MPC is to *avoid* receding horizon control because the closed-loop properties are unnecessary complicated to analyze (not to mention systematic synthesis of controllers). Nevertheless, in application of closed-loop predictive control to continuous processes a receding horizon *implementation* is needed. Central lies the requirement that if the data of the optimization problem does not change over a sample, then the optimal feedforward and feedback controller solution coincides with the previous solution. Therefore, the receding horizon implementation is related only to derive a consistent feedback mechanism in a dynamic programming sense. The key to the solution is duality that shows that the optimal feedback controller of the closed-loop MPC problem is also optimal for *some* LQG control law. From this it follows that the recursive solution is obtained via a state-feedback (in addition to the future output feedback) to account for all past measurements. This state-feedback gain is easily computed using the same numerical methods as for the output feedback controller.

Application to nonlinear process systems The developed tools are all based on linear time-varying dynamics as perturbation models along trajectories of nonlinear models. In analogy of sensitivity methods in nonlinear programming, a sequential optimization procedure is proposed in which the closed-loop MPC problem is interpreted as a sub-optimization problem. The optimal feedforward trajectory and feedback response to measurements is added to the results of the previous iteration as in application of linear time-varying MPC to nonlinear systems. The advantage in this approach is that the quality of the solution improves at each cycle due to the receding horizon implementation. To illustrate the control methodology, the constrained finite horizon LQG controller has been applied to a HDPE polymerization reactor. A realistic process model for the HDPE process was available as a set of differential algebraic equations in a professional generic modelling environment. The simulations on the HDPE case confirmed the superior constrained handling for disturbance scenarios compared to open-loop MPC.

9.2 Outlook and recommendations

In this thesis a new perspective on model predictive control has been given opening the way to new theoretic and applied research.

Recommendations for applied research. The first recommendation is to apply the closed-loop predictive techniques to a real system to reveal its value. Presumably the best way to proceed is build a constrained finite horizon LQG controller for a system that is currently controlled by a Kalman filter based linear MPC. In that case, the Kalman filter can be reused to compute the state-estimate. The quadratic weighting matrices are then used to compute the output feedback controller K and state-feedback controller L (Chapter 7) as well as the back-off vector. Then, on-line the feedforward is computed using the same quadratic regulatory objective function as for the MPC. Closed-loop performance can then be improved by exploiting the 2 degrees of freedom since different quadratic weighting matrices can be used for the feedforward and feedback respectively for optimal tracking and disturbance rejection. The optimal steady-states can still be generated by a real time economic optimization layer and fed to the controller optimization problem. A second step is then to shift the economic tasks such as constraint pushing to the control level. In a third step the feedback controllers can be included in the on-line optimization by the methods discussed in this thesis leading to the full closed-loop MPC implementation. Here the freedom in convex objective functions should be exploited to add some robustness to the feedback controller. This can be achieved by adding some closed-loop variance penalty to the linear objective functions to reduce the extremity of purely economically optimal solutions.

Recommendations for theoretic research. In the past decades MPC applications have largely driven theoretic control research focussed on understanding receding horizon control. In closed-loop predictive control an explicit feedback control law is used which should substantially reduce the complexity of the current Lyapunov based results to get more direct and practical tests. Because of the back-off that is used, the connection of the finite horizon solution to the asymptotic results (Chapter 6) should provide the infinite horizon extensions in which the converged closed-loop ellipsoids are natural candidates for terminal sets. Other interesting research directions are related to other types of disturbances and plant uncertainty to arrive at robust model predictive control. The direct feedback gives explicit control over the disturbance responses of the system and other performance criteria can be explored. Because any observer can be used to render the closed-loop affine in the controller parameters, it is natural to break with the traditional use of the Kalman filter in advanced process control and to move on to other observer-based output feedback controllers. This allows for plant-model mismatch as then the stringent requirement that the observer is a perfect representation of the plant dynamics can be relaxed. Reduced models seem to fit in this approach as well, which can be very important because model adequacy is much easier established than model accuracy in closed-loop. The derivation of recursive solutions are expected to be obtained as for the quadratic case using the optimality condition and the existing results in control theory.

References

Andersen, E.D., Roos, C. and Terlaky, T. (2000). On implementing a primal-dual interior-point method for conic quadratic optimization. *Technical Report W-274*, Helsinki School of Economics and Business Administration, Department of Economics and Management Science.

Anderson, B.D.O. (1998). From Youla-Kučera to identification, adaptive and nonlinear control *Automatica*, 34(12):1485–1506.

Aström, K.J. and Wittenmark, B. (1990). Computer-controlled systems: Theory and design. *Prentice Hall*, USA.

Athans, M. and Schweppe, F.C. (1965). Gradient matrices and matrix calculations, *Technical Note 1965-53*, MIT, Lincoln Laboratory.

Athans, M. (1971). The role and use of the stochastic Linear-Quadratic-Gaussian problem in control system design. *IEEE Trans. on Automatic Control*, 16(6):529–552.

Bar-Shalom, Y. and Tse, E. (1974). Dual effect, certainty equivalence and separation in stochastic control. *IEEE Trans. on Automatic Control*, 19(5):494-500.

Bard, Y. (1974). Nonlinear parameter estimation, *Acadmic Press*, USA.

Batina, I., Stoorvogel A.A. and Weiland, S. (2002). Optimal control of linear stochastic systems with state and input constraints, *Proc. IEEE Conf. Decision and Control*, 41:1564-1569.

Bemporad, A. (1998). Reducing conservatism in predictive control of constrained systems with disturbances, *Proc. IEEE Conference on Decision and Control*, 37:1384–1389.

Bemporad, A. and Morari, M. (1999). Robust model predictive control: A survey. *in: Robustness in identification and control Ed: Garulli, A., Tesi, A., Vicino, A., 207-226.*

Ben-Tal, A. and Nemirovski, A. (1999). Robust solutions of uncertain linear programs, *Operation Research Letters*, 25(1):1-13.

Ben-Tal, A. and Nemirovski, A. (2001). Lectures on modern convex optimization. Analysis, algorithms, and engineering applications. *MPS-SIAM Series on Optimization*, USA.

Bode, H.W. (1945). Network analysis and feedback amplifier design *Bell System Technical Journal*, 19: 421-454.

Boyd, S. and Vandenberghe, L. (2002). Convex optimization, *Lecture Notes Stanford University*, USA.

Breakwell, J.V., Speyer, J.L. and Bryson, A.E. (1962). Optimization and Control of Nonlinear Systems Using the Second Variation, *SIAM Journal of Control*, Ser. A., 1(2):193–223.

Brewer, J.W. (1978). Kronecker products and matrix calculus in system theory, *IEEE Trans. on Circuit and Systems*, 25(9):772–781.

Bryson Jr., A.E. (1996). Optimal Control-1950 to 1985, *IEEE Control Systems Magazine*, 16(3):26–33.

Campo, P.J. and Morari, M. (1987). Robust model predictive control, *Proc. American Control Conference*, 6:1021–1026.

Chmielewski, D. and Manousiouthakis, V. (1996). On constrained infinite-time quadratic optimal control, *Systems and Control Letters*, 29:121–129.

Choi, K.Y. and Ray, W.H. (1992). The dynamic behaviour of fluidized bed reactors for solid catalyzed gas phase olefin polymerization, *Chemical Engineering Science*, 40(12):2261–2279, 1992.

Cox, H. (1964). On the estimation of state variables and parameters for noisy dynamical systems, *IEEE Trans. on Automatic Control*, 9(1):5–12.

Cutler, C.R., and Ramaker, B.L. (1980). Dynamic matrix control: A computer algorithm, *Proc. Joint Automatic Control Conference.*

Dabbene, F., Gay, P. and Polyak, B.T. (2003). Recursive algorithms for inner approximation of convex polytopes, *Automatica*, 39:1773–1781.

Desoer, C.A., Liu, R., Murray, J. and Saeks, R. (1980). Feedback system design: The fractional representation approach to analysis and synthesis, *IEEE Trans. on Automatic Control*, 25(3):399–412.

Doyle, J.C. (1978). Guaranteed margins for LQG regulators, *IEEE Trans. on Automatic Control*, 23(4):756–757.

Doyle, J.C. and Stein, G. (1979). Robustness with observers, *IEEE Trans. on Automatic Control*, 24(4):607–611.

Doyle, J.C. and Stein, G. (1983). Multivariable feedback design: concepts for a classical/modern synthesis, *IEEE Trans. on Automatic Control*, 26(1):4-16.

Doyle, J.C., Glover, K. Khargonekar, P.P. and Francis, B.A. (1989). State-space solutions to standard $\mathbf{H}_2$ and $\mathbf{H}_\infty$ control problems, *IEEE Trans. on Automatic Control*, 34(8):831–847.

Doob, J.L. (1953). Stochastic Processes, *John-Wiley & Sons*, USA.

Dreyfus, S.E. (1962). Some types of optimal control of stochastic systems, *SIAM Journal of Control*, Ser. A., 2(1):121–135.

Forbes, J.F., Marlin, T.E. and MacGregor, J.F. (1994). Model adequacy requirements for optimizing plant operations, *Computers and Chemical Engineering*, 18(6):497–510.

Furuta, K. and Wongsaisuwan, M. (1993). Closed-form solutions to discrete time LQ optimal control and disturbance attenuation, *Systems & Control Letters* 20(6):427–437.

Furuta, K. (1994). Alternative solution of the discrete time Kalman filter, *Systems & Control Letters* 22(6):429–435, 1994.

Furuta, K. and Wongsaisuwan M. (1995). Discrete-time LQG dynamic controller design using plant Markov parameters, *Automatica* 31(9):1317-1324, 1995.

Garcia, C.E., and Morari, M. (1982). Internal Model Control. 1. A unifying review and some new results, *Ind. Eng. Chem. Process Des. Dev.*, 21:308-323.

Garcia, C.E., Morshedi, A.M. and Morari, M. (1986). Quadratic programming solution of dynamic matric control, *Chemical Engineering Communications*, 46:73-87.

Garcia, C.E., Prett, D.M. and Morari, M. (1989). Model Predictive Control: Theory and practice - a survey, *Automatica*, 25(3):335-348.

Gattu, G. and Zafiriou, E.(1992) Nonlinear quadratic dynamic matrix control with state estimatation, *Ind. Eng. Chem. Process Des. Dev.*, 31(4):1096-1104.

Geering, H.P. (1976). On calculating gradient matrices, *IEEE Trans. on Automatic Control*, 21(4):615–616.

Gilbert, E.G. (1963). Controllability and observability in multivariable linear systems, *SIAM Journal of Control*, Ser. A., 1(2):128–151.

Gilbert, E.G. and Tan, K.T. (1991). Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *IEEE Trans. on Automatic Control*, 36(9):1008–1020.

Gökcek, C., Kabamba, P.T. and Meerkov, S.M. (2001). An LQR/LQG theory for systems with saturating actuators, *IEEE Trans. on Automatic Control*, 46(10):1529-1542.

