LOWER LIMIT ON CONTROLLER GAIN FOR ACCEPTABLE DISTURBANCE REJECTION

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Abstract: The objective of almost all controller tuning rules found in the literature, going back to the classic PID rules of Ziegler an Nichols (1942), is to get the "fastest" possible closed-loop response, subject to maintaining stability with reasonable robustness margins. This gives a maximum limit on the controller gain. In practice, however, we often want control to be as smooth and "slow" as possible, subject to satisfying some minimum performance requirements. This gives a minimum limit on the controller gain, and the goal of this paper is to derive this minimum limit, when the performance requirements is to achieve a specified level of disturbance rejection. Together with the more traditional tunings rules this results in a *range* for the acceptable controller gain.

Keywords: Process control, PID tuning, averaging control, measurement noise

1. INTRODUCTION

The objective of almost all PID tuning rules found in the literature, e.g., (Ziegler and Nichols, 1942) (Cohen and Coon, 1953) (Astrom and Hagglund, 1995), (Rivera *et al.*, 1986), is to get the "fastest" possible closed-loop response, subject to maintaining stability with reasonable robustness margins. The model-based direct synthesis approaches of Rivera *et al.* (1986) and Smith and Corripio (1985) contain the closed-loop time constant τ_c as a tuning parameter, but also in these works the emphasis is to obtain a lower bound on τ_c (fast response). To obtain stability and robustness, the value of τ_c is limited by the effective time delay θ of the process, and typically a value $\tau_c = \theta$ is selected (Skogestad, 2003). For processes with a small effective delay this may lead to an unnecessary fast response, and a larger value of τ_c (slower response) should be used. However, the response cannot be too slow, because otherwise we do not achieve acceptable performance. In this paper we assume that the main performance specification is that the disturbance effect on the output should be bounded.

In summary, the goal of this paper is the to derive conditions for the "slowest possible" response (upper bound on closed-loop time constant τ_c; lower bound on controller gain K_c), subject to achieving acceptable disturbance rejection. There has been work along these lines in the literature on controllability analysis and decentralized control (Hovd and Skogestad, 1992) (Hovd and Skogestad, 1994) (Skogestad and Postlethwaite, 1996), but the implications of these results on controller tuning have not been considered.

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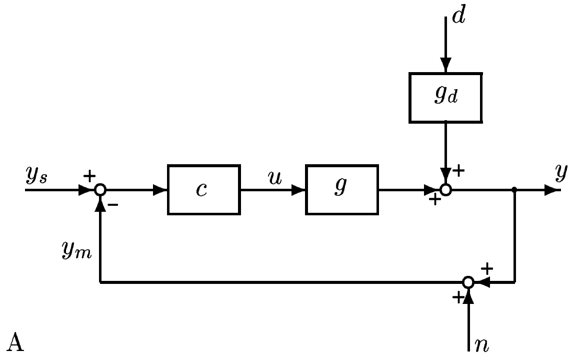

Fig. 1. Block diagram of feedback control system.

2. DERIVATION OF LOWER LIMIT ON CONTROLLER GAIN

The linear transfer function model in deviation variables is written (Figure 1)

$$y = g(s)u + g_d(s)d \tag{1}$$

where u is the manipulated input (controller output), d the disturbance, y the controlled output, $g(s)$ the process transfer function, and $g_d(s)$ the disturbance transfer function model. The Laplace variable s is often omitted to simplify notation.

With feedback control we have $u = c(s)(y_s - y)$, where $c(s)$ is the feedback controller and we in the following do not consider setpoint changes, i.e. $y_s = 0$. The effect of the disturbance d on the control output y under closed-loop control is then

$$y = \frac{g_d(s)}{1+L(s)} d = S(s) g_d(s) \cdot d \tag{2}$$

where $S = 1/(1+L)$ is the sensitivity function and $L(s) = g(s)c(s)$ is the loop gain.

We consider the following performance requirement:

- The (steady-state) output variation y in (2) should be less than $|y_{max}|$ in response to any sinusoidal disturbance of magnitude $|d_0|$.

For simplicity we assume that the values of $|y_{max}|$ and $|d_0|$ are constant, independent of frequency. From (2) the performance requirement $|y(j\omega)| \le y_{max}$ then gives

$$|S(j\omega)| \cdot |g_d(j\omega)| \cdot |d_0| \le |y_{max}|$$

or equivalently

$$|1 + L(j\omega)| \ge \frac{|g_d(j\omega)| \cdot |d_0|}{|y_{max}|} = |G_d(j\omega)| \tag{3}$$

where we have introduced the scaled disturbance gain

$$G_d \overset{\text{def}}{=} g_d \cdot \frac{|d_0|}{|y_{max}|} \tag{4}$$

The requirement (3) is illustrated in Figure 2.

We define the bandwidth ω_B as the frequency where $|S| = 1/|1+L|$ first crosses 1 from below, and ω_d as the highest frequency where $|G_d|$ crosses 1, i.e. $|G_d(j\omega_d)| = 1$. From (3) and Figure 2 we must require $\omega_B \ge \omega_d$, that is, ω_d provides a lower limit on the closed-loop bandwidth for acceptable disturbance rejection.

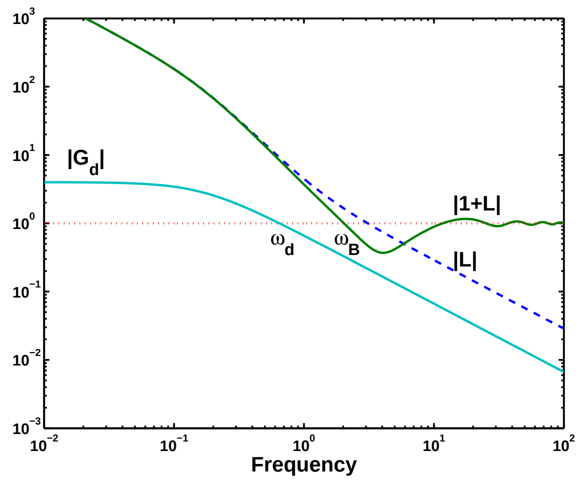

Fig. 2. Performance requirement $|1 + L| \ge |G_d|$ (3) is satisfied at all frequencies.
Data: $g_d = g = 4\frac{e^{-0.25s}}{6s+1}$, $|y_{max}| = 1, |d_0| = 1$, PI-control with $K_c = 4.313$ and $\tau_I = 0.82$.

At low frequencies $\omega \le \omega_B$ [rad/s], within the closed-loop bandwidth, we have $|L| \gg 1$ and (3) gives $|L| \ge |G_d|$, which gives the following lower limit on the frequency-dependent controller gain for acceptable disturbance rejection

$$|c(j\omega)| \ge \frac{|g_d(j\omega)| \cdot |d_0|}{|g(j\omega)| \cdot |y_{max}|}; \quad \omega < \omega_B \tag{5}$$

which may be rewritten as

$$|c(j\omega)| \ge \frac{|u_0(j\omega)|}{|y_{max}|}; \quad \omega < \omega_B \tag{6}$$

where $|u_0(j\omega)| \overset{\text{def}}{=} \frac{|g_d(j\omega)\cdot|d_0|}{|g(j\omega)|}$ is the magnitude of the input change needed to reject the disturbances at frequencies where $|L| \gg 1$. This interpretation follows since at low frequencies $y \approx 0$ and from (1) the required input to reject the disturbance is $u_0 = -(g_d/g)d_0$. From (6) we derive the following useful rule at lower frequencies where control is effective:

- The minimum controller gain at a given frequency is approximately equal to input change required for disturbance rejection divided by the allowed output variation.

As expected, tight control (with $|y_{max}|$ small) requires a large controller gain $|c|$, as does a large disturbances (with $|u_0|$ large).

2.1 Load disturbance

For the special (and very common) case of an input (load) disturbance ($g_d = g$) the required input change equals the disturbance magnitude, $|u_0| = |d_0|$, and the bound (5) becomes

$$\text{Load disturbance}: |c(j\omega)| \geq \frac{|d_0|}{|y_{max}|}; \ \omega < \omega_B \quad (7)$$

where $|d_0|$ is the magnitude of the input (load) disturbance. This bound is illustrated in Figure 3 for a PI- and PID-controller.

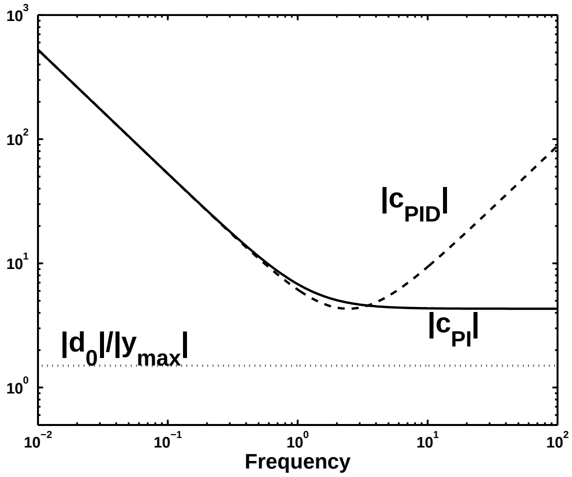

Fig. 3. Controller gain $|c|$ as a function of frequency for PI- and PID-controller.
Data PI-controller: $K_c = 4.31, \tau_I = 0.82$; PID-controller: $K_c = 4.31, \tau_I = 0.82, \tau_D = 0.20$.

Both for a PI-controller and for a PID-controller [2]

$$c_{PIDS}(s) = K_c \left(1 + \frac{1}{\tau_I s} + \tau_D s\right) \quad (8)$$

the minimum value of the controller gain $|c(j\omega)|$ as a function of frequency is always equal to K_c (independent of the values of τ_I and τ_D) (see also Figure 3):

$$\min_w |c_{PID}(j\omega)| = K_c$$

For a well-tuned PI- and PID-controller, ω_B is about at the frequency where the controller gain reaches it minimum, and from (7) we then get the following bound in order to achieve acceptable disturbance rejection with PI- and PID-control:

$$\text{Load disturbance}: \quad K_c \geq \frac{|d_0|}{|y_{max}|} \quad (9)$$

For PID tuning rules that are parameterized in terms of a single tuning parameter, like IMC-PID(Rivera *et al.*, 1986) or SIMC-PID(Skogestad, 2003), we can from the value of K_c obtain the tuning parameter (e.g. τ_c) and from this obtain the remaining controller parameters (τ_I and τ_D). For example, the SIMC PI-tunings(Skogestad, 2003) for a first-order delay process

$$g(s) = k\,\frac{e^{-\theta s}}{\tau_1 s + 1} \quad (10)$$

are

$$K_c = \frac{1}{k} \cdot \frac{\tau_1}{\tau_c + \theta} \quad (11)$$

$$\tau_I = \min\left(\tau_1, 4(\tau_c + \theta)\right) \quad (12)$$

and with a given value of K_c, we can obtain τ_c from (11) and then obtain τ_I from (12).

3. PI-EXAMPLE

Consider a first-order with delay process with time constant $\tau_1 = 6$ and time delay $\theta = 0.25$:

$$g(s) = 4\,\frac{e^{-0.25s}}{6s + 1} \quad (13)$$

The performance requirement is that the output deviation should stay within $\pm|y_{max}| = 1$ in response to a step input (load) disturbance of magnitude $|d_0| = 1$, which from (9) requires $K_c \geq |d_0|/|y_{max}| = 1$ (for a sinusoidal disturbance). It is also desirable that control is as smooth as possible, which means that we want K_c as small as possible.

Tuning for fast response. The "closed-loop" Ziegler-Nichols (ZN) settings for this process are

$$K_c = 4.313, \ \tau_I = 0.82 \quad (14)$$

We note that K_c is 4.3 times the minimum required value, so we expect that the output response is much better than the requirement. This is confirmed both by the frequency plot in Figure 2, as well as the time response to a unit step input disturbance in Figure 4. The output deviation in Figure 4 is less than 0.2, well below $|y_{max}| = 1$. However, because of the high controller gain, the input usage and also the output response is sensitive to measurement noise n on y (dashed line in Figure 4).

Tuning for smooth response. The above response is unnecessary fast so the controller gain may be reduced. We choose $K_c = |d_0|/|y_{max}| = 1$. With $k = 4, \tau_1 = 6, \theta = 2.5$ we get from (11) that $K_c = 1$ corresponds to $\tau_c = 1.25$, and from (12) we obtain the following SIMC-settings:

$$K_c = 1; \quad \tau_I = 6 \quad (15)$$

[2] In this paper we consider the "ideal" PID controller in (8) and the ZN-settings are assumed to be given for this form.

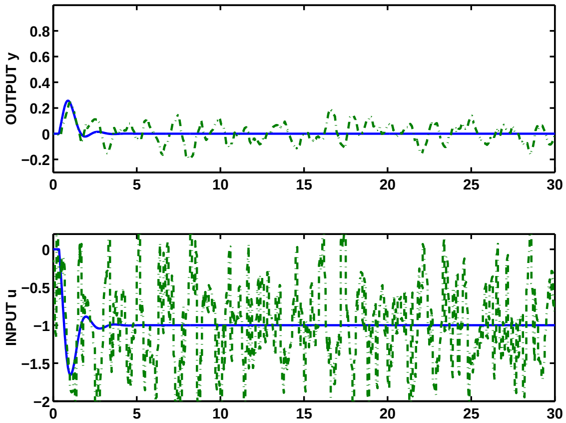

Fig. 4. Response to step load disturbance with "fast" ZN-PI controller (14).
Dashed line: With measurement noise. Solid line: No noise

The corresponding disturbance response in Figure 5 has a maximum output deviation of about 0.7, which is below $|y_{max}| = 1$, and input usage is smooth with no sensitivity to noise. Thus, this tuning is preferred in practice.

Remark: We may reduce K_c further below 1 and still achieve an output deviation less than $y_{max} = 1$. The reason why (9) is not tight in this case, is mainly that the expression is derived for a sinusoidal disturbance whereas we here consider a step disturbance.

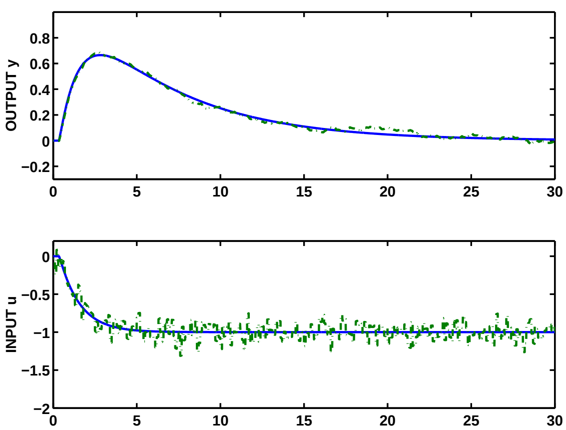

Fig. 5. Response to step load disturbance with "smooth" SIMC PI controller (15).
Dashed line: With measurement noise. Solid line: No noise

4. DISCUSSION

4.1 Averaging level control

A well-known case where a low controller gain is desired is for "averaging level control" where we use a tank in order to smoothen flow disturbances. Here the main control objective is to have smooth input usage (smooth flow variations), subject to the requirement of stabilizing the system and keeping the level within bounds when there are flow disturbances. In (9), $|d_0|$ is the magnitude of the flowrate change ($|\Delta q|$) and $|y_{max}|$ is the allowable level change ($|\Delta h_{max}|$). From (9) the minimum controller gain for averaging level control is

$$K_c \geq \frac{|\Delta q|}{|\Delta h_{max}|} \tag{16}$$

which agrees with the value normally recommended (e.g. (Marlin, 2000)). The process transfer function $g(s)$ from u (flowrate q) to y (level h) is close to integrating (with τ_1 in (10) very large) and can be written

$$g(s) = \frac{k'}{s} e^{-\theta s}$$

where $k' = k/\tau_1$ is the slope of the response. From the SIMC-rule for the controller gain in (11) we get $\tau_c + \theta = 1/(K_c k')$, which upon substitution into (12) gives the integral time

$$\tau_I = \frac{4}{K_c k'} \tag{17}$$

which agrees with the industrially recommended value in Fruehauf *et al.* (1994).

4.2 Controllability implications

An approximate maximum value of the controller gain is achieved by selecting the desired closed-loop response time τ_c in (11) equal to zero. This gives the "maximum" controller gain

$$K_{c,max} = \frac{1}{k}\,\frac{\tau_1}{\theta} = \frac{1}{k'\theta} \tag{18}$$

If the "maximum" controller gain in (18) is smaller than the "minimum" controller gain computed above, then the process is not controllable – at least not with PID control with reasonably robust tunings. In words, the speed of response required for disturbance rejection is faster than what can be achieved with the given time delay. For example, for a load disturbance the minimum controller gain $K_{c,min}$ is given by (9), and requiring $K_{c,max} \geq K_{c,min}$ for controllability gives an upper bound on the allowed delay

$$\theta \leq \frac{|y_{max}|}{|d_0|}\frac{\tau_1}{k}$$

The right hand side represents the minimum response time, and we note, as expected, that a small response time is required if we have a tight performance requirement ($|y_{max}|$ small), a large disturbance ($|d_0|$ large), or a "fast-acting" disturbance ($k' = k/\tau_1$ large).

4.3 Generalization to multivariable systems

The results in this paper can be directly generalized to decentralized control of multivariable systems by introducing the closed-loop disturbance gain (Hovd and Skogestad, 1992) (Hovd and Skogestad, 1994) (Skogestad and Postlethwaite, 1996).

5. CONCLUSION

The requirement of acceptable disturbance rejection (output deviation less than $|y_{max}|$ in response to a sinusoidal disturbance of magnitude $|d_0|$), results in a lower limit (5) on the controller gain. In words, the minimum controller gain at a given frequency is approximately equal to input change required for disturbance rejection divided by the allowed output variation. For a load disturbance and PI or PID control this requirement becomes $K_c \geq |d_0|/|y_{max}|$ (9).

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