Optimalty of SVD controllers

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Technical Report: 18 April 1996, Revised: 27 Dec. 1996

Abstract

Plant structure is utilized for the simplification of system analysis and controller synthesis. For plants where the directionality is independent of frequency, the singular value decomposition (SVD) is used to decouple the system into nominally independent subsystems of lower dimension. In H_2- and H_∞-optimal control, the controller synthesis can thereafter be performed for each of these subsystems independently, and the resulting overall SVD controller will be optimal (the same will hold for any norm which is invariant under unitary transformations). In H_∞-optimal control the resulting controller is also *super-optimal*, as a controller of dimension $n \times n$ will minimize the norm in n directions. For robust control in terms of the structured singular value, μ, the SVD controller is optimal for a practically relevant class of block diagonal structures and uncertainty and performance weights. The results are applied to the ill-conditioned distillation case study of Skogestad et al. (1988), where it is shown that an SVD controller is μ-optimal for the case of unstructured input uncertainty.

1 Introduction

In this paper we study SVD controllers which we define to have the form

$$K(s) = V\Sigma_K(s)U^H \tag{1}$$

Here $\Sigma_K(s)$ is a diagonal matrix with real rational transfer functions on the diagonal, and U and V are real unitary singular vector matrices which are derived from a singular value decomposition (SVD) of the plant $G(s)$. Here H denotes Hermitian (complex conjugate transpose) which for real matrices is equal to the transpose, i.e., $U^H = U^T$.

SVD controllers have been studied previously by Hung and MacFarlane [19] and Lau et al. [21]. In both these references the SVD structure is essentially used to counteract interactions at one given frequency, as the problems considered are such that U and V change with frequency. However, in this paper we consider a class of problems for which U and V are constant at all frequencies and can be chosen to be real. Restricting our attention

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to these cases allows us to address the optimality of the SVD controller for H_2-, H_∞- and μ-optimal control. To be more specific, we consider plants $G(s)$ of dimension $n \times n$ which can be decomposed into

$$G(s) = U\Sigma_G(s)V^H; \quad \Sigma_G(s) = \mathrm{diag}\{\sigma_{Gi}(s)\} \tag{2}$$

where the output and input rotation matrices, U and V, are constant real unitary (i.e. orthonormal) matrices, and $\Sigma_G(s)$ is a diagonal matrix with real rational transfer functions on the diagonal. The requirement for $\Sigma_G(s)$ to have rational transfer function elements arises because we use state-space based controller synthesis methods, and need the elements to be realizable. Restricting U and V to be real means that the controller $K(s)$ will always be realizable provided $\Sigma_K(s)$ is realizable.

Eq. (2) is the singular value decomposition of the plant $G(s)$ with the slight modification that the diagonal elements of $\Sigma_G(s)$, which we will refer to as *singular values*, have *phase*, and without necessarily requiring that the singular values in $\Sigma_G(s)$ are ordered according to their magnitudes. At a given frequency any transfer function can be decomposed into its singular value decomposition, but we are here assuming that the rotation matrices U and V are independent of frequency. In this case the singular value decomposition can be used to decompose the plant into n "subplants" $\sigma_{Gi}(s)$ (the diagonal elements of $\Sigma_G(s)$). To simplify the presentation, we consider in this paper only SISO subplants, but it is straightforward to generalize the results to cases where unitary transformations decompose the plant into MIMO subplants, that is, $\Sigma_G(s)$ is block-diagonal (see [17] for details).

Two contributions of this paper are to show that under certain mild conditions on the control problem weights, the optimal controller for a plant of the form in (2) is an SVD-controller, and that the controller design can be simplified for such problems. The basis for these results is that the H_2- and H_∞-norms

$$\begin{aligned} \|M(s)\|_2 &\equiv \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Trace}\left(M^H(j\omega)M(j\omega)\right)d\omega}; \\ \|M(s)\|_\infty &\equiv \sup_\omega \overline{\sigma}(M(j\omega)) \end{aligned}$$

are invariant to unitary scalings. To make use of this property we need that not only the plant, but the control problem as a whole (including the weights) can be "diagonalized" by unitary matrices. For the diagonalized problem we then find that a diagonal controller is optimal, and when putting things together we obtain an SVD controller. Furthermore, controller design is simplified since the elements of the diagonal controller can be obtained by performing controller synthesis on n *independent* subsystems involving $\Sigma_G(s)$. We show that in the H_∞ case the resulting controller is *super-optimal*, as the norm is minimized in the worst direction for each of these subsystems.

Although these results in hindsight may seem straightforward, they have not to our knowledge been presented before in the control literature, at least not in this general form. This is somewhat surprising since plants of the form in (2) are common in practical applications. The most important subclass is probably symmetric circulant plants, where we in addition have that the input and output rotation matrices are equal (i.e., $V = U$) and are also equal to the eigenvector matrix. We treated this subclass in detail in a previous paper [17], and this paper generalizes the results to a broader class of problems.

The most significant new contribution in this paper is to show that the SVD controller may be optimal also when we consider H_∞ robust performance (i.e., μ-optimal control) and have model uncertainty which allows for plants which may not be of the form in (2) (although the nominal plant is of this form). In particular, we find that with some mild conditions on the weights the result holds for any combination of "full-block" (unstructured) uncertainty, and for repeated diagonal complex uncertainty. In the paper we show that it may also apply in special cases to general diagonal uncertainty. Again, we find significant simplifications in controller synthesis (of the μ-optimal controller), though in this case the subsystems cannot always be considered independently.

2 Examples of Plants Described by SVD

In this section we provide examples of plants which can be expressed in the form given in (2). The multivariable directionality of these plants, as expressed by the two singular vector matrices U and V, does not change with frequency, and U and V are real. The following two classes of plants are of special interest in applications:

A. Plants with scalar dynamics multiplied by a constant matrix. Let

$$G(s) = k(s)A \tag{3}$$

where A is a constant real matrix. One example is the simplified distillation column model studied by Skogestad et al. (1988) studied in an example towards the end of the paper. Plant models of this form occur frequently in practice, at least in the chemical process industries, where the control engineer often chooses to work with very crude models.