Golub,G.H. and van Loan, C.F. (1996). Matrix computations, *The Johns Hopkins University Press*, USA.

Heaton, A. (1996). An introduction to industrial chemistry, *Blackie Academic and Professional*, 3rd edition, UK.

Hindi, H. and Boyd, S. (1998). Analysis of linear systems with saturation using convex optimization, *Proc. IEEE Conf. Decision and Control*, 37:903–908.

Hindi, H., Hassibi, B. and Boyd, S. (1998b). Multiobjective $\mathbf{H}_2/\mathbf{H}_\infty$-optimal control via finite dimensional Q-parameterization and linear matrix inequalities, *Proc. American Control Conference*, 5:3244–3249.

Horn, R.A. and Johnson, C.R. (1999), Matrix analysis, *Cambridge University Press*, UK.

Ingham, J., Dunn, I.J., Heinzle, E. and Prenosil, J.E. (1994). Chemical engineering dynamics, modeling with PC simulation, *VCH*, Germany.

Isidori, A. (1985). Nonlinear control systems, An introduction, *Springer-Verlag*, Germany.

Jazwinsky, A.H. (1970). Stochastic processes and filtering theory, *Academic Press*, USA.

Kailath, T. (1968). An innovations approach to least-squares estimation part I: Linear filtering in additive white noise, *IEEE Trans. on Automatic Control*, 13(6):646–655.

Kadam, J.V., Schlegel, M., Marquardt, W., Tousain, R.L., van Hessem, D.H., van den Berg, J, Bosgra, O.H. A two-level strategy of integrated dynamic optimization and control of industrial processes - a case study, In: J. Grievink, J. v. Schijndel (Eds.): *Escape*, Elsevier, 12: 511-516.

Kalman, R.E.(1960). A new approach to linear filtering and prediction problems, *Trans. ASME, Journal of Basic Engineering*, 82:35-45.

Kalman, R.E. and Bucy, R.S. (1961). New results in linear filtering and prediction theory, *Trans. ASME, Journal of Basic Engineering*, Ser. D., 83:95-107.

Kalman, R.E. (1963). Mathematical description of linear dynamical systems, *SIAM Journal of Control*, Ser. A., 1(2):152–192.

Kaminski, P.G., Bryson JR., A.E. and Schmidt, S.F. (1971). Discrete square root filtering: A survey of current techniques, *IEEE Trans. on Automatic Control*, 16(6):727–736.

Karmarkar, N.(1984). A new polynomial-time algorithm for linear programming, Combinatorica, 4:373–395.

Kingman, J.F.C. and Taylor, S.J. (1966). Introduction to measure and probability, *Cambridge University Press*, UK.

Kozub, D.J. and Macgregor, J.F. (1992). State estimation for semi-batch polymerization, *Computers and Chemical Engineering*, 47:1047–1062.

Kučera, V. (1974). Algebraic theory of dicrete optimal control for multivariable systems, *Kybernetika*, Suppl. 10:1974,11:1975,12:1976.

Kwakernaak, H. and Sivan, R. (1972). Linear optimal control systems, *Wiley-Intersience*, USA.

Lee, J.H. and Ricker, N.L. (1994). Extended Kalman filter based nonlinear model predictive control, *Ind. & Eng. Chem. Res.*, 33(6):1530–1541.

Lee, J.H. and Yu, Z. (1997). Worst-case formulations of model predictive control for systems with bounded parameters, *Automatica*, 33(5):763–781.

Lewis, F.L. (1986). Optimal estimation with an introduction to stochastic control theory, *John Wiley & Sons*, USA.

Ljung, L. (2003). Challenges of nonlinear identification, *Bode Lecture, IEEE Conf. Decision and Control.*

Ljung, L. (1987). System identification - theory for the user, *Prentice-Hall*, USA.

Lobo, M.S., Vandenberghe, L., Boyd, S. and Lebret, H. (1998). Applications of second-order cone programming, *Linear Algebra and Applications*, 284:193-228.

Luenberger, D.G.(1969). Optimization by vector space methods, *John Wiley & Sons*, USA.

Luenberger, D.G. (1973). Introduction to linear and nonlinear programming, *Addison-Wesley Publishing Company*, USA.

Lundström, P., Lee, J.H., Morari, M. and Skogestad, S. (1995) Limitations of dynamic matrix control, *Computers and Chemical Engineering*, 19(4):409-421.

Maciejowski, J.M. (1994). Multivariable feedback design, *Addisson-wesley Publishing Company*, UK.

Masubuchi, I., Ohara, A. and Suda, N. (1998). LMI-based controller synthesis: A unified formulation and solution, *Int. J. Robust Nonlinear Control*, 8(8):669-686.

The MathWorks Ltd (2002). MATLAB/Simulink UserGuide.

Mayne, D.Q., Rawlings, J.B., Rao, C.V., Scokaert, P.O.M. (2000). Constrained model predictive control: stability and optimality, *Automatica*, 36:789–814.

MacAuley, K.B., MacGregor, J.F. and Hamielec, A.E. (1990). A kinetic model for industrial gas-phase ethylene copolymerization, *AIChE Journal*, 36(6):837–850.

MacAuley, K.B. and MacGregor, J.F. (1991). On-line inference of polymer properties in an industrial polyethylene reactor, *AIChE Journal*, 37(6):825–835.

MacAuley, K.B.and MacGregor, J.F. (1992). Optimal grade transitions in a gas phase polyethelene reactor, *AIChE Journal*, 38(10):1564–1576.

MacAuley, K.B.and MacGregor, J.F. (1993). Nonlinear product property control in industrial gas-phase polyethylene reactors, *AIChE Journal*, 39(5):855-866.

