Selection of Controlled Variables: Maximum Gain Rule and Combination of Measurements

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The appropriate selection of controlled variables is one of the most important tasks in plantwide control. In this paper, we consider the selection of secondary temperature measurements for indirect composition control of distillation columns. The maximum gain rule (maximize minimum singular value of scaled gain matrix) is compared to the exact local method, and it is found that the gain rule should be used with care for ill-conditioned plants like distillation columns. We also consider the use of optimal combinations of measurements to further reduce the effect of disturbances and implementation errors.

1. Introduction

The selection of controlled variables is one of the most important tasks in control structure design[1] because this choice can limit the operational (economic) performance of the whole control system. This problem is combinatorial in nature and has been addressed by many authors.[2−6]

For problems where the constraints are active, it is clear that the active constraints should be selected as controlled variables. On the other hand, if the optimum is unconstrained, the choice of controlled variables is much more difficult because there are no limit on the possible variables or combinations.

In this paper, we consider the problem of indirect control, where the issue is to select secondary variables ($c = y_2$) such that we indirectly achieve good control of the primary variables (y_1) in spite of disturbances (d) and implementation errors (n_{y_2}). In our case, the primary variables (y_1) are the product compositions in the distillation column and the candidate secondary measurements ($c = y_2$) include temperatures, flows, and flow ratios. Figure 1 shows the block diagram representation of indirect control, where the matrix H represents the selection/combination of measurements y.

$$c = Hy_m = H(y + y^n) \tag{1}$$

We assume that the number of controlled variables (n_c) is equal to the number of inputs (n_u), so that the controller K in Figure 1 is square. We consider two cases for the selection of controlled variables $c = Hy$:

1. Control individual measurements (H is then a selection matrix where each row has one 1 element and the rest are 0s).

2. Control measurements combinations (H is then a full static matrix).

The issue in the first case is to select from the set of candidate measurements y a subset of n_c measurements. For the distillation example, we have $n_u = 2$ inputs (L and V) and we want to select $n_c = n_u = 2$ temperature measurements. In the second case, the truly optimal solution is to use all the measurements (about 50 for the distillation example). However, to simplify one normally first selects a subset of the candidate measurements and then finds $n_c = n_u = 2$ linear combinations of the selected measurements (as given by H).

Moore[3] proposed to select individual controlled variables (measurements) c using a singular value decomposition (SVD) analysis of the steady-state gain matrix G_{all} from the inputs u to all the candidate measurements y. After decomposing the gain matrix $G_{\text{all}} = U\Sigma V^{\text{T}}$, he proposed to use the orthonormal matrix U (matrix of left singular vectors) to locate the most sensitive measurements (with largest absolute values), which should be used as controlled variables.

On the basis of the idea of self-optimizing control,[7] Halvorsen et al.[8] derived rigorously the closely related method of selecting controlled variables that maximize the minimum singular value, $\underline{\sigma}(G')$, of the appropriately scaled gain matrix from inputs u to the selected outputs c. The "maximum gain rule"[8] is the following: *Select controlled variables c such that we maximize the minimum singular value of the scaled gain matrix* G', $\underline{\sigma}(G')$, *where*

$$G' = S_1 G S_2 \tag{2}$$

Here, G is the steady-state gain matrix from manipulated variables u to selected controlled variables c.

$$c = Gu \tag{3}$$

where $G = HG^y$, and S_1 and S_2 are the output and input scalings, respectively.

The first part of this paper analyzes the maximum gain rule by comparing it with the exact local method of Halvorsen et al.[8]

The second part is related to selecting the *optimal combination* of variables that minimizes the effects of disturbances and implementation error.[6,9,10] One approach is the two-step procedure of Alstad and Skogestad,[10] where in the first step the maximum gain rule is used to select the best set of measurements, and in the second step we find H. For finding H, Hori et al.[9] presented a simple method that achieves perfect indirect control (with zero disturbance loss), and the same result can be obtained from the nullspace method of Alstad and Skogestad.[10] However, these methods do not consider implementation (measurement) errors y^n. This weakness is overcome with the "exact local method"[8] for which Alstad et al.[11] derived an analytical solution for the optimal H.

All these methods are compared on a binary distillation case study, where $U = [L\ V]$, $y_1 = [x_{\text{top}}^{\text{H}}\ x_{\text{btm}}^{\text{L}}]$, and y includes temperatures and/or flows. The distillation column was selected because it is "ill-conditioned" with a large variation in gain depending on the input direction, and the scaling S_2 is expected

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to be more critical in such cases. The objective function J to be minimized is the relative steady-state deviation from the desired setpoint,

$$J = \Delta X^2 = \left(\frac{x^{\mathrm{H}}_{\mathrm{top}} - x^{\mathrm{H}}_{\mathrm{top},s}}{x^{\mathrm{H}}_{\mathrm{top},s}}\right)^2 + \left(\frac{x^{\mathrm{L}}_{\mathrm{btm}} - x^{\mathrm{L}}_{\mathrm{btm},s}}{x^{\mathrm{L}}_{\mathrm{btm},s}}\right)^2 \quad (4)$$

where $x^{\mathrm{H}}_{\mathrm{top}}$ is the composition of the heavy key-component (H) in the top of the column and $x^{\mathrm{L}}_{\mathrm{btm}}$ is the composition of the light key-component (L) in the bottom.