B. Circulant symmetric plants. This class of plants was treated in detail in our previous paper [17], and short statement of the main properties are given here. Plants with symmetric circulant transfer matrices are common in practice, and include a large number of processes with some symmetric spatial arrangement. Examples include paper machines where edge effects are neglected [22, 36], dies for plastic films [28], and multizone crystal growth furnaces [1]. The general form of a circulant matrix C of dimension $n \times n$ is:

$$C(s) = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_n \\ c_n & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_n & c_1 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_1 \end{bmatrix} \tag{4}$$

In general, all circulant matrices can be diagonalized by the same unitary matrix, namely the Fourier matrix F, i.e. $C(s) = F^H \Lambda_C(s) F$. Thus it follows that plants described by circulant transfer function matrices (4) have the same structure as in (2), except that the Fourier matrix is complex while we require U and V in (2) to be real. However, if we consider *symmetric* circulant plants for which

$$c_k = c_{n-k+2}; \quad k = 2, 3, \cdots, \nu \tag{5}$$

where $\nu = n/2$ for even n and $\nu = (n+1)/2$ for odd n, then we can choose the eigenvector matrix to be real, i.e.

$$C(s) = R^T \Lambda_C(s) R \tag{6}$$

where R is a real matrix. This is on the form in (2) with $U = V = R^T$. The real eigenvector matrix R may be obtained as given in [17].

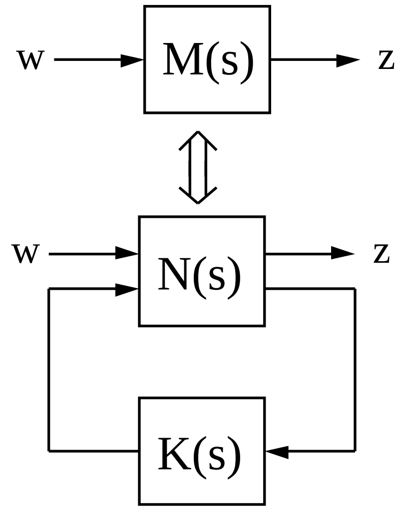

Figure 1: Expressing $M(s)$ as a linear fractional transformation of the controller $K(s)$.

Parallel plants. The matrix C in (4) is called *parallel* if

$$c_2 = c_3 = \cdots = c_n \tag{7}$$

Parallel transfer function matrices occur frequently in the process industries, and arise whenever there are identical units in parallel which interact with each other. Examples are found in distribution networks, when there are parallel units (e.g. reactors, compressors, pumps, heat exchangers) in a chemical plant [31, 32, 17], for electric power systems [26, 27], for adhesive coating processes [5], or for communication between ships [16].

Remark: The set of plants given by (2) is more general than the two classes A and B given above, since the first class only includes plants for which the diagonal elements of $\Sigma_G(s)$ have the same dynamic behavior, and the second class only includes plants for which the unitary rotation matrices U and V are equal.

3 SVD Control Problem

In this section we consider plants which can be decomposed into $G(s) = U\Sigma_G(s)V^H$ (as shown in Eq. 2) and define more exactly the class of control problems covered by the results of this paper.

A general control problem is depicted in Fig. 1 where we have $z = M(s)w$. Here w represents some external input signals (e.g. disturbances, noise, references), and z represents the external output signals (e.g. control error, input signals) which we want to keep small. The overall transfer function $M(s)$ depends on the controller $K(s)$ and the controller synthesis problem is then: $\min_K \|M\|$. Typical choices of norm include the H_2- or the H_∞-norm (or possibly the structured singular value for the case with model uncertainty). In this paper we consider control problems where $M(s)$ may be written as a linear fractional transformation (LFT) of the controller $K(s)$ as shown in Fig. 1. We now define the general class of *SVD problems* which are covered by the results of this paper.

Definition 1 (SVD problem) *Consider a $n \times n$ plant $G(s) = U\Sigma_G(s)V^H$, where U and V are real orthogonal matrices and $\Sigma_G(s)$ is a diagonal transfer function matrix. Consider a* *control problem where the objective is to design a feedback controller $K(s)$ which minimizes a unitary invariant norm of*

$$M(s) = W_O(s)M_0(s)W_I(s),$$

where

$$M(s) = F_l(N(s), K(s)) = N_{11}(s) + N_{12}(s)K(s)\left[I - N_{22}(s)K(s)\right]^{-1} N_{21}(s). \qquad (8)$$

The interconnection matrix *$N(s)$ is a function of the plant model and the weights, but is independent of the controller K.*

The weighting matrices $W_O(s)$ and $W_I(s)$ are defined to be block-diagonal matrices with each block having dimensions compatible with the dimensions of the subblocks containing $G(s)$ and $K(s)$ in $M_0(s)$:

$$W_O(s) = \text{diag}\{W_{Oi}(s)\}; \quad W_{Oi}(s) = U_{Oi}\Sigma_{W_{Oi}}(s)V_{Oi}^H$$

$$W_I(s) = \text{diag}\{W_{Ii}(s)\}; \quad W_{Ii}(s) = U_{Ii}\Sigma_{W_{Ii}}(s)V_{Ii}^H,$$

and V_{Oi} and U_{Ii} satisfying

- *$V_{Oi} = U$ when $W_{Oi}(s)$ premultiplies $G(s)$ in subblocks of $M_0(s)$;*
- *$V_{Oi} = V$ when $W_{Oi}(s)$ premultiplies $K(s)$ in subblocks of $M_0(s)$;*
- *$U_{Ii} = V$ when $W_{Ii}(s)$ postmultiplies $G(s)$ in subblocks of $M_0(s)$;*
- *$U_{Ii} = U$ when $W_{Ii}(s)$ postmultiplies $K(s)$ in subblocks of $M_0(s)$.*

The terms "premultiply" and "postmultiply" are used in a general sense, for instance, in the formula $W_O(I + GK)^{-1}W_I$, the weight W_O premultiplies G and W_I postmultiplies K. There are no requirements on the other matrices in the weights, other than U_{Oi} and V_{Ii} being unitary and $\Sigma_{W_{Ii}}(s)$ and $\Sigma_{W_{Oi}}(s)$ being diagonal.