Michalska, H. and Mayne, D.Q. (1995). Robust receding horizon control of constrained nonlinear systems, *IEEE Trans. on Automatic Control*, 38:1623–1632.

Meier III, L., Larson, R.E. and Tether, A.J. (1971). Dynamic programming for stochastic control of discrete systems, *IEEE Trans. on Automatic Control*, 16(6):767–775.

Moore, B.C. (1981). Principal component analysis in linear systems: controllability, observability and model reduction, *IEEE Trans. on Automatic Control*, 26(1):17–32.

Morari, M. and Lee, J.H. (1997). Model Predictive Control: past, present and future, *Proc. ESCAPE 7*.

Muske, K.R.(1995). Linear Model Predictive Control of Chemical Processes, *PhD. Thesis*, University of Texas at Austin.

Nesterov, Y.E. and Todd, M.J. (1998). Primal-Dual interior point methods for self-scaled cones, *SIAM Journal of Optimization*, 8(2):324-364.

Nett, C.N., Jacobson, C.A. and Balas, M.J. (1984). A connection between state-space and doubly coprime fractional representations. *IEEE Trans. on Automatic Control*, 29(9):831–832.

Nevistic, V. (1997). MPC with linear time-varying prediciton models, *Technical report*, Automatic Control Laboratory, ETH Zurich.

Newton, G.C., Guild, L.A. and Kaiser J.F.(1957). Analytical design of feedback systems, *John Wiley & Sons*, USA.

Papoulis, A. (1965). Probability, random variables and stochastic processes, *McGraw-Hill Kogakusha*, Japan.

Li, P., Wendt, M. and Wozny, G. (2002) A probablistically constrained model predictive controller, *Automatica*, 38(7):1171-1176.

Platzman, L.K. and Johnson, T.L. (1976). A Linear-Quadratic control problem with innovations-feedthrough solution, *IEEE Trans. on Automatic Control*, 21(5):721-725.

Polderman, J.W. and Willems, J.C. (1998). Introduction to mathematical systems theory - a behavioral approach, *Springer-Verlag*, USA.

Post, M. and Ludlage, J.L. (1993). Advanced control of a fluidized bed ethylene polymerization reactor, *Technical report*, 93/022/MP, IPCOS.

Prett, D.M. and Gilette, R.D. (1980). Optimization and constrained multivariable control of a catalytic cracker unit, *Proc. Joint Automatic Control Conference.*

Process Systems Enterprise (2003). gPROMS Advanced User Guide, UK.

Qin, S.J. and Badgewell, T.A. (2003). A survey of industrial model predictive control technology, *Control Engineering Practise*, 11(7):733-764.

Rawlings, J.B., Meadows, E.S. and Muske, K.R. (1993). Nonlinear model predictive control: a tutorial and survey, *IFAC Advanced control of Chemical Processes*, pages 185–197.

Rawlings, J.B. (2000). Tutorial overview of model predictive control, *IEEE Control Systems Magazine*, 20(3):38–43.

Rhodes, I.B. (1971). A tutorial introduction to estimation and filtering, *IEEE Trans. on Automatic Control*, 16(6):688–706.

Richalet, J., Rault, A., Testud, J.L. and Papon, J. (1978). Model predictive heuristic control: applications to industrial processes, *Automatica*, 12:413–482.

Robertson, D.G. and Lee, J.H. (1995). A least squares formulation for state estimation, *Journal of process control*, 5(4):291–299.

Robertson, D.G., Lee, J.H. and Rawlings, J.B. (1996). A moving horizon-based approach for least-squares estimation, *Aiche Journal*, 42(8):2209–2224.

Rockafellar R.T. (1970). Convex analysis, *Princeton Mathematical Series*, USA.

Roos, C., Terlaky, T. and Vial, J.-Ph (1997), Theory and algorithms for linear optimization. An interior point approach. Series in discrete mathematics and optimization, *John Wiley & Sons*, USA.

Ruben, H. (1960). Probability content of regions under spherical normal distributions I, The Annals of Mathematical Statistics, 31: 598–618.

Rudin, W. (1976). Principles of mathematical analysis, McGraw-Hill Book Company,

Scherer, C.W. (1995). Multiobjective $\mathbf{H}_2/\mathbf{H}_\infty$ control, *IEEE Trans. Automatic Control*, 40(6):1054–1062.

Scherer, C.W., Gahinet, P. and Chilali, M. (1997). Multi-objective output-feedback control via LMI optimization, *IEEE Trans. on Automatic Control*, 42:896–911.

Scherer, C.W. (1999). LMIs in controller analysis and synthesis, *Lecture Notes and Slides*, Delft Center for Systems and Control.

Scherer, C.W. (1999). Lower bounds in multi-objective $\mathbf{H}_2/\mathbf{H}_\infty$ problems, *Proc. IEEE Conference on Decision and Control.*

Scherer, C.W. (2000). An efficient solution to multi-objective control problems with LMI objectives, *Systems & Control Letters*, 40(1):43–57.

Scokaert, P.O.M. and Rawlings, J.B. (1998). Constrained linear quadratic regulation, *IEEE Trans. on Automatic Control*, 43(8):1163–1169.

Scokaert, P.O.M. and Mayne, D.Q.(1998). Min-max feedback model predictive control for constrained linear systems, *IEEE Trans. on Automatic Control*, 43(8):1136–1142.

Soeterbeek, A.R.M. (1990). Predictive control, a unified approach, *PhD. Thesis*, Delft University of Technology.

Safonov, M.G. and Athans, M. (1977). Gain and phase margin of multiloop LQG controllers, *IEEE Trans. on Automatic Control*, 22(2):173-178.

Speyer, J.L., Deyst, J. and Jacobson, D.H. (1974). Optimization of stochastic linear systems with additive measurement and process noise using exponential performance criteria, *IEEE Trans. on Automatic Control*, 19(4):358–366.

Sturm, J. (1999). Using sedumi 1.02, a Matlab toolbox for optimization over symmetrical cones, *Optimization Methods and Software*, 11(12):625–635, http://fewcal.kub.nl/sturm/software/sedumi.html.

Sussmann, H.J. (1972). Minimal realizations of nonlinear systems, *Geometric Methods in Systems Theory*, ed. Mayne and Brockett, D. Riedel, The Netherlands.

Striebel, C. (1965). Sufficient statistics in the optimal control of stochastic systems, *Journal of Mathematical Analysis and Applications*, 12:576–592.

Tousain, R.L. (2000). General purpose dynamic optimization with gPROMS. Speeding up large-scale dynamic optimizaion for NMPC, *Technical Report A-891*, Delft Center for Systems and Control.

Tousain, R.L. (2002). Dynamic optimization in business-wide process control, *PhD. Thesis*, Delft Center for Systems and Control.

Tse, E. (1971). On the optimal control of stochastic linear systems, *IEEE Trans. on Automatic Control*, 16(6):776–785.

Van Brempt,W. (2000). HDPE6.gPROMS, Userguide, version 1.2, *Technical report*, IPCOS b.v.

Van De Water, H. and Willems, J.C. (1981). The certainty equivalence property in stochastic control theory, *IEEE Trans. on Automatic Control*, 26(5):1080–1087.

Van Hessem, D.H. and Tousain, R.L. (2001). Implementation of the INCOOP base-case reference arthictecture, and supplement Updated reference architecture (2002). *Technical Reports INCOOP-TRP-TUD-B-0026/0029.*

Van Hessem, D.H., Scherer, C.W. and Bosgra, O.H. (2001). LMI-based closed-loop economic optimization of stochastic process operation under state and inputs constraints. *Proc. IEEE Conf. Decision and Control*, 40:4228–4223.

Van Hessem, D.H. and Bosgra, O.H. (2002). Closed-loop stochastic dynamic process optimization under input and state constraints. *Proc. American Control Conference*, 20:2023–2028.

Van Hessem, D.H. and Bosgra, O.H. (2002). A conic reformulation of model predictive control including bounded and stochastic disturbances under state and input constraints. *Proc. IEEE Conf. Decision and Control*, 41:4643-4648.

Van Hessem, D.H. and Bosgra, O.H. (2003). A full solution to the constrained stochastic closed-loop MPC problem via state and innovations feedback and its receding horizon implementation, *Proc. IEEE Conf. Decision and Control*, 42:929-934.

Van Hessem, D.H. and Bosgra, O.H. (2004). Closed-loop stochastic model predictive control in a receding horizon applied to a continuous gas-phase polymerization reactor example *To be presented at the American Control Conference.*

Willems, J.C. (1971). Least squares stationary optimal control and the algebraic Riccati equation, *IEEE Trans. on Automatic Control*, 16(6):621–634.

Werner, D. (2000). Funktionalanalysis (in German), *Springer-Verlag*, Germany.

Wonham, W.M. (1968). On the separation theorem of stochastic control, SIAM Journal of Control, 6(2):312-326.

Wright, S.J. (1997). Primal-dual interior-point methods, SIAM, USA.

Youla, D.C., Jabr, H.A., and Bongiorno JR., J.J. (1976). Modern Wiener-Hopf design of optimal controllers-Part I: The single-input-output case, *IEEE Trans. on Automatic Control*, 21(3):3–13.

Youla, D.C., Jabr, H.A. and Bongiorno JR., J.J. (1976). Modern Wiener-Hopf design of optimal controllers-Part II: The multivariable case, *IEEE Trans. on Automatic Control*, 21(3):319–338.

Zames, G. (1981). Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms and approximate inverses, *IEEE Trans. on Automatic Control*, 26(2):301–320.

Zames, G. and Francis, B.A. (1983). Feedback, minimax sensitivity and optimal robustness, *IEEE Trans. on Automatic Control*, 28(5):585–601.

Zheng, A. and Morari, M. (1995). Control of Linear Systems with Constraints, *Proc. American Control Conference*, 14:3704–3708.

Zhou, K., Doyle, J.C. and Glover, K. (1996). Robust and Optimal Control, *Prentice Hall*, USA.

Ziegler, J. and Nichols, N. (1942) Optimum settings for automatic controllers, *Trans. ASME, Journal of Dynamic Systems, Measurement and Control*, 64:759-768.

List of Symbols

Symbols in Systems, Signals and Control

A	System matrix at time k
A_k	System matrix at time k
A_c	System matrix of controller
A_{cl}	System matrix of closed-loop
A_Q	State-space matrix of Q FIR parameter
$\mathcal{A}$	Transformed closed-loop system matrix
B	Input matrix maps u to x
B_k	System matrix at time k
B_c	System matrix of controller
B_{cl}	System matrix of closed-loop
B_k^w	Input matrix maps w to x at time k
B_Q	State-space matrix of Q FIR parameter
$\mathcal{B}_j$	Transformed closed-loop system matrix
C	Output matrix maps x to y
C_c	Output matrix of controller
C_{cl}	Output matrix of closed-loop
C_k	Output matrix maps x to y at time k
C_Q	State-space matrix of Q FIR parameter
C_k^z	Output matrix maps x to z
$\mathcal{C}_j$	Transformed closed-loop system matrix
d	Deterministic bounded disturbance
D	Feedthrough matrix maps u to y
	Bounded disturbance set
D_c	Feedthrough matrix of controller
D_{cl}	Feedthrough matrix of closed-loop
D_k	Feedthrough matrix maps u to y at time k
D_Q	State-space matrix of Q FIR parameter
D_k^w	Output matrix maps w to y
D_k^z	Feedthrough matrix maps u to z
$\mathcal{D}_j$	Transformed closed-loop system matrix
e_k	Estimation error at time k
f	Nonlinear system eqautions
F	Feedback Gain

G_{uy}	Dynamic system maps $\mathbf{u}$ to $\mathbf{y}$ (same for other signals)
G_{uy}^k	Lower-block matrix of G_{uy}
I	Identity system
k	Discrete time index
K_k	Feedback controller in predictive problems
K_v	Innovations feedback controller in predictive problems
K_y	Output feedback controller in predictive problems
$\mathbf{K}_k$	Set of non-anticipating controllers
L_k	Lifted State-feedback gain
M	Feedthrough matrix maps v to u
M_l	Markov parameters
N_k	Kalman gain/optimal filter gain
N_Q	Static output feedback gain
n_u	Number of control inputs
n_w	Number of disturbance inputs
n_x	Number of states
n_y	Number of measured outputs
n_z	Number of controlled variables
Q	Youla-parameter in Q-parameterization
Q_k	Weighting matrix LQG
R_k	Weighting matrix LQG
t	Time
T_1, T_2, T_3	Dynamic systems in affine parameterization of closed-loop
u_k	Manipulated variable/control input
v	Algebraic variables
v_k	Innovations
w_k	Disturbance input/process disturbance
W	Covariance matrix disturbances
x_k	State variable/latent variable
y_k	Process variable/measured output
z_k	Controlled variable/performance output
z	Complex variable in Z-transform
$\Phi_{i,j}$	State transition matrix discrete time dynamics
$\Phi_{i,j}^e$	State transition matrix discrete time observer dynamics
$\Phi_{i,j}^c$	State transition matrix discrete time controller dynamics

Symbols in Optimization Problems

A	Matrix in least squares problem
$\hat{\mathbf{A}}$	Transformed controller parameters
$\mathcal{A}$	Operator in linear equality constraints
B	Matrix in least squares problem

$\hat{\mathbf{B}}$	Transformed controller parameters
b	Vector in linear equality constraints
c	Vector in linear objectives
C	Matrix in least squares problem
$\hat{\mathbf{C}}$	Transformed controller parameters
$\hat{\mathbf{D}}$	Transformed controller parameters
E_i	Block structured matrix with identity matrix on location i
$\mathcal{E}$	Ellipsoid
F_P	Factor of the matrix P
g	Vector in inequality constraint definition
g_j	j^{th} element of g
h_j	Row j in H
H	Matrix in inequality constraint definition
I	Identity matrix
j	index of constraints
J	Objective function
J_{RH}	Objective function receding horizon
K	Cone in optimization problems
K^*	Polar cone in optimization problems
$\mathbf{L}$	Lagrangian
$\mathbf{L}_+^n$	n-dimensional Lorentz cone
m	Number of contrstraints
M	Matrix gradient
M_k	Optimization variable includes state and innovations feedback
$\mathbf{p}$	Generic vector of optimization variables
$\mathcal{P}$	Polytope generated by linear inequality constraints
Q_k	Weighting matrix LQG at time k
R_k	Weighting matrix LQG
$\mathbf{R}^n$	n-dimensional Euclidean space
$\mathbf{R}_+^n$	Standard positive cone in $\mathbf{R}^n$
S_0	Matrices in Kronecker implemetation of CLMPC
S_1	Matrices in Kronecker implemetation of CLMPC
S_2	Matrices in Kronecker implemetation of CLMPC
S_+^n	Positive Semi-Definite Cone
r	Multiplier vector in back-off term
s	Slack variables (for inequality constraints)
	Sensitivity in optimization
x	Primal variables
X	Vector space
	Free matrix in generic least squares problem
y	Dual variable
β	Penalty on constraint relaxation
γ	Objective value
ϵ	Upperbound for sub-optimal controllers
ζ	Free variable

η	Lagrange multiplier
λ	Vector valued Lagrange multiplier
Λ	Matrix valued Lagrange multiplier
Λ_{ij}	Sub-matrix of Λ
μ	Upper bounds
ν	Back-off vector

Symbols in Probability and Stochastics

$\mathbf{b}$	Step-shaped disturbance (bias)
E	Mathematical expectation
F_P	Factor of the covariance matrix P
F_W	Factor of the covariance matrix W
f_x	Probability density function of random variable
F_x	Distribution function of random variable x
I	Identity map
P	Probability of event
P_k	Covariance matrix of state
$P_k^{\hat{x}}$	Covariance matrix of state-estimate
P_k^e	Covariance matrix of estimation error
$\mathbf{r}$	Ramp-shaped disturbance
R	Covariance matrix
$\mathbf{R}^n$	n-dimensional Euclidean space
$\mathbf{R}^n_+$	Standard positive cone in $\mathbf{R}^n$
V_k	Joint variance matrix state-estimate and estimation error
W_k	Variance matrix of Gaussian disturbances
Z	Covariance matrix performance outputs
α	Certainty level in probability constraints
μ	Mean
σ	Standard deviation
Σ	Diagonal matrix with singular values
ξ	Generic element sample space

Symbols used in examples

B	Bottom flow
c	Fraction component in liquid
c_F	Feed composition
D	Distillate flow

F	Feedflow
	Force
J	Block matrix in difference operation
k	Stiffness spring
L	Reflux flow
m	Mass
M	Mass hold up on trays
R	Recycle ratio
T	Matrix for difference operation
v	Fraction component vapor
	velocity
V	Vapor flow
q	Thermal quality of the feed
x	Position
z_{lo}	Lowerbounds in column example
z_{up}	Upperbounds in column example

List of Sub/Superscripts and Operations

det	Determinant
inf	Infimization
max	maximization
$\mathcal{T}_v(T)$	Truncated Toeplitz matrix of system $T(z)$
x^c	Control error
x^e	Estimation error
x_k	Value of x at discrete time t_k
x^l	Variable of nonlinear system in iteration l
x_r^l	Reference trajectory in of nonlinear system in iteration l
x_{nl}	State of actual nonlinear process
x^r	Predicted signal
x^r	Reference signal
$\hat{x}_k$	A priori estimate of x
$\hat{x}_{k\|k-1}$	A priori estimate of x given all data up to $k-1$
$x^\star$	Optimal argument optimization problem
x^+	Extended state for rate-penalties
$\mathbf{x}$	Stacked vector of x_k for several time instances
$\\|X\\|_F$	Frobenius norm
$\rho(X)$	Spectral radius
$\otimes$	Kronecker product

List of Abbreviations

APC	Advanced Process Control
CLMPC	Closed-loop MPC
CFHLQG	Constrained Finite Horizon LQG
$CFHLQG^A$	Feedback subproblem
$CFHLQG^B$	Feedforward subproblem
CV	Controlled Variable
DAE	Differential Algebraic Equations
DMC	Dynamic Matrix Control
EKF	Extended Kalman Filter
FHLQG	Finite Horizon LQG
FIR	Finite Impulse Response
GPC	Generalized Predictive Control
HDPE	High Density PolyEthylene
INCOOP	INtegration of COntrol and plant-wide OPtimization
LP	Linear Program
LQG	Linear Quadratic Gaussian
LQR	Linear Quadratic Regulator
LTI	Linear Time Invariant
LTV	Liner Time Varying
LV	Latent Variable
MAC	Model Algorithmic Control
MPHC	Model Predictive Heuristic Control
MIMO	Multiple Input Multiple Output
MPC	Model Predictive Control
MV	Manipulated Variable
ODE	Ordinary Differential Equations
PI	Proportional+Integral
PV	Process Variable
QDMC	Quadratic Dynamic Matrix Control
QP	Quadratic Program
RTO	Real Time Optimization
SDP	Semi-Definite Program
SISO	Single Input Single Output
SOCP	Second-Order Cone Program
SSP	Steady State Problem
SQP	Sequential Quadratic Programming

Samenvatting

De chemische proces industrie wordt door de toenemende concurrentie op de wereldmarkt gedwongen om de efficiëntie van de procesoperatie te verhogen. Vanwege de hoge investeringskosten die nodig zijn voor het bouwen van nieuwe fabrieken is het uiterst interessant om de bestaande fabrieken anders te opereren. Het is daarom wenselijk dat fabrieken bijvoorbeeld een grotere variëteit aan chemische producten produceren of sneller wisselen tussen productspecificaties of operatie strategiën. Om aan de eisen van de markt te voldoen is onderzoek en ontwikkeling een noodzakelijke eis om producten, productie processen en operationele strategieën te vernieuwen. In bijna alle gevallen spelen modellen van verschillende complexiteit van fabrieken of processen een belangrijke rol in de analyse van technologische problemen en de synthese van oplossingen. Modellen formaliseren de huidige status van onze systeem kennis, wijzen op zwakke plekken tijdens validatie, leiden tot nieuw onderzoek, zijn goede dragers van kennis over lange perioden en maken multidisciplinaire samenwerking tussen ingenieurs en wetenschappers mogelijk.

Model-gebaseerde voorspellende regelaars (MPC) zijn een voorbeeld van zeer succesvol gebruik van modellen in de verbetering van procesoperatie. Het schept de mogelijkheid voor multivariabele procesregeling waarin veel ongelijkheidsbegrenzingen kunnen worden beschouwd. Het grote aantal bedrijven dat MPC technologie aanbiedt en het grote aantal succesvolle implementaties in de laatste twintig jaar wijzen erop dat MPC in een volwassen status verkeert, maar het gebrek aan voorspelbaarheid van het gesloten-lus gedrag legt het sterke tekort aan theoretische ontwikkeling bloot. De grootste structurele limitatie van MPC is dat het een open-loop voorspellende regelmethode is met de paradoxale eigenschap dat toekomstige verstoringen en toekomstige procesmetingen niet worden beschouwd. Het impliciete karakter dat volgt uit toepassing van het voortschrijdende horizon principe is moeilijk te analyseren en systematische instelling van de MPC met ongelijkheidsbegrenzingen is ronduit onmogelijk zonder zeer uitvoerige simulatie studies. Het is niet bekend hoe de regelaarinstellingen gekozen moeten worden om de storingsgevoeligheid van het proces te beïnvloeden hetgeen één van de basis eigenschappen van iedere systematische ontwerpmethode van terugkoppelregelingen is. Daarom zijn er geen eenvoudige methodes om voorspellende regelaars voor gewenst gesloten-lus gedrag te ontwerpen, om de optimale bescherming tegen overschrijding van begrenzingen te bieden of om het proces tegen de limiet aan te opereren. De realiteit van moderne geavanceerde proces regeling, waar zowel ongelijkheidsbegrenzingen *en* stochastische verstoringen een centrale rol spelen, wijst er op dat er een fundamentele noodzaak is om het MPC probleem zodanig te herformuleren dat deze begrenzingen en verstoring *gelijktijdig* worden beschouwd, terwijl de dynamische economie van de fabriek wordt geoptimaliseerd.

In dit proefschrift wordt een nieuw raamwerk voor geavanceerde proces regeling gepresenteerd. Het doel is de ontwikkeling van een strategie voor geavanceerde procesregeling waarin voorspellende regelaars systematisch worden ontworpen zonder uitvoerige simulatie studies. Dit betekent dat het voortschrijdende horizon principe voor het genereren van terugkoppeling volledig wordt vervangen door een directe terugkoppeling. Dit voorkomt alle typische complicaties in analyse en synthese van voorspellende regelaars en maakt het mogelijk om een regelsysteem te ontwerpen met *gegarandeerde* prestaties. Dit creëert de mogelijkheid om ontwerpmethoden voor hoge-prestatie regelingen te combineren met de geavanceerde procesregelingen om economisch te optimaliseren onder ongelijkheidsbegrenzingen.

In de ontwikkeling van een efficiënte structuur die aan de bovenstaande eisen voldoet moeten een aantal stappen tegelijkertijd worden gezet. Cruciaal is het meenemen van toekomstige verstoringen en toekomstige proces metingen naast de traditioneel aanwezige gemanipuleerde en geregelde variabelen. De beschikbaarheid van toekomstige metingen en gemanipuleerde variabelen wordt gebruikt om een expliciete regelwet op te stellen zodat een parameterisatie van realiseerbare storingsgevoeligheidsfuncties ontstaat. Vanwege de onbegrensdheid van Gaussische verstoringen worden de ongelijkheidsbegrenzingen met een bepaalde zekerheid afgedwongen. De minimaal benodigde veiligheidsmarges tot deze begrenzingen worden dan bepaald door de keuze van de regelaar en de tweede-orde statistische eigenschappen van de verstoringen. Bovenop deze terugkoppelingsstructuur wordt een voorwaartse koppeling gebruikt voor alle deterministische taken zoals transities naar optimale stationaire toestanden of veranderingen in productspecificaties en procesbelasting. Nadat de veiligheidsmarge van de ongelijkheidsbegrenzingen zijn afgetrokken wordt de voorwaartse koppeling gevonden door het oplossen van een deterministisch dynamisch optimalisatie probleem dat simultaan met de terugkoppelwet word geïmplementeerd zodat het proces in operatie gegarandeerd binnen de begrenzingen blijft.

Twee model-gebaseerde voorspellende regelaars zijn ontwikkeld die bovenstaande strategie implementeren. De volledige oplossing wordt gegeven door de zogenaamde gesloten-lus model-gebaseerde voorspellende regelaar (closed-loop MPC) waarin de terugkoppeling, de veiligheidsmarge en het voorwaarts gekoppelde traject simultaan worden geoptimaliseerd. Een voorspellende formulering van de Youla-Kučera parameterisatie van de gesloten-lus maakt dit probleem convex zodat het kan worden opgelost met behulp van moderne optimalisatie algoritmes. Een versimpeling van deze procedure is gegeven door het eindige horizon kwadratisch probleem (constrained finite horizon LQG control). In dit geval wordt het probleem in tweeën gesplitst. In een eerste stap een vaste geschikt gekozen terugkoppelwet berekend die de veiligheidsmarge vastlegt gevolgd door een tweede stap waarin het voorwaartse gekoppelde traject met deze vast gekozen marges wordt geoptimaliseerd. Het voordeel van deze aanpak is dat de complexiteit van de berekening nagenoeg gelijk is aan die van standaard open-lus MPC en dat maakt deze aanpak geschikt voor soortgelijke problemen. Voor continue processen worden beide oplossingen in een voortschrijdende horizon *implementatie* gezet zonder hieraan terugkoppeling te ontlenen. De optimale implementatie wordt verkregen door een voorspellende toestandsterugkoppeling naast de uitgangsterugkoppeling te plaatsen, zodanig dat het volledige optimalisatie probleem altijd een vast aantal vrijheidsgraden heeft. Beide technieken zijn gebaseerd op lineaire tijdsvariërende systemen en kunnen daarom worden toegepast op zowel lineaire als niet-lineaire dynamische systemen. Toepassing van de voorgestelde regelstrategie op een gesimuleerde niet-lineaire industriële polymerisatie reactor laat veelbelovende resultaten zien die verder toegepast en theoretisch onderzoek naar gesloten-lus voorspellende regelmethoden motiveren.

Curriculum Vitae

Dennis van Hessem was born on April 9, 1973 in Dalfsen, Overijssel, The Netherlands. He finished his pre-university school (VWO) in 1992 at the Atheneum J.C. de Glopper in Capelle aan den IJssel. In August 1992 he started his study Mechanical Engineering at Delft University of Technology, where he graduated in Januari 1999 in Systems and Control. He wrote his MSc. thesis on process modelling for which research was conducted at Unilever Research in Vlaardingen. From March 1999 to September 2003, he was a Ph.D. student Mechanical Engineering in the Systems and Control Group, where he conducted his research within the EU sponsored INCOOP project. The results of this research have been reported in this thesis. In October 2003 he started working in the process systems engineering department of Shell International Chemicals at the Shell Research and Technology Centre in Amsterdam.



ISBN 90-407-2489-X

TU Delft

Delft University of Technology

Delft University Press