2. Maximum Gain Rule

The *maximum gain rule* states that one should select controlled variables to maximize the minimum singular value of $G' = S_1 G S_2$. The minimum singular value has the monotonic property, which means that we can use a branch and bound algorithm to search for the configuration with the largest minimum singular value, thus avoiding the evaluation of all possible configurations.[12]

To derive the maximum gain rule, and in particular to derive the correct scaling S_2, we consider the loss L as the difference between the actual value of the cost function $J(u,d)$, obtained with a specific control strategy, e.g. with the controlled variable c is constant, and the truly optimal value of the cost function $J_{\mathrm{opt}}(d)$, that is,

$$L(u,d) = J(u,d) - J_{\mathrm{opt}}(d) \quad (5)$$

In our case, with indirect control, $J_{\mathrm{opt}}(d) = 0$; see eq 4. The second-order expansion of the loss function is[8]

$$L(u,d) = \frac{1}{2}(c - c_{\mathrm{opt}}(d))^{\mathrm{T}} G^{-\mathrm{T}} J_{uu} G^{-1}(c - c_{\mathrm{opt}}(d)) \quad (6)$$

where J_{uu} is the matrix of second derivatives of J with respect to u (Hessian matrix) and G is the gain matrix from the inputs u to c. Defining $z = J_{uu}{}^{-1/2} G^{-1}(c - c_{\mathrm{opt}}(d))$, eq 6 becomes[8]

$$L(u,d) = \frac{1}{2}\|z\|_2^2 \quad (7)$$

Introducing $e_c = c - c_{\mathrm{opt}}(d)$ and *assuming* that each controlled variable c_i is scaled such that the sum of its optimal range (c_{opt}) and its implementation error (n^y) is unity, i.e., for combined errors the 2-norm is less or equal to 1, Halvorsen et al.[8] show that the worst-case loss is

$$L_{\max} = \max_{\|e_c'\|_2 \le 1} \frac{1}{2}\|z\|_2^2 \le \frac{1}{2(\underline{\sigma}(S_1 G J_{uu}{}^{-1/2}))^2} \quad (8)$$

To minimize the loss $L_{\max}$, the (correct) maximum gain rule is then to maximize $\underline{\sigma}(S_1 G S_2)$, where $S_2 = J_{uu}{}^{-1/2}$.

2.1. Output Scaling (S_1). An important part of the maximum gain rule is to scale the output variables appropriately, i.e., such that $\|e_c'\|_2 \le 1$ holds. The outputs are scaled with respect to their "span", which is the sum of

1. optimal variation (c_{opt}) due to disturbances d
2. effect of implementation error (c^n) and we have

$$S_1 = \mathrm{diag}\{1/\mathrm{span}(c_i)\} \quad (9)$$

where

$$\mathrm{span}(c_i) = |\mathrm{opt\ var}| + |\mathrm{implem\ error}| = |c_{\mathrm{opt}}|_i + |c^n|_i \quad (10)$$

From a linear model, the optimal variation may be obtained as follows. Write the linear steady-state model as follows:

$$y_1 = G_1 u + G_{d1} d \quad (11)$$

$$c = Gu + G_d d \quad (12)$$

where y_1 are the primary variables, c are the measurements (candidate controlled variables), u are manipulated variables, and d are disturbances. In the presence of disturbances (d), perfect control of the primary variables ($y_1 = 0$) is obtained with

$$u_{\mathrm{opt}} = -G_1{}^{-1} G_{d1} d \quad (13)$$

The resulting optimal variation of the measurements (c) is

$$c_{\mathrm{opt}} = (-G G_1{}^{-1} G_{d1} + G_d) d \quad (14)$$

Remark: We here use perfect control ($y_1 = 0$) as the reference for computing the optimal variation. This is recommended even for cases where some manipulated variables are kept constant (e.g., reflux L is constant) such that $y_1 = 0$ is not possible, because perfect control of y_1 is in any case the objective.

2.2. Input scaling (S_2). From eq 8, the best (correct) input "scaling" for the maximum gain rule is to select $S_2 = J_{uu}{}^{-1/2}$. The term "scaling" is a bit misleading because $J_{uu}{}^{-1/2}$ is generally not a diagonal matrix.

Simplified Scaling (Assuming J_{uu} Unitary). In many cases,[8] it is assumed that $S_2 = J_{uu}{}^{-1/2}$ is unitary, which is referred to as simplified scaling. For the minimum singular value, the property $\underline{\sigma}(AB) \ge \underline{\sigma}(A)\underline{\sigma}(B)$ holds for any nonsingular matrices A and B of any dimension.[13] From eq 8, we then have that $\underline{\sigma}(G') = \underline{\sigma}(S_1 G J_{uu}{}^{-1/2}) \ge \underline{\sigma}(J_{uu}{}^{-1/2})\underline{\sigma}(S_1 G)$. Here, $\underline{\sigma}(J_{uu}{}^{-1/2}) = 1/\bar{\sigma}(J_{uu})^{1/2}$, and we derive from eq 8 an upper bound for the maximum loss

$$L_{\max} = \max_{\|e_c'\|_2 \le 1} L \le \frac{\bar{\sigma}(J_{uu})}{2(\underline{\sigma}(S_1 G))^2} \quad (15)$$

To analyze the tightness of this bound, we derive a lower bound from the inequality[13] $\bar{\sigma}(AB) \ge \underline{\sigma}(A)\bar{\sigma}(B)$. From eq 8, we then have $\bar{\sigma}(S_1 G J_{uu}{}^{-1/2}) \ge \underline{\sigma}(J_{uu}{}^{-1/2})\bar{\sigma}(S_1 G)$, and using $\bar{\sigma}(S_1 G) = 1/\underline{\sigma}(S_1 G)$, we then find

$$\frac{\underline{\sigma}(J_{uu})}{2(\underline{\sigma}(S_1 G))^2} \le L_{\max} \le \frac{\bar{\sigma}(J_{uu})}{2(\underline{\sigma}(S_1 G))^2} \quad (16)$$

That is, for systems that can be scaled such that J_{uu} has a small condition number ($\gamma(J_{uu}) = \bar{\sigma}(J_{uu})/\underline{\sigma}(J_{uu})$), the lower and upper bounds are close. In particular, if J_{uu} is unitary then $\gamma(J_{uu}) = 1$ and simplified scaling is exact.

3. Exact Local Method

To evaluate the maximum gain rule presented in the last section, we calculate the maximum loss for each set of controlled variables using the exact local method.[8] The exact value of the worst-case local loss is

$$L_{\max} = \max_{\|e_c'\|_2 \le 1} L = \frac{(\bar{\sigma}([M_d\ M_{n^y}]))^2}{2} \quad (17)$$

where

$$M_d = J_{uu}{}^{1/2}(J_{uu}{}^{-1} J_{ud} - G^{-1} G_d) W_d \quad (18)$$

$$M_{n^y} = J_{uu}{}^{1/2} G^{-1} W_n \quad (19)$$

M_d represents the loss in the primary variables caused by disturbances, and M_{n^y} represents the loss caused by implementation error.

The magnitude of the disturbances and implementation error enter into the diagonal matrices W_d and W_n, respectively. Note

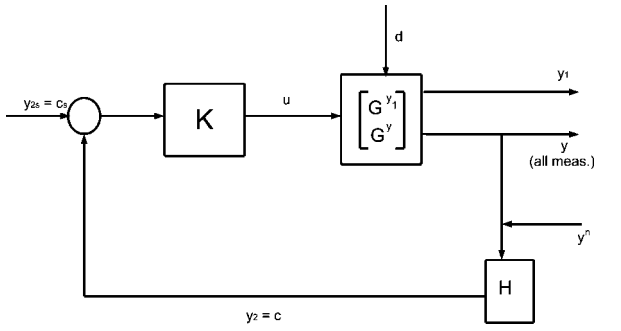

Figure 1. Block diagram of indirect control with selection of measurements.

that the elements in W_n are equal to $|c^n|_i$ introduced earlier. Normally, the source of the implementation error is the measurement error for y and we have

$$W_n = HW_{n^y} \tag{20}$$

where W_{n^y} is a diagonal matrix of the expected measurement errors for y; see y^n in eq 1.

For indirect control, consider a quadratic cost function

$$J = y_1^{\mathrm{T}} Q y_1 + u^{\mathrm{T}} R u \tag{21}$$

where Q and R are weighting matrices (both are symmetric positive-definite). From eq 11, we then have

$$J_{uu} = 2(G_1^{\mathrm{T}} Q G_1 + R) \tag{22}$$

$$J_{ud} = 2G_1^{\mathrm{T}} Q G_{d1} \tag{23}$$

For the distillation example, with the cost function in eq 4, the weighting matrices are $Q = \mathrm{diag}([1/(x^{\mathrm{H}}_{\mathrm{top},s})^2\ 1/(x^{\mathrm{L}}_{\mathrm{btm},s})^2])$ and $R = 0$.

We here use the exact local method as an analysis tool. One may also use the exact local method to select optimal measurements. However, minimizing $\underline{\sigma}(G')$ requires the evaluation of fewer combinations, and each evaluation is also less time-consuming.

4. Case Study: Distillation Column

The variable selection methods (maximum gain rule and exact local method) are applied to a binary distillation column ("column A" of Skogestad and Postlethwaite).[13] The chemical components are denoted L (light) and H (heavy). The mixture is assumed to be ideal with a constant relative volatility ($\alpha_{\mathrm{LH}} = 1.5$). The main disturbances are the feed flow rate (F), feed enthalpy (q_{F}), and feed composition (z_{F}). The feed is about 50% mol of light component ($z_{\mathrm{F}} = 0.5$). The primary controlled variables are the product compositions

$$y_1 = \begin{bmatrix} x_{\mathrm{top}} \\ x_{\mathrm{btm}} \end{bmatrix}$$

The objective of the column is to get a top product with 99% light component (and 1% heavy) ($x^{\mathrm{H}}_{\mathrm{top},s} = 0.01$), and a bottom product with 1% light component ($x^{\mathrm{L}}_{\mathrm{btm},s} = 0.01$). The column has 41 stages (including the reboiler and the total condenser), and these stages are numbered from bottom to top (see Figure 2).

This is a conventional distillation column with a given feed and pressure, and it has four degrees of freedom: reflux flow rate (L), vapor boilup (V), distillate flow rate (D), and bottoms flow rate (B), i.e., $u_0 = [L\ V\ D\ B]^{\mathrm{T}}$. We need to stabilize two liquid levels, and for this, we selected B and D. This gives the conventional LV-configuration, where L and V are the remaining steady-state degrees of freedom for composition control, $u = [L\ V]$. Note that the steady-state gain matrix G_1 from $u = [L\ V]$ to y_1 is generally ill-conditioned. In our case, the gain matrices are

$$G_1 = \begin{bmatrix} 1.085 & -1.098 \\ 0.875 & -0.862 \end{bmatrix}$$

and

$$G_{d1} = \begin{bmatrix} 0.586 & 1.119 & 1.092 \\ -0.394 & -0.881 & -0.868 \end{bmatrix}$$

so

$$J_{uu}^{-1/2} = \begin{bmatrix} 0.263 & 0.259 \\ 0.259 & 0.262 \end{bmatrix} \quad \text{and}$$

$$J_{ud} = 10^4 \begin{bmatrix} 1.961 & 3.970 & 3.889 \\ -1.996 & -3.976 & -3.895 \end{bmatrix} \tag{24}$$

The condition number $\gamma = \bar{\sigma}/\underline{\sigma}$ is the same for both G_1 and $J_{uu}^{-1/2}$ in this case, $\gamma(G_1) = \gamma(J_{uu}^{-1/2}) = 145.6$ (they are the same because $R = 0$ and $Q = \beta I$, where β is any constant different from zero). Note that an unitary matrix has a condition number of 1, so J_{uu} is far from unitary. The minimized condition number (with respect to diagonal scaling) is also large, $\gamma^*(G_1) = \min_{D_1,D_2} \gamma(D_1 G_1 D_2) = 141.9$.