Remark 1. The definition of an SVD control problem may seem restrictive and complicated, but the conditions on the weights are satisfied for most problems with a plant on the form $G(s) = U\Sigma_G(s)V^H$.

Remark 2. Essentially, the weights must be consistent with the plant $G(s)$, such that, after substituting $G(s) = U\Sigma_G(s)V^H$ and $K(s) = V\Sigma_K(s)U^H$ into $M_0(s)$, the unitary matrices U and V are canceled by the weights when forming $M(s)$, in the sense that we can write $M(s) = U_O\tilde{M}(s)V_I^H$ where all the blocks of $\tilde{M}(s)$ are diagonal. A simple example is given below. A similar transformation may be used to obtain a block diagonal $\tilde{N}(s)$, but since $N(s)$ is independent of the controller we do not need to assume an SVD-controller to achieve this. This is important when proving that the SVD-controller is actually optimal (see next section).

Remark 3. Scalar times identity weights, $W_i(s) = w_i(s)I$ always satisfy the conditions of an SVD problem since $w_i(s)I = Uw_i(s)U^H = Vw_i(s)V^H$.

Mixed Sensitivity Example.

Consider the well-known *mixed sensitivity* problem for which

$$M(s) = \begin{bmatrix} W_1(s)T(s) \\ W_2(s)S(s) \end{bmatrix} \qquad (9)$$

where $S(s) = (I + G(s)K(s))^{-1}$ is the sensitivity and $T(s) = G(s)K(s)(I + G(s)K(s))^{-1}$ is the complementary sensitivity. In terms of the notation in Definition 1,

$$M_0 = \begin{bmatrix} T(s) \\ S(s) \end{bmatrix}; \quad W_O(s) = \text{diag}\{W_1(s), W_2(s)\}; \quad W_I(s) = I = UU^H$$

Here $W_1(s)$ and $W_2(s)$ are weighting matrices which are selected by the designer to achieve the desired control performance. As assumed throughout this paper, the plant is $G(s) = U\Sigma_G(s)V^H$. To get an SVD problem, we must assume that $W_1(s)$ and $W_2(s)$ are of the form

$$W_1(s) = U_1\Sigma_{W_1}(s)U^H; \quad W_2(s) = U_2\Sigma_{W_2}(s)U^H \tag{10}$$

where U_1 and U_2 are unitary matrices but may otherwise be chosen freely. (We note again that we may always choose $W_1(s)$ and $W_2(s)$ as a scalar times identity weight). Consider an SVD-controller $K(s) = V\Sigma_K(s)U^H$ and introduce $U_O = \text{diag}\{U_1, U_2\}$ and $V_I = U$. We then find that writing $M(s) = U_O\tilde{M}(s)V_I$ yields a block-diagonal $\tilde{M}$

$$\tilde{M}(s) = \begin{bmatrix} \Sigma_{W_1}(s)\Sigma_T(s) \\ \Sigma_{W_2}(s)\Sigma_S(s) \end{bmatrix} \tag{11}$$

where $\Sigma_T = \Sigma_G\Sigma_K(I + \Sigma_G\Sigma_K)^{-1}$ and $\Sigma_S = (I + \Sigma_G\Sigma_K)^{-1}$.

4 H_2- and H_∞-Optimal Control

In this section we consider H_2- and H_∞-optimal control. The results also apply to any other norm which is invariant under unitary transformations.

Theorem 1 (H_2- and H_∞-Optimality) *Consider an SVD problem (Definition 1). Then*

1. *There exists an SVD controller that is $H_2(H_\infty)$-optimal.*
2. *The optimal controller can be computed by designing n independent SISO $H_2(H_\infty)$-optimal controllers, one for each of the SISO subplants of the plant.*
3. *For H_∞-optimal control, this controller is super-optimal, that is, the H_∞-objective is optimized in n directions.*

Proof:

1. Express the matrix whose H_2- or H_∞-norm we want to minimize as a Linear Fractional Transformation (LFT) of the controller $K(s)$ to obtain the interconnection matrix $N(s)$ (see Eq. (8) and Fig. 1).
2. For an SVD problem, $N(s)$ will be such that there exists block-diagonal unitary matrices

$$U_W = \text{diag}\{\text{diag}\{U_{Oi}\}, U\}; \qquad V_W = \text{diag}\{\text{diag}\{V_{Ii}\}, V\} \tag{12}$$

such that

$$\tilde{N}(s) = U_W^H N(s) V_W \tag{13}$$

is a matrix consisting of diagonal subblocks (as illustrated in the upper part of Fig. 2). The proper rearrangement of the inputs and outputs of $\tilde{N}(s)$ (i.e. permutations) yields a permuted $\tilde{N}$ which is block diagonal matrix as illustrated in the bottom of Fig. 2. Note that the matrices needed for these permutations are unitary.

$$
\begin{bmatrix}
a_1 & & & & b_1 & & & \\
& a_2 & & & & b_2 & & \\
& & \ddots & & & & \ddots & \\
& & & a_n & & & & b_n \\
c_1 & & & & d_1 & & & \\
& c_2 & & & & d_2 & & \\
& & \ddots & & & & \ddots & \\
& & & c_n & & & & d_n \\
e_1 & & & & f_1 & & & \\
& e_2 & & & & f_2 & & \\
& & \ddots & & & & \ddots & \\
& & & e_n & & & & f_n
\end{bmatrix}
$$

$$\Updownarrow \text{ Permutations}$$

$$
\begin{bmatrix}
a_1 & b_1 & & & & & \\
c_1 & d_1 & & & & & \\
e_1 & f_1 & & & & & \\
& & a_2 & b_2 & & & \\
& & c_2 & d_2 & & & \\
& & e_2 & f_2 & & & \\
& & & & \ddots & & \\
& & & & & a_n & b_n \\
& & & & & c_n & d_n \\
& & & & & e_n & f_n
\end{bmatrix}
$$

Figure 2: Top: $\tilde{N}$ for a case with 3×2 main blocks. Bottom: $\tilde{N}$ permuted to have the n independent synthesis subproblems along the main diagonal. From the bottom matrix it is apparent that the controller design problem consists of n independent subproblems.

3. The control problem in terms of permuted $\tilde{N}(s)$ is the same as the original one. This follows since the H_2- and H_∞-norms are invariant to pre- and postmultiplication with unitary matrices. Also note that N is independent of the controller $K(s)$ and that the unitary matrices V_W and U_W used to transform $N(s)$ into $\tilde{N}(s)$ are independent $K(s)$. No assumption about the structure of the controller $K(s)$ is therefore necessary at this point.

4. The diagonal structure of the permuted $\tilde{N}(s)$ means that the controller synthesis problem is decomposed into n independent subproblems: Any off-diagonal block of the controller $\tilde{K}(s)$ will only affect the input to a subplant *for whose output it has no measurement.* Therefore any off-diagonal block of the optimal $\tilde{K}(s)$ can be taken to be zero. This is equivalent to saying that a decentralized controller is optimal for a decentralized plant with decentralized weight (cost) functions. Although this statement seems intuitively obvious, we have included a detailed proof in the appendix.

5. To recover the corresponding controller $K(s)$ for the original problem, we note from the lower right parts of U_W and V_W that U and V are the unitary matrices used to diagonalize the lower right part of N. (This is because for one degree of freedom feedback control the lower right part of N is equal to the plant G.) Thus, if we refer to the optimal diagonal controller for $\tilde{N}$ as $\Sigma_K(s)$, then the optimal controller for the original problem is $K(s) = V\Sigma_K(s)U^H$, which is an SVD controller.

6. For $\tilde{N}(s)$ the control problem consists of n independent synthesis problems of lower dimension, and the controller $\tilde{K}$ is obtained by minimizing the appropriate norm for each separate subproblem. In particular, for the H_∞-case the H_∞-norm is minimized in n directions, which is referred to by many researchers [20, 15, 35] as *super-optimality.* □

Remark 1. In general, the solution to the H_∞ controller synthesis problem is non-unique [12], since many controllers many achieve the optimum H_∞ norm in the worst direction, while doing equally well or better in the other directions. Super-optimality [20, 15, 35] is achieved when the H_∞-norm is optimized not only in the *worst* direction, but in n directions.

Remark 2. The interconnection matrix $\tilde{N}(s)$ in Fig. 2 has the same number of states as $N(s)$ in Fig. 2, and the number of states of $\tilde{N}(s)$ equal the sum of the number of states of each diagonal block of $\tilde{N}(s)$. Thus the number of states of the controller resulting from collecting the SISO controllers in $\tilde{K}(s)$ will equal the number of states in $\tilde{N}(s)$, which is equal to the number of states of a controller based on regular H_∞ synthesis. That is, for this class of problems super-optimality does not require a controller with a higher number of states.

Remark 3. In general we solve n independent synthesis subproblems of low dimension. In some cases the problem is even further reduced in size since some of these subproblems are identical. For example, for the case of symmetric circulant systems we need only solve $(n+1)/2$ SISO problems for odd n and $n/2+1$ problem for even n. For the case of parallel processes we need only solve two independent subproblems (since $n-1$ subproblems are identical). For details see [17].

Remark 4. The theorem may be generalized to cases where the subplants $\sigma_{Gi}(s)$ are matrices. For example, see [17] who considered the special case of symmetric circulant plants.

Mixed Sensitivity Example (continued).

For this example the interconnection matrix becomes

$$N(s) = \begin{bmatrix} 0 & W_1(s)G(s) \\ W_2(s) & -W_2(s)G(s) \\ I & -G(s) \end{bmatrix} \tag{14}$$

Since the H_∞- and H_2-norms are unitary invariant, we can from Eq.(12) use $U_W = \text{diag}\{U_o, U\} = \text{diag}\{U_1, U_2, U\}$ and $V_W = \text{diag}\{V_I, V\} = \text{diag}\{U, V\}$ to scale the output and input of $N(s)$, respectively, to give an *equivalent* optimal control problem with

$$\tilde{N}(s) = U_W^H N(s) V_W = \begin{bmatrix} 0 & \Sigma_{W_1}(s)\Sigma_G(s) \\ \Sigma_{W_2}(s) & -\Sigma_{W_2}(s)\Sigma_G(s) \\ I & -\Sigma_G(s) \end{bmatrix}; \qquad \tilde{K}(s) = UK(s)V^H. \tag{15}$$

The transformed interconnection matrix $\tilde{N}(s)$ in (15) consists of diagonal subblocks (similar to the upper matrix in Fig. 2, with $a_j = 0$, $e_j = 1$ and $f_j = -\sigma_{Gj}$), and we may permute the order of the inputs and outputs such that we get a block-diagonal matrix (similar to the lower matrix in Fig. 2) for which the optimal controller $\tilde{K}(s)$ is diagonal. To find the optimal $\tilde{K} = \Sigma_K(s)$ we need only solve n subproblems of smaller dimension. The H_∞-norm of the overall system $M(s)$ is equal to the maximum H_∞-norm of the n subproblems (we now use the fact that the optimal controller is an SVD controller)

$$\|M(s)\|_\infty = \max_{j\in\{1,\dots,n\}} \left\| \begin{matrix} \sigma_{W_1j}(s)\sigma_{Gj}(s)\sigma_{Kj}(s)/(1+\sigma_{Gj}(s)\sigma_{Kj}(s)) \\ \sigma_{W_2j}(s)/(1+\sigma_{Gj}(s)\sigma_{Kj}(s)) \end{matrix} \right\|_\infty \tag{16}$$

whereas the H_2 norm of $M(s)$ is equal to the sum of the H_2 norm of the subproblems

$$\|M(s)\|_2 = \sum_{j=1}^{n} \left\| \begin{matrix} \sigma_{W_1j}(s)\sigma_{Gj}(s)\sigma_{Kj}(s)/(1+\sigma_{Gj}(s)\sigma_{Kj}(s)) \\ \sigma_{W_2j}(s)/(1+\sigma_{Gj}(s)\sigma_{Kj}(s)) \end{matrix} \right\|_2 \tag{17}$$

5 μ-Optimal Control

In this section we shall generalize the H_∞-problem studied above to the design of robust optimal controllers. This control problem results when we introduce model uncertainty and want to minimize the H_∞-norm for robust performance, or alternatively want to optimize robust stability.

5.1 The Structured Singular Value

The structured singular value, μ, is used as a means of taking uncertainty in a feedback system explicitly into account. Readers not familiar with the structured singular value are referred to [10]; only a very brief introduction will be given here. The uncertainties in the system are modeled with H_∞ norm-bounded perturbation blocks with weights to normalize the maximum singular value of each perturbation block to unity. The block diagram for the feedback system is then rearranged to give an interconnection matrix $M(s)$ and a block-diagonal matrix Δ with the perturbation blocks along the diagonal (see Fig. 3). If Δ is a

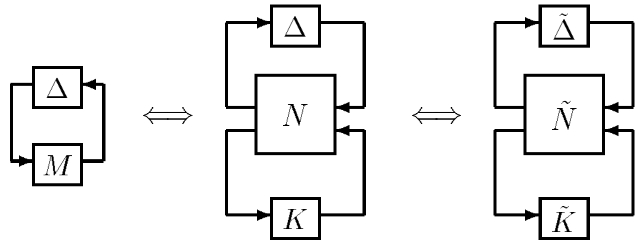

Figure 3: Equivalent representations of system M with perturbation Δ.

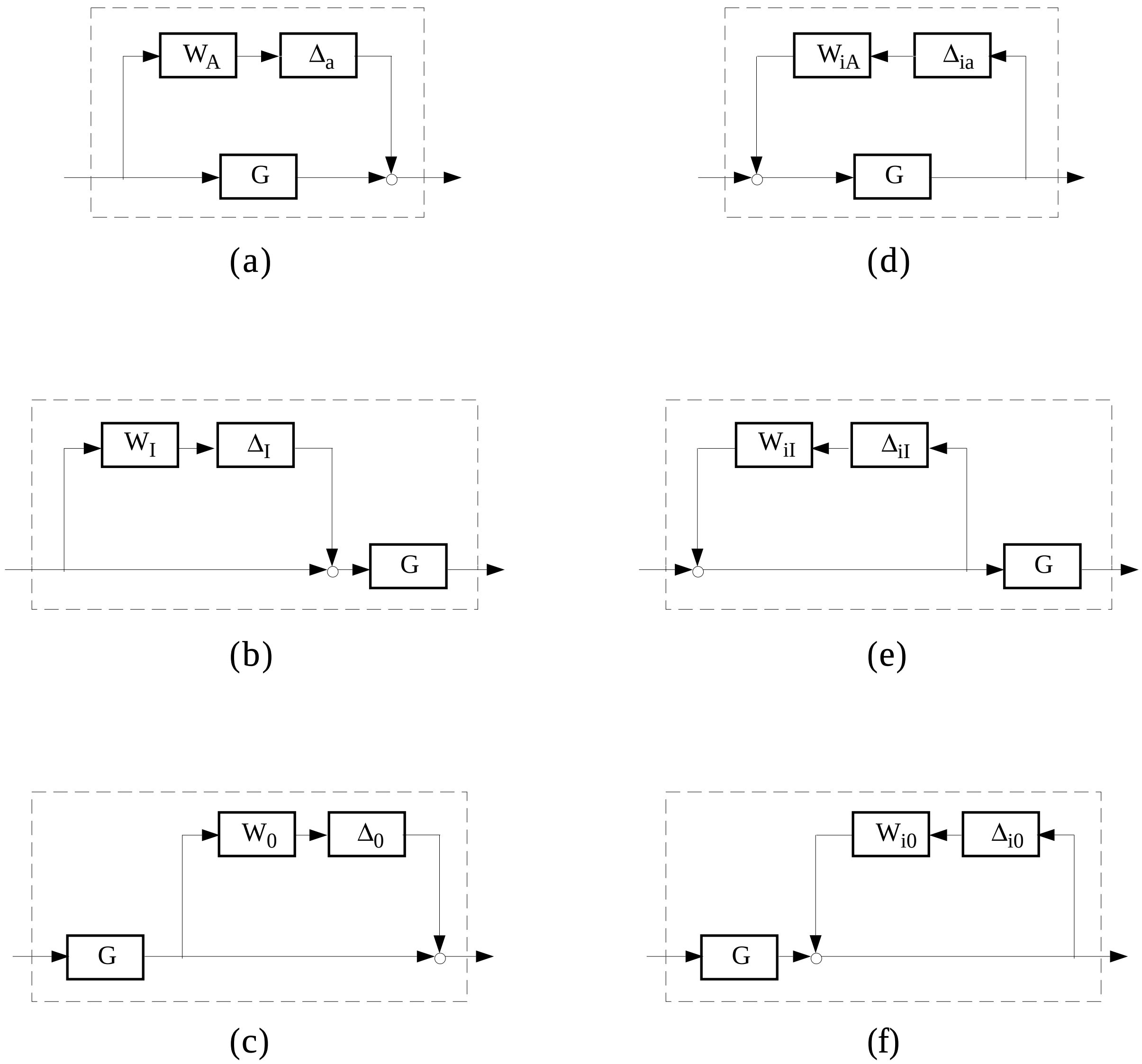

Figure 4: Various sources of uncertainties: (a) additive, (b) multiplicative input, (c) multiplicative output, (d) inverse additive, (e) inverse multiplicative input, (f) inverse multiplicative output.

full matrix (i.e., Δ has no structure), the controller synthesis problem is a H_∞ problem, and is covered by the results of the previous section. Otherwise, the structured singular value is needed to account for the uncertainty in a non conservative manner.

It is important to note that Δ often has two levels of structure. First, Δ is often composed of *subblocks* Δ_i *of the same size as* G

$$\Delta = \mathrm{diag}\{\Delta_i\} \tag{18}$$

These subblocks may represent different *sources* of uncertainty in the system as illustrated in Fig. 4. For example, actuator uncertainty is located at the input of the plant and is commonly modeled as multiplicative input uncertainty, i.e., $\Delta_i = \Delta_I$. Second, each subblock Δ_i may have structure to reduce conservatism. For example, actuators may not influence each other, so uncertainty associated with these actuators would be described by a *diagonal* Δ_i. The most common (and useful) structures for the subblocks Δ_i are:

- Full block uncertainty: Δ_i is a full matrix of the same dimension as the plant $G(s)$.
- Independent diagonal uncertainty: $\Delta_i = diag\{\delta_{ij}\},\ j = 1, \ldots, n$ is a diagonal matrix with the same dimension n as the plant $G(s)$.
- Repeated diagonal uncertainty: $\Delta_i = \delta_i I$, i.e., a scalar uncertainty δ_i multiplied with an identity matrix of the same dimension as the plant $G(s)$.

The structured singular value with respect to the uncertainty structure Δ is defined as

$$\mu(M) \equiv \begin{cases} 0 \text{ if there does not exist } \Delta \text{ such that } \det(I + M\Delta) = 0 \\ \left[\min\limits_{\Delta}\{\bar{\sigma}(\Delta)\ |\det(I + M\Delta) = 0\}\right]^{-1} \quad \text{otherwise} \end{cases} \tag{19}$$

Thus, μ^{-1} is the smallest value for $\bar{\sigma}(\Delta)$ for which there exists a Δ that makes the feedback system consisting of M and Δ (in Fig. 3) unstable. If μ^{-1} is larger than the magnitude of the uncertainties in the system for all frequencies then the uncertainties cannot destabilize the system, and the system is said to have robust stability. Usually, weights are used to normalize the perturbations[1] and the system is robustly stable if $\mu^{-1} > 1$, or equivalently, $\mu < 1$. Similarly, a system is said to have robust performance if it fulfills the specified performance criteria for any allowable uncertainty. Testing robust performance involves a μ test of increased dimension compared to the robust stability test.

Currently no simple computational method exists for exactly calculating μ in general, and recent work suggests that an efficient exact method may not be possible [6]. However, when the perturbations are complex then reasonably tight upper and lower bounds can be derived from the following potentially loose bounds:

$$\rho(M) \leq \mu(M) \leq \bar{\sigma}(M) \tag{20}$$

To get tighter bounds scalings are included in the bounds and optimized over. For example, the tight upper bound is

$$\mu(M) \leq \inf_D \bar{\sigma}(DMD^{-1}) \tag{21}$$

[1] In general, there may be weights on both the inputs and the outputs of the perturbation blocks. Usually, one of these weights can be chosen to be an identity matrix

where D is an invertible matrix with a structure such that $D^{-1}\Delta D = \Delta$. For example, $D = dI$ if Δ is a full matrix, and D is a full matrix if Δ is repeated diagonal ($\Delta = \delta I$). For complex uncertainties the upper bound (21) is equal to μ for three or fewer full blocks [10], and usually within 1-2% when all there are no repeated blocks [2]. Even in the case where there are repeated blocks, which can give a larger difference between μ and its upper bound, the fact that μ is NP-hard motivates designing the controller based on the readily computable upper bound. A controller which minimizes the upper bound for μ in Eq. (21) will be said to be DMD^{-1}*-optimal*.

The upper bound for μ is of interest in its own right for several reasons. One reason is that the goal of the more popular procedure for designing robust controllers, called DK-iteration, is to minimize the upper bound. Another reason is that, when all the uncertainties are full and complex, the upper bound is a necessary and sufficient condition for robustness to *arbitrarily-slow time-varying linear uncertainty* (see [29] for details). It can be argued that this uncertainty description may be more useful for practical control problems.

The standard DK-iteration procedure [11] attempts to find the DMD^{-1}*-optimal* controller. DK-iteration involves alternating between the following two steps until the upper bound is no longer minimized.

D Step: Find $D(s)$ to minimize frequency-by-frequency the upper bound on μ in (21).

K Step: Scale the controller design problem with $D(s)$, and design an H_∞-optimal controller for the scaled design problem DMD^{-1}.

Although convergence to the global optimum is not guaranteed, DK-iteration appears to work well [11].

5.2 μ-Optimality of SVD controllers

In Section 4 we showed that an SVD controller was optimal for SVD problems involving the H_2- or H_∞-norm. The proof involved showing that for SVD problems the interconnection matrix could be pre- and postmultiplied by block-diagonal unitary matrices to arrive at an equivalent interconnection matrix $\tilde{N}$ which consists of diagonal subblocks (as in Fig. 2), and that the control problem in terms of $\tilde{N}$ is equivalent to the original problem since the H_2- and H_∞-norms are unitary invariant.

This simple approach does not directly apply when we want to minimize μ or its upper bound ($\overline{\sigma}(DMD^{-1})$). For μ, we must make sure that pre- and postmultiplying of the interconnection matrix by unitary matrices does not alter the structure of the uncertainty Δ. Similarly, for the DMD^{-1}-problem we must make sure that pre- and postmultiplying with unitary matrices does not alter the structure of the scaling matrices D. Therefore, additional conditions on the uncertainty weights have to be imposed to ensure that the structures of Δ and D remain unchanged.

We shall first consider the important case where all the uncertainty blocks are full matrices (unstructured uncertainty), and then generalize the results to a larger class of uncertainty.

5.2.1 Full block uncertainties

Theorem 2 (DMD^{-1}-Optimality for full block uncertainty) *Consider a SVD control problem with $M(s) = W_O(s)M_0(s)W_I(s)$ as in Definition 1, and multiple sources of uncertainty $\Delta = \mathrm{diag}\{\Delta_i\}$ (as illustrated in Fig. 3) where each uncertainty Δ_i is a full block of the* *same size as the plant G. Consider the problem of finding a controller $K(s)$ that minimizes* $\sup_{\omega} \min_{D} \| DMD^{-1} \|_\infty$ *where $D^{-1}\Delta D = \Delta$. Then*

1. *There exists an SVD controller which is optimal.*
2. *If DK-iteration is used to obtain the optimal controller, the* **K** *step (with fixed D) consists of n independent SISO H_∞-optimal control problems, one for each of the SISO subplants σ_{Gi} of $G(s)$.*

Proof: Let N denote the interconnection matrix corresponding to M. N has a block structure corresponding to the uncertainties Δ_i. With fixed D-scales we may absorb D and D^{-1} into N to get

$$N_D = \hat{D} N \hat{D}^{-1}; \quad \hat{D} = \mathrm{diag}\{D, I\}$$

We are then left with an H_∞-problem in terms of N_D. Since all uncertainty blocks Δ_i are full, the D-scales are of the form $D = \mathrm{diag}\{D_i\}$, $D_i = d_i I_i$. Then the only difference between N and N_D will be that the offdiagonal blocks are multiplied by scalars. Thus, the "structure" of each block in N_D will be the same as in N, and we can use the same transformation $\tilde{N}_D(s) = U_W^H N(s) V_W$ as in the proof of Theorem 1, to obtain a $\tilde{N}_D$ with diagonal blocks (as in the upper part of Figure 2). As in Theorem 1 subsequent permutations yield a block-diagonal $\tilde{N}_D$ (as in the lower part of Figure 2). It then follows that for a fixed D an SVD-controller is optimal and can be obtained by solving n independent SISO H_∞-problems. Since an SVD-controller is optimal for any fixed D this structure must also be optimal for the optimal D.
□

Note that for full-block uncertainty no additional requirements on the weights are required, besides those given already for SVD-problems. For example, we may use scalar times identity weights, $W_i(s) = w_i(s)I$ to represent the magnitude of each uncertainty. This is the weight most commonly used in applications.

Also note that an SVD controller is optimal in this case, in spite of the fact that full block (unstructured) uncertainty will allow for plants which cannot be written on the form $G(s) = U\Sigma_G(s)v^H$ (though the nominal plant is on this form).

5.2.2 Generalization: Robust SVD Problem

Here we want to generalize the result in Theorem 2 to a larger class of uncertainty. To this effect we define a subset of SVD problems which have additional conditions on the weights.

Definition 2 Robust SVD Problems. *Consider an SVD problem with $M(s) = W_O(s)M_0(s)W_I(s)$ as in Definition 1, and multiple sources of uncertainty $\Delta = diag\{\Delta_i\}$, as illustrated in Fig. 3. In addition to the requirements of Definition 1, the weights $W_{Oi} = U_{Oi}\Sigma_{W_{Oi}}(s)V_{Oi}^H(s)$ and $W_{Ii} = U_{Ii}\Sigma_{W_{Ii}}(s)V_{Ii}^H(s)$ related to each Δ_i should fulfill the following:*

1. *$U_{Oi} = V_{Ii}$ for all repeated diagonal uncertainty, $\Delta_i = \delta_i I$*
2. *$U_{Oi} = V_{Ii} = I$ for all independent diagonal uncertainty, $\Delta_i = \mathrm{diag}\{\delta_{ik}\}, k = 1, \ldots, n$*

For a full Δ_i no additional assumptions on the weights are necessary.

Now we show that for this class of problems the interconnection matrix N can be pre- and postmultiplied by block-diagonal unitary matrices to arrive at an equivalent interconnection matrix $\tilde{N}$ which consists of diagonal subblocks (as in Fig. 2).

Lemma 1 *Let $\tilde{N}$ be defined as in Eqs. (12) and (13). For μ-optimality and DMD^{-1}-optimality of Robust SVD problems (Definition 2), the "diagonalized" control problem is* equivalent *to the original problem, in the sense that*

$$\min_K \mu\left(F_l(N,K)\right) = \min_{\tilde{K}} \mu\left(F_l(\tilde{N},\tilde{K})\right) \tag{22}$$

$$\min_K \inf_D \left(DF_l(N,K)D^{-1}\right) = \min_{\tilde{K}} \inf_D \left(DF_l(\tilde{N},\tilde{K})D^{-1}\right) \tag{23}$$

where both μ problems are with respect to the uncertainty in the original control problem, and the structure of the D matrices in both DMD^{-1}-problems is compatible with this uncertainty.

Proof: In the block diagram for the system, replace G with $U\Sigma_G(s)V^H$, and substitute in the weights $W_{Ii}(s)$ and $W_{Oi}(s)$. Rearranging the block diagram (see Fig. 3) gives $\tilde{N}$ with diagonal subblocks (similar to the top matrix in Fig. 2) with the subblocks of $\tilde{\Delta}$ given by $\tilde{\Delta}_i = V_{Ii}^H \Delta_i U_{Oi}$. Note that under the assumptions on U_{Oi} and V_{Ii} in Definition 2

1. $\tilde{\Delta}_i$ is full if and only if Δ_i is full;
2. $\tilde{\Delta}_i$ is repeated diagonal if and only if Δ_i repeated diagonal;
3. $\tilde{\Delta}_i$ is independent diagonal if and only if Δ_i independent diagonal.

Thus in Fig. 3 the middle block diagram is equivalent to the rightmost block diagram.

A similar argument holds with regard to the upper bound of μ. Under the assumptions on U_{Ii} and V_{Oi}, for each diagonal or full block Δ_i the corresponding D_i and its inverse commute with U_{Ii} and V_{Oi}. For repeated diagonal blocks the U_{Ii} and V_{Oi} can be absorbed into the D_i. □

Remark 1. Requirement 1 in Definition 2 for repeated diagonal blocks holds regardless of the uncertainty's location when the plant is described by a normal transfer function matrix (e.g., symmetric circulant plants) and the weights are repeated diagonal.

Remark 2. Requirement 1 also always holds for multiplicative or inverse multiplicative (see Fig. 4) repeated diagonal uncertainty with repeated diagonal weights. Intuitively, it is not surprising that this uncertainty does not prevent the system from being "diagonalized", as this uncertainty can not change the structure of G. For example, for the multiplicative case

$$G(s)(I+w(s)\delta I) = (I+w(s)\delta I)G(s) = (1+w(s)\delta)U\Sigma_G(s)V^H = U[(1+w(s)\delta)\Sigma_G(s)]V^H \tag{24}$$

Thus, if the nominal plant G is within one of the two classes of plants described in Section 2, then the plant with the repeated diagonal uncertainties (and weights) will be within the same class and have the same singular vector matrices U and V.

Remark 3. Requirement 2 in Definition 2 on the weights for independent diagonal uncertainty is very restrictive. For example, it allows for scalar times identity weights only for cases when U or V are equal to the identity matrix (that is, the inputs or outputs to the plant are naturally aligned in the direction of the singular values). One example of a plant with $V = I$ is the DV configuration for composition control of distillation columns studied by Skogestad et al. [34]. This means that in most cases with diagonal uncertainty we cannot assume that an SVD-controller is optimal.

Theorem 3 (μ-Optimality) *Consider a Robust SVD problem where the objective is to minimize* $\sup_\omega \mu(M)$. *Assume that all uncertainty blocks* Δ_i *are diagonal (repeated diagonal uncertainty or independent diagonal uncertainty) except possibly one full block. Then*

1. *There exists an SVD controller which is optimal.*
2. *The* μ*-optimal control problem decouples into* n *independent SISO* μ*-optimal control problems, one for each of the SISO subplants of the plant.*
3. *For the case where one of the uncertainties is a full block, the full block can be replaced by a diagonal (repeated or independent) block without affecting the value of the* μ *objective.*

Proof: If all uncertainty blocks Δ_i are diagonal (including repeated diagonal uncertainty), then the system consists of independent subsystems. If one uncertainty block is full, then the diagonal uncertainty blocks can be absorbed into the interconnection matrix to get a "reduced" $\tilde{N}$ which still consist of diagonal subblocks after absorbing the diagonal uncertainty blocks. Whatever the values of the diagonal blocks, we know from Thm. 1 that an SVD controller is optimal for this "reduced" control problem. Thus an SVD controller is optimal for the original μ problem.

When all of the uncertainties are diagonal, the μ-optimal control problem decouples into n independent SISO μ-optimal control problems. When one uncertainty block is full, then consider the "reduced" $\tilde{N}$ described above. As the M-matrix for the "reduced" control problem is diagonal, its spectral radius is equal to its maximum singular value. Hence the full block uncertainty for the "reduced" control problem can be replaced by a repeated scalar or independent scalar diagonal uncertainty block without changing the value of the "reduced" μ objective function. Since the worst-case full block uncertainty can be taken to be diagonal for all values of the other diagonal uncertainty blocks, this must also be true for the worst-case diagonal uncertainty blocks. Hence the full block can be replaced by a repeated or independent scalar diagonal block for the transformed ($\tilde{N}$) control problem, without affecting the value of the worst-case μ objective. Hence the original μ-optimal control problem decouples into n independent SISO μ-optimal control problems, one for each of the SISO subplants of the plant.

□

Remark to Theorem 3. In Theorem 3, the weights corresponding to the full block uncertainty do *not* need to satisfy the restrictive assumptions for a repeated or independent scalar uncertainty block in a robust SVD problem (in Definition 2). This is because the full block uncertainty is not replaced by a diagonal uncertainty until the "transformed" N-tilde matrix has been constructed. The proof shows that replacing the one full block with a diagonal block does not change the value of the worst-case μ objective for the "transformed" μ problem. Lemma 1 which is applied to the *original* μ problem with one full block and multiple diagonal uncertainty blocks implies that this worst-case μ objective is equal to the objective of the original μ problem. In other words, the steps to handling the full block are:

original μ problem with one full block and multiple diagonal blocks (where the robust SVD problem has been defined for these uncertainty blocks)

$\Updownarrow$

"transformed" μ problem with multiple diagonal blocks and one full block

$\Updownarrow$

"transformed" μ problem with multiple diagonal blocks and the one full block replaced by a diagonal block

$\Updownarrow$

independent μ-synthesis problems

The weights for the original μ problem need only satisfy the assumptions required by a robust SVD problem with the original uncertainty description. It is these assumptions on the weights that resulted in the "transformed" μ problem. Because the full block is replaced by a diagonal block only *after* the "transformed" μ problem has been formed, stricter assumptions on the weights need not be assumed for the full block uncertainty.

Theorems 2 and 3 complement each other in that Theorem 3 handles one form of uncertainty (diagonal) and Theorem 2 handles another (full). By assuming μ is equal to its upper bound we can handle both types of uncertainty.

Theorem 4 (μ- and DMD^{-1}-Optimality) *Consider a Robust SVD control problem (Definition 2), and assume that μ is equal to its upper bound (21). Then*

1. *There exists an SVD controller which is μ-optimal.*
2. *For the DK-iteration procedure the* **K** *step consists of n independent SISO H_∞-optimal control problems, one for each of the SISO subplants of the plant.*
3. *For repeated diagonal uncertainty: D_i can be taken to be* diagonal *rather than full in the* **D** *step.*

Proof:

1. All diagonal blocks (repeated or independent) can be absorbed into the interconnection matrix $\tilde{N}$ without changing its structure. By Thm. 2 an SVD controller is optimal for this "reduced" control problem for all values of the diagonal blocks. Thus an SVD controller is optimal for the original μ problem.
2. For independent diagonal and full block Δ_i, D_i is diagonal and cannot induce interaction between individual subproblems. This also holds for D_i corresponding to repeated diagonal $\Delta_i = \delta_i I$. To see this, again consider the "reduced" control problem. If the D_i corresponding to the repeated diagonal blocks introduced interaction between subproblems, they would effectively allow for a larger class of uncertainty than the original uncertainty description.
3. Scalings D_i which do not cause interactions between subproblems are parametrized by unitary times diagonal matrices. The unitary matrices do not affect the value of the H_∞-norm, so can be ignored. $\square$

Remark. The assumption that μ is equal to its upper bound is not restrictive. This equality always holds when all uncertainty subblocks Δ_i are full and three or less, or when one block is full and one is repeated diagonal, and has been found to approximately hold (within 1-2%) for all problems of practical interest [2].

5.3 DK-Iