Tuning PI controllers based on $\mathcal{H}_\infty$ Weighted Sensitivity

S. Alcántara[1] S. Skogestad[2] C. Grimholt[2] C. Pedret[1] R. Vilanova[1]

[1]Department of Telecommunications and Systems Engineering, Autonomous University of Barcelona, Spain

[2]Department of Chemical Engineering, Norwegian University of Science and Technology, Trondheim, Norway

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Outline

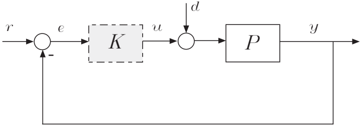
Introduction

- Most of the controllers in industry are of PID type, and among them, many are PI controllers or have the derivative action switched off.
- There are plenty of tuning rules in the literature for PI controllers. For stable processes, a remarkably simple and effective one is that given by the SIMC rule, suggested by Skogestad.
- SIMC is based on conventional IMC-PI, but increases the integral gain to improve disturbance rejection.
- A limitation of the SIMC rule is the restriction to stable plants only.

Goals

- Derive simple PI tuning rules (**also for unstable plants**) analytically along the lines of SIMC.
- This time, instead of basing the derivation on IMC, an $\mathcal{H}_\infty$ weighted sensitivity problem is posed.
- In addition to the common λ parameter, a new one (γ) is introduced to adjust the tradeoff between set-point (*servo*) and disturbance (*regulatory*) responses.

Weighted Sensitivity



A basic problem in $\mathcal{H}_\infty$ is the weighted sensitivity problem:

$$\min_{K\in\mathcal{C}} \|\mathcal{N}\|_\infty = \min_{K\in\mathcal{C}} \|WS\|_\infty \qquad (1)$$

where

- S is the sensitivity function: $S = \frac{1}{1+PK}$, being P the plant under control and K the feedback controller.
- $\mathcal{C}$ denotes the set of stabilizing feedback controllers.
- W is a Minimum-Phase (MP) weight responsible for the shaping of S (*the key point*).

Weighted Sensitivity

Lemma

*Assume that P is purely rational (i.e., there is no time delay in P), and that W is a MP weight **including the unstable poles of** P. Then, the optimal weighted sensitivity subject to internal stability ($K \in \mathcal{C}$) is given by*

$$\mathcal{N}^o = \rho \frac{q(-s)}{q(s)} \tag{2}$$

where ρ and $q = 1 + q_1 s + \cdots + q_{\nu-1} s^{\nu-1}$ (Hurwitz) are uniquely determined by the interpolation constraints:

$$W(z_i) = \mathcal{N}^o(z_i) \qquad i = 1 \ldots \nu, \tag{3}$$

being $z_1 \ldots z_\nu$ ($\nu \geq 1$) the RHP zeros of P.

Weighted Sensitivity

Consider the following factorizations:

$$P = \frac{n_p}{d_p} = \frac{n_p^+ n_p^-}{d_p^+ d_p^-} \qquad W = \frac{n_w}{d_w} = \frac{n_w}{d_w' d_p^+} \tag{4}$$

Then,

$$S = \mathcal{N}^o W^{-1} = \rho \frac{q(-s) d_w}{q(s) n_w} \tag{5}$$

$$T = 1 - \mathcal{N}^o W^{-1} = \frac{n_p^+ \chi}{q(s) n_w} \tag{6}$$

$$K = \left(\frac{1 - \mathcal{N}^o W^{-1}}{\mathcal{N}^o W^{-1}} \right) P^{-1} = \frac{d_p^- \chi}{\rho n_p^- q(-s) d_w'} \tag{7}$$

Posing the problem

Our final objective is the derivation of simple PI tuning rules using a weighted sensitivity problem. To this aim, we need to specify:

- A simple model for P
- A simple weight W

Model

We adopt a First Order plus Time Delay (FOPTD) model:

$$P = K_g \frac{e^{-sh}}{\tau s + 1} \tag{8}$$

where

- K_g is the (zero frequency) gain.
- h is the (effective) time delay.
- τ is the time constant of the process.
  - In particular, τ may be negative to account for unstable plants.
  - Integrating plants can be treated considering the limit $\tau \to \infty$.

Weight (*key point*)

The following (**possibly unstable**) weight is proposed

$$W = \frac{(\lambda s + 1)(\gamma s + 1)}{s(\tau s + 1)} \tag{9}$$

where $\lambda > 0$ and $\gamma \in [\lambda, |\tau|]$ are used as tuning parameters.
Rationale: Start considering $\lambda = 0$, then

- If $\gamma = |\tau| \rightarrow |W| = |1/s| \rightarrow \min_{K \in \mathcal{C}} \|S\|_\infty$ subject to integral action (*Servo-type* design)
- If $\gamma = \lambda \rightarrow |W| = \frac{1}{K_g}|P/s| \rightarrow \min_{K \in \mathcal{C}} \|PS\|_\infty$ subject to integral action (*Regulator-type* design)

As we increase λ, the minimization of $|S|$ at higher frequencies is emphasized, preventing large peaks on S at the expense of closed-loop bandwidth ($\approx$ IMC).

Solving the problem

In order to apply the Lemma, we need to approximate the time delay, so we finally take

$$P \approx K_g \frac{-sh+1}{\tau s+1} \tag{10}$$

Now, applying the Lemma, the optimal weighted sensitivity function is

$$\mathcal{N}^o = \rho = \frac{(\lambda+h)(\gamma+h)}{\tau+h} \tag{11}$$

The corresponding feedback controller is

$$K = \frac{\chi}{K_g \rho s}, \; \chi = 1 + \frac{\tau(h+\lambda+\gamma)-\lambda\gamma}{\tau+h} s \tag{12}$$

Solution

The PI tuning rules

Considering a PI controller

$$K = K_c \left(1 + \frac{1}{T_i s}\right) \quad (13)$$

the following tuning rule is obtained:

Model	K_c	T_i	
$K_g \frac{e^{-sh}}{\tau s+1}$	$\frac{1}{K_g} \frac{T_i}{\lambda+\gamma+h-T_i}$	$\frac{\tau(h+\lambda+\gamma)-\lambda\gamma}{\tau+h}$	$\lambda > 0, \gamma \in [\lambda, \lvert\tau\rvert]$

The PI tuning rules

Let us analyze the tuning rules for the extreme values of γ.

- $\gamma = |\tau|$ (Servo)

Model	K_c	T_i
$K_g \frac{e^{-sh}}{\tau s+1}$	$\frac{1}{K_g} \frac{T_i}{\lambda+\lvert\tau\rvert+h-T_i}$	$\frac{\tau(h+\lambda+\lvert\tau\rvert)-\lambda\lvert\tau\rvert}{\tau+h}$

In particular, note that when $\tau > 0$ (stable case):
$K_c = \frac{\tau}{K_g(\lambda+h)}$, $T_i = \tau$ (the well-known IMC rule).

- $\gamma = \lambda$ (Regulator)

Model	K_c	T_i
$K_g \frac{e^{-sh}}{\tau s+1}$	$\frac{1}{K_g} \frac{\tau}{\lambda+h} \left(\frac{h+2\lambda-\lambda^2/\tau}{h+\lambda} \right)$	$\frac{\tau(h+2\lambda)-\lambda^2}{\tau+h}$

Guidelines for tuning λ, γ

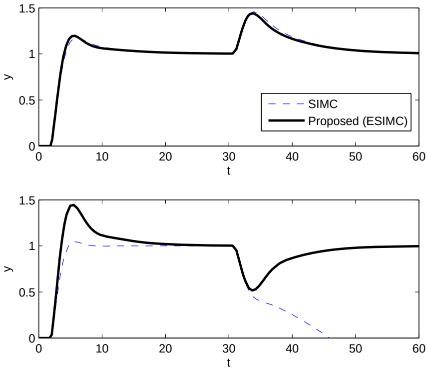
It has been found that a good interval for λ is given by $[h, 2h]$, regarding γ:

- If $h/|\tau| \geq 1$, choose $\gamma = |\tau|$
- As $h/|\tau| \to 0$, select $\gamma \to \lambda$ (otherwise the disturbance response would be very sluggish)
- For a good trade-off between servo/regulator responses, a simple approach is to *copy* SIMC by selecting $T_i = \min\{|\tau|, 4(\lambda + h)\}$. This yields:

$$\gamma = \max\left\{|\tau|, (3\tau + 4h)\frac{\lambda + h}{\tau - \lambda}\right\} \tag{14}$$

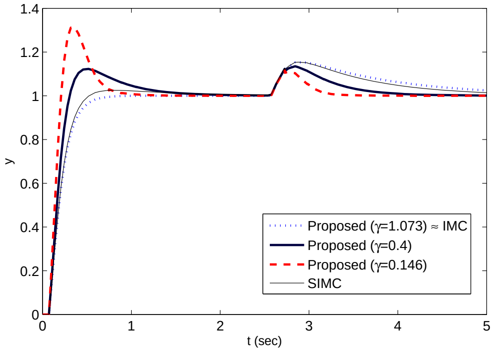
The resulting tuning behaves approximately like SIMC but also applies to slow unstable processes.

Comparison with SIMC



Time responses for $P = \frac{e^{-s}}{20s+1}$ (top) and $P = \frac{e^{-s}}{-20s+1}$ (bottom).

Comparison with SIMC



Time responses for $P = \frac{e^{-0.073s}}{1.073s+1}$. Does SIMC yield the best tradeoff? The proposed method may be used to revisit the SIMC rules, including unstable plants in the discussion.

Conclusions

- An analytical $\mathcal{H}_\infty$-based design procedure has been presented and applied to the tuning of *PI* controllers.
- Coprime factorizations are avoided in the unstable plant case.
- The resulting tuning rules are instructive (implying both *robustness/performance* and *servo/regulator* issues) and may be used for teaching purposes.

S. Alcántara, W.D. Zhang, C. Pedret, R. Vilanova, S. Skogestad, **IMC-like Analytical $\mathcal{H}_\infty$ design with S/SP mixed sensitivity consideration: Utility in PID tuning guidance**, Journal of Process Control, 21 (4), 554-563 (2011).

Corrected version reprinted in: 21 (6), 976-985 (2011).

Thank you for your attention !