Our objective is to use the two available degrees of freedom to keep the top and bottom compositions (primary variables y_1) close to their optimal values; see eq 4. As compositions are difficult to measure (due to long time delays, high cost, etc.), we want to use indirect control where temperatures and/or flows (including flow ratios L/D, L/F, V/B, V/F) are used as secondary controlled variables (c). For simplicity, we assume that the

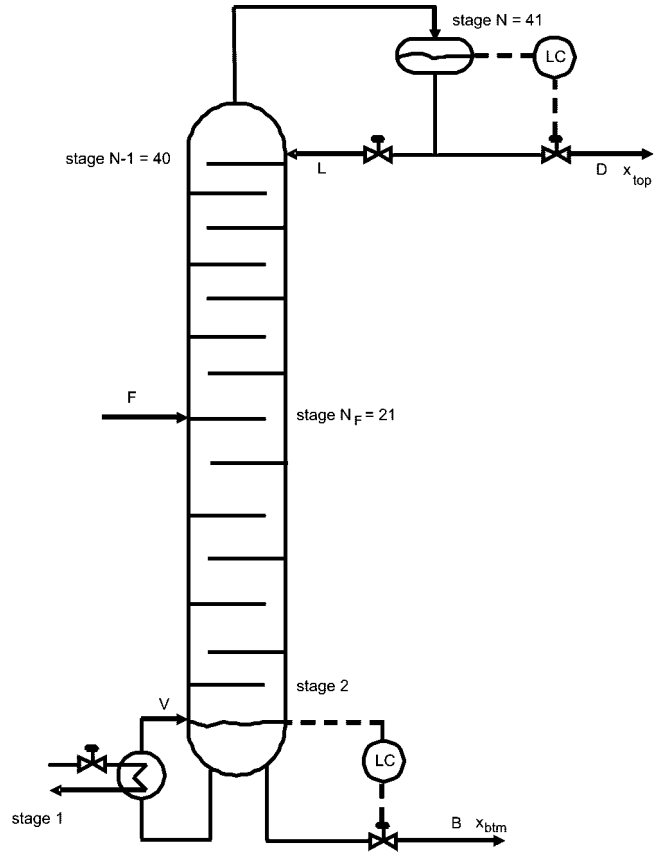

Figure 2. Distillation column.

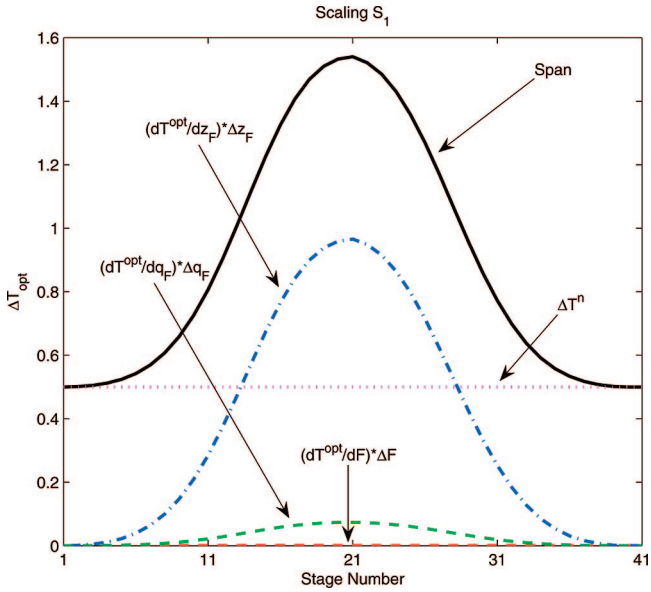

Figure 3. Optimal temperature variations and total span (output scaling S_1).

temperature T_i (°C) on each stage i can be calculated as a linear function of the liquid composition in each stage[14]

$$T_i = 0x_{A,i} + 10x_{B,i} \tag{25}$$

This may seem unrealistic, but results using detailed models show that this is actually of minor importance.[15] To compare the maximum gain rule with the exact method, we use the maximum composition deviation:[15]

$$\Delta X_{\max} = \sqrt{J_{\max}} = \sqrt{L_{\max}} \tag{26}$$

where $L_{\max}$ is calculated from eq 17 (exact method) or estimated from eqs 8 (maximum gain rule) and 15 (simplified maximum gain rule).

4.1. Output Scaling (S_1). The output scaling is $S_1 = 1/(|\Delta c_{\text{opt}}| + \Delta c^n)$ where

$$|\Delta c_{\text{opt}}| = \left|\frac{\mathrm{d}c_{\text{opt}}}{\mathrm{d}z_{\text{F}}}\right|\Delta z_{\text{F}}^{\text{E}} + \left|\frac{\mathrm{d}c_{\text{opt}}}{\mathrm{d}F}\right|\Delta F^{\text{E}} + \left|\frac{\mathrm{d}c_{\text{opt}}}{\mathrm{d}q_{\text{F}}}\right|\Delta q_{\text{F}}^{\text{E}} + |\Delta c^n| \tag{27}$$

Here, $\Delta z_{\text{F}}^{\text{E}}$, ΔF^{E}, and $\Delta q_{\text{F}}^{\text{E}}$ are the expected (typical) disturbances and Δc^n is the expected implementation/measurement error for controlling the measurements. For temperatures, the implementation error ($\Delta c^n = \Delta T^n$) is assumed to be the same for all stages ($\Delta T^n = \pm 0.5$ °C), while the implementation error is ±10% for flow rates and ±15% for flow ratios. The expected magnitudes of the disturbances are 20% for feed rate F, 10% for feed composition z_{F}, and 10% for feed enthalpy q_{F}.

The resulting optimal variations Δc_{opt} for temperatures along the column obtained using eq 14 are shown in Figure 3. Temperatures on stages close to the feed stage are more sensitive to disturbances, while the stages close to the ends are affected more by the implementation error (the optimal variation is zero at the ends). Also, the figure shows that the main disturbance is in the feed composition. The optimal variation in temperature with feed flowrate is zero, because we have assumed constant efficiency for the column. This result confirms what is reported in the literature.[16]

4.2. Input scaling (S_2). In this section, we want to compare the maximum gain rule (section 2) and the simplified maximum gain rule (section 2.2) with the exact local method (section 3). The three methods were compared using the composition deviations

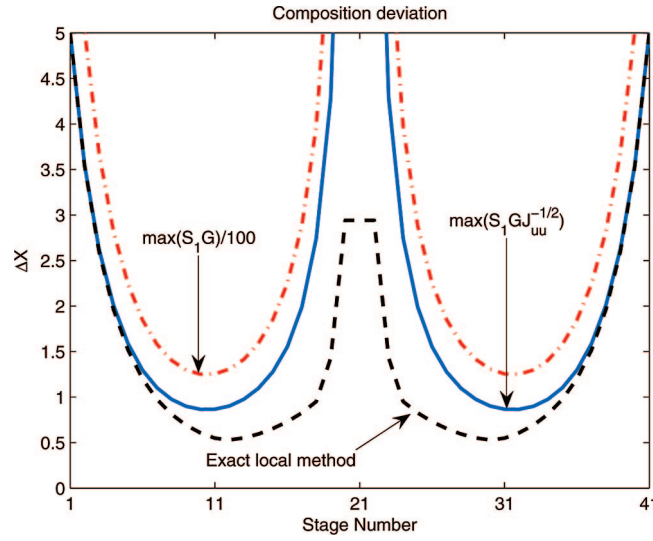

Figure 4. Comparison of composition deviation estimates ($S_1GJ_{uu}^{-1/2}$ (eq 8), S_1G (eq 15), and exact local method (eq 17)).

estimated by eqs 8 (maximum gain rule, $S_1GJ_{uu}^{-1/2}$), 15 (simplified maximum gain rule, S_1G), and 17 (exact local method). Figure 4 shows the results with two symmetrically located temperatures as controlled variables. A "good" location should have a composition deviation ΔX of about 1 or less. This figure shows that the three methods give almost the same best temperatures, but note that the estimated ΔX is a factor 100 times higher when we use $\underline{\sigma}(S_1G)$ (simplified maximum gain rule). This happens because, as shown earlier, the condition number for this process is very large (145.6).

In Table 1, we consider the more general case where the candidates two controlled variables are any combination of temperatures and flows (including flow ratios L/D, L/F, etc.). Table 1 shows that the simplified maximum gain rule using $\underline{\sigma}(S_1G)$ gives again completely wrong (too high) estimates for ΔX in most cases. The simplified method gives that the best control configuration with the largest value of $\underline{\sigma}(S_1G)$ is to keep L/F and V/B constant. However, this is actually a poor choice with an exact loss of 18.60.

On the other hand, with the correctly scaled gain matrix $G' = S_1GJ_{uu}^{-1/2}$, the results with the maximum gain rule are very close to the exact method. The upper bound on ΔX from eq 8 is at most a factor 2.008 higher than the correct ΔX from eq 17. The (correct) maximum gain rule gives $T_9 - T_{32}$ as the best set of controlled variables (exact loss of 0.675). This is close to the optimum (minimum) steady-state composition deviation (ΔX) of 0.530, which is obtained when we control temperatures on stages 12 and 30, that is, with the temperatures symmetrically located on each side of the feed stage. Thus, although the maximum gain rule using $G' = S_1GJ_{uu}^{-1/2}$ is not exact, it gives results in terms of temperature selection that are very similar to the exact method. It is thus suitable for screening of candidate controlled variables.

5. Linear Combination of Measurements

Thus far, we have considered single measurements as controlled variables. Another option is to use combinations of measurements $c = Hy$, where H is a combination rather than a selection matrix. The goal of using several measurements is to further reduce the effect of disturbances and implementation errors. The use of using multiple temperatures to control distillation columns has been suggested also by other authors.[17,18]

For a given set of measurements (y_s), the combination matrix H can be evaluated in two different ways:

1. *Nullspace method*: Perfect disturbance rejection[9] with the assumption of no measurement/implementation error gives

$$H = \tilde{G}_1 \tilde{G}'^{-1} \tag{28}$$

where $\tilde{G}_1 = [G_1\ G_{d1}]$ and $\tilde{G}' = W_{n_y}{}^{-1}\tilde{G} = W_{n_y}{}^{-1}[G\ G_d]$. It is here assumed that the number of measurements is $n_y = n_u + n_d$, such that $\tilde{G}'$ is invertible.

For the *LV*-distillation column case study, we here consider only temperatures as possible measurements and we do not consider the feed flowrate as a disturbance because its optimal variation in temperature is zero (see section 4.1). Then, we have 2 inputs ($n_u = 2$), 2 disturbances (z_F, q_F, i.e. $n_d = 2$), and we need to select four temperatures ($n_y = 4$) to obtain perfect disturbance rejection.

2. *Exact local method*: The combined effect of disturbances and implementation error is minimized by finding the matrix H that minimize the 2-norm of $M = [M_d\ M_n]$ in eq 17. It can be shown that this is equivalent to minimizing $\|H\tilde{F}\|_2^2$ subject to $HG^y = J_{uu}{}^{1/2}$, where $\tilde{F} = [FW_d\ W_{n^y}]$ and $F = -(G^y J_{uu}{}^{-1} J_{ud} - G_d^y)$. From this, one can derive an analytic expression for H[11]

$$H^T = (\tilde{F}\tilde{F}^T)^{-1} G^y (G^{yT}(\tilde{F}\tilde{F}^T)^{-1} G^y)^{-1} J_{uu}{}^{1/2} \tag{29}$$

Equation 29 holds for any number of measurements (n_y).

Note that the solutions in eq 28 and 29 are not unique, so if H is an optimal solution, another optimal solution is $H_1 = DH$, where D is a nonsingular matrix of dimension $n_u \times n_u$.

5.1. Two-Step Procedure. To avoid the evaluation of the exact loss (eq 17) for all possible combinations of measurements, Alstad et al.[11] suggested a two-step approach to obtain the combination of variables.

1. Select the set of measurements applying the suboptimal method of Alstad et al.[11] to maximize the minimum singular value of the scaled matrix $\tilde{G}'$. The hope is that the scaling S_1 and the maximum gain rule should result in a set which is also insensitive to implementation error.

2. Find H from eq 28 or 29.

5.1.1. Step 1: Selection of Measurements. The suboptimal approach of maximizing $\bar{\sigma}(\tilde{G}')$, where $\tilde{G}' = S_1\tilde{G} = [S_1G\ S_1G_d]$ was applied to selecting four temperatures in the distillation column. The temperatures selected are on stages 9, 16, 24, and 33.

5.1.2. Step 2: Perfect Disturbance Rejection. For the distillation column example with (from step 1)

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = H \begin{bmatrix} T_9 \\ T_{16} \\ T_{24} \\ T_{33} \end{bmatrix} \tag{30}$$

we find that zero disturbance loss is obtained with

$$H = \tilde{G}_1 \tilde{G}'^{-1} = \begin{bmatrix} -0.7068 & 0.0444 & 0.0911 & -0.0663 \\ 0.0442 & -0.0376 & -0.0766 & 0.6903 \end{bmatrix} \tag{31}$$

As the H-matrix is not unique, we have normalized it so that its 2-norm is equal to 1, i.e., $\|H\|_2 = 1$.

The maximum loss, which in this case is caused by measurement error only, is 0.822 (calculated by exact local method). Surprisingly, this loss is larger than the value of 0.530 obtained with the best two single temperatures (see Table 1). This is because the measurement selection in step 1 is suboptimal and because the implementation error is not considered when using eq 28.

5.1.3. Step 2: Exact Local Method. Next, eq 29, which also accounts for implementation error, was used to obtain the optimal combination of the four temperatures in eq 30. The resulting H-matrix is

$$H = \begin{bmatrix} 0.4751 & 0.3980 & -0.2433 & -0.0474 \\ -0.0770 & 0.1070 & -0.2456 & -0.6905 \end{bmatrix} \tag{32}$$

which gives a maximum loss of 0.582. In this case, the loss due to disturbance is 0.370 (it is nonzero since we do not have perfect disturbance rejection) and the loss due to implementation error is 0.516. The total loss of 0.582 is less than with perfect disturbance rejection (0.822), but again is larger than the best combination of two temperatures (0.530). This is because the measurement selection in step 1 is suboptimal.

To correct for this, we tried another method for selecting measurements in step 1 ("optimal rule" in the work of Alstad et al.),[11] where we minimize $\bar{\sigma}(\tilde{J}[G\ G_d]^{-1} W_{n^y})$. However, applying this to the distillation example gives temperatures T_{11}, T_{19}, T_{21} and T_{33}, and the maximum loss is 1.0226, which is even worse than before. The conclusion is therefore that the two-step procedure performs poorly for our distillation case study.

5.2. One-Step Procedure with Four Measurements. We here use the exact local method to directly find the truly optimal combination in one step. There are

Table 1. Steady-State Composition Deviation ΔX with Control of Individual Measurements (c)

		Maximum Gain Rule				
configuration	exact method	assume J_{uu} unitary (simplified)		$S_2 = J_{uu}{}^{-1/2}$ (correct)		
fixed variables (c)	ΔX (eq 17)	$\underline{\sigma}(S_1G)$	est. ΔX (eq 15)	$\underline{\sigma}(S_1GJ_{uu}{}^{-1/2})$	est ΔX (eq 8)	ΔX (eq 8)/ΔX (eq 17)
$T_{12}-T_{30}$	0.530	1.508	131	0.783	0.903	1.704
$T_{12}-T_{29}$	0.541	1.442	137	0.752	0.941	1.739
$T_{14}-T_{28}$	0.595	1.241	159	0.645	1.100	1.849
T_9-T_{32}	0.675	1.548	127	0.792	0.893	1.323
$T_{15}-T_{26}$	0.706	0.956	206	0.499	1.417	2.007
$T_{15}-L/F$	0.916	1.531	129	0.607	1.164	1.271
$T_{16}-V/F$	1.148	1.125	175	0.498	1.419	1.236
$T_{19}-L$	1.223	0.815	242	0.400	1.767	1.445
$T_{15}-L/D$	1.321	0.727	272	0.342	2.067	1.565
$T_{22}-V$	1.470	0.639	309	0.305	2.320	1.578
$T_{24}-V/B$	1.711	0.571	345	0.261	2.712	1.585
T_1-T_{41}	5.000	0.271	728	0.141	5.000	1.000
$L/D-V/B$	15.80	0.878	225	0.040	17.80	1.127
$L/F-V/B$	18.60	1.603	123	0.028	25.60	1.376
$L-B$	21.10	0.805	245	0.020	35.20	1.668
$D-V$	21.20	0.634	311	0.020	35.20	1.660
$L/F-V/F$	90.00	1.600	124	0.007	109.0	1.211

$$\begin{bmatrix} 41 \\ 4 \end{bmatrix} = \frac{41 \times 40 \times 39 \times 38}{4 \times 3 \times 2} = 101\,270$$

possible combinations of four temperatures, and we obtained the best set of four temperatures (T_{10}, T_{11}, T_{31}, and T_{32}) by applying the exact local method to all combinations. However, note that Kariwala[6] has derived a branch and bound algorithm that could alternatively have been used to find the best solution in a more efficient way.

The corresponding optimal combination matrix from eq 29 is

$$H_1 = DH = \begin{bmatrix} 0.4087 & 0.4962 & 0.1880 & 0.1349 \\ -0.2269 & -0.2980 & -0.4831 & -0.3994 \end{bmatrix} \quad (33)$$

The maximum loss in this case is 0.440 (0.302 due to disturbances and 0.407 due to implementation error), which is now smaller than the best combination of two temperatures (0.530).

It is interesting to note that it is optimal to select neighboring temperatures (T_{10}/T_{11} and T_{31}/T_{32}), and it is also interesting to note that this location is close to the optimal with only two measurements (T_{12} and T_{30}). Apparently, neighboring temperatures are optimal for reducing the effect of measurement errors.

Note that H is not unique and by choosing

$$D = \begin{bmatrix} 1 & 0.389 \\ -0.555 & -1 \end{bmatrix}$$

we find that another optimal combination matrix is

$$H_1 = DH = \begin{bmatrix} 0.320 & 0.380 & 0 & -0.020 \\ 0 & 0.023 & 0.379 & 0.325 \end{bmatrix} \quad (34)$$

which confirms that the optimal combination is approximately to control an average of T_{10} and T_{11} and an average of T_{31} and T_{32}.

5.3. All Measurements. To compare with the theoretically optimal solution, we consider the case where we combine all 41 temperatures. The matrix H was calculated using the analytical solution for the exact local method, eq 29, resulting in a maximum loss of 0.226 (0.067 due to disturbance and 0.216 to implementation error), which is the lowest possible.

The results for all the cases studied are summarized in Table 2.

Table 2. Composition Deviation ΔX for Various Configurations: Summary

	ΔX		
controlled variables c_1-c_2	disturbance	meas noise	total
$T_{12}-T_{30}$	0.376	0.513	0.530
H [T_9, T_{16}, T_{24}, T_{33}] (31 from eq 28)	0	0.822	0.822
H [T_9, T_{16}, T_{24}, T_{33}] (32 from eq 29)	0.370	0.516	0.582
H [T_{10}, T_{11}, T_{31}, T_{32}] (33 from eq 29)	0.302	0.407	0.440
H [all temperatures] (from eq 29)	0.067	0.216	0.226

6. Summary and Conclusions

The problem of controlled variable selection is the following: We have n_u independent variables (inputs) u and we want to select from the candidate measurements y a set of $n_c = n_u$ controlled variables $c = Hy$. The objective is that with constant setpoints (c_s) for these controlled variables, the cost J should be close to its optimal value, in spite of unknown disturbances and implementation/measurement errors. In this paper, the cost function J is the setpoint deviation for the primary variables y_1 (indirect control); see eq 4.

The number of candidate measurements y is generally larger or equal to the number of inputs, because we generally include in the set of candidate measurements y also the inputs u. In practice, we very rarely use all measurements. Thus, finding $c = Hy$ consists of two steps:

Step 1. Select a subset of n_y measurements to be used for control.

Step 2. Find H.

In Step 2, there are two main cases:

(a) $n_y = n_c$: control of individual measurements (no further work is required to find H, e.g., we may select $H = I$).

(b) $n_y > n_c$: control measurements combinations. Here, H is a full matrix and two methods have been proposed to find H:

1. Perfect disturbance rejection (measurement errors neglected): H is computed from eq 28. In eq 28, it is assumed that $n_y = n_u + n_d$, so that $\tilde{G}$ is invertible, but more generally, the extended nullspace method may be used.[11]

2. Minimize the combined effect of disturbance and measurement errors (exact local method): H computed from eq 29.[11]

For a specific choice of controlled variables, $c = Hy$, one may use the exact local method, eq 17, to compute the resulting loss $L_{\max}$ for combined disturbances and measurement errors. An upper bound on $L_{\max}$ is provided by eq 8, from which we get the maximum gain rule: Select measurements to maximize $\underline{\sigma}(S_1 G J_{uu}^{-1/2})$, where G is the steady-state gain matrix from u to $c = Hy$ and $S_1 = \mathrm{diag}\{1/\mathrm{span}(c_i)\}$, where $\mathrm{span}(c_i) = |c_{i,\mathrm{opt}}| + |c^{n_i}|$.

Note that efficient branch and bounds algorithms have been derived for both selecting individual measurements according the maximum gain rule and for selecting optimal measurement combinations according to the exact local method.[19,20]

Several interesting insights can be derived from the results in this paper:

1. For ill-conditioned plants, like distillation columns, one needs to include in the maximum gain rule the correct input scaling $S_2 = J_{uu}^{-1/2}$ (where J_{uu} is the Hessian of the cost function). The simplified maximum gain rule, where we maximize $\underline{\sigma}(S_1 G)$, should be avoided for ill-conditioned plants (where the condition number of J_{uu} is large). This is clearly illustrated by the results in Table 1.

2. Table 1 shows that the upper bound on $L_{\max} = \Delta X^2$ in eq 8, which is the basis for the maximum gain rule, provides a good estimate. It is within a factor $2.01^2 = 4.03$ of the correct value for the distillation column example.

3. The two-step procedure, where we first select measurements according to the method suggested by Alstad et al.,[11] did not perform well for the distillation example. For the case with four measurements ($n_y = 4$), this method selects uncorrelated measurements along the entire column (T_9, T_{16}, T_{24}, T_{33}), whereas the optimal method for this particular example, with the effect of measurement errors taken properly into account, is to use correlated measurements from the middle of each section (T_{10}, T_{11}, T_{31}, T_{32}).

4. The benefit of adding measurements beyond the minimum ($n_y = 2$) is small for this particular example and control of two individual measurements (T_{12}, T_{30}) is acceptable. From Table 2, we find that this gives a product composition deviation ($L_{\max}^{1/2}$) of 0.530, whereas the best with four temperatures is 0.440 and that with 41 temperatures is 0.226.

In conclusion, the maximum gain rule (maximize $\underline{\sigma}(S_1 G S_2)$ with $S_2 = J_{uu}^{-1/2}$) can be used for selecting individual measurements as controlled variables ($n_y = n_c$), and even better results can be obtained with the exact local method, eq 17. For measurement combinations ($n_y > n_c$), the optimal H for the exact

local method is given by eq 29. Kariwala and Cao[6] have derived efficient branch and bound methods that avoid the need to evaluate all possible combinations.

Literature Cited

(1) Yi, C. K.; Luyben, W. L. Evaluation of plant-wide control structures by steady-state disturbance sensitivity analysis. *Ind. Eng. Chem. Res.* **1995**, *34* (7), 2393.

(2) Mellefont, D. J.; Sargent, R. W. H. Selection of measurements for optimal feedback control. *Ind. Eng. Chem. Process Des. Dev.* **1978**, *17* (4), 549.

(3) Moore, C. F. Selection of controlled and manipulated variables. *Practical Distillation Control*, Luyben, W. L., Ed.; Van Nostrand Reinhold: New York, 1992.

(4) Chmielewski, D. J.; Palmer, T.; Manousiouthakis, V. On the theory of optimal sensor placement. *AIChE J.* **2002**, *48* (5), 1001.

(5) Bian, S.; Henson, M. A. Measurement selection for on-line estimation of nonlinear wave models for high purity distillation columns. *Chem. Eng. Sci.* **2006**, *61*, 3210.

(6) Kariwala, V. Optimal measurement combination for local self-optimizing control. *Ind. Eng. Chem. Res.* **2007**, *46* (11), 3629.

(7) Skogestad, S. Plantwide control: the search for the self-optimizing control structure. *J. Process Control* **2000**, *10* (5), 487.

(8) Halvorsen, I. J.; Skogestad, S.; Morud, J.; Alstad, V. Optimal selection of controlled variables. *Ind. Eng. Chem. Res.* **2003**, *42* (14), 3273.

(9) Hori, E. S.; Skogestad, S.; Alstad, V. Perfect steady-state indirect control. *Ind. Eng. Chem. Res.* **2005**, *44* (4), 863.

(10) Alstad, V.; Skogestad, S. Null space method for selecting optimal measurement combinations as controlled variables. *Ind. Eng. Chem. Res.* **2007**, *46* (3), 846.

(11) Alstad, V.; Skogestad, S.; Hori, E. S. Optimal measurement combinations as controlled variables. *J. Process Control*, in press.

(12) Cao, Y.; Rossiter, D.; Owens, D. H. Globally optimal control structure selection using branch and bound method. *Preprints of DYCOPS'5*, Corfu, Greece, June 7−10, 1998.

(13) Skogestad, S.; Postlethwaite, I. *Multivariable feedback control-analysis and design*, 2nd ed.; John Wiley and Sons: London, UK, 2005.

(14) Skogestad, S. Dynamics and control of distillation columns: a tutorial introduction. *Chem. Eng. Res. Des.* **1997**, *75* (A6), 539.

(15) Hori, E. S.; Skogestad, S. Selection of control structure and temperature location for two-product distillation columns. *Chem. Eng. Res. Des.* **2007**, *85* (A3), 293.

(16) Luyben, W. L. Effect of feed composition on the selection of control structures for high-purity binary distillation. *Ind. Eng. Chem. Res.* **2005**, *44* (20), 7800.

(17) Yu, C. C.; Luyben, W. L. Use of multiple temperatures for the control of multicomponent distillation columns. *Ind. Eng. Chem. Process Des. Dev.* **1984**, *23* (3), 590.

(18) Chew, M.; Jones, W. E.; Wilson, J. A. Distillation column control using the whole temperature profile. *Distillation and Absorption*, London, UK, Sept 4−6, 2006.

(19) Cao. Y.; Kariwala, V. Bidirectional branch and bound for controlled variable selection. I- Principles and minimum singular value criterion. *Comput. Chem. Eng.*, in press.

(20) Kariwala, V.; Cao, Y. Bidirectional branch and bound for controlled variable selection. II- Exact local method for self-optimizing control. *Comput. Chem. Eng.*, submitted for publication.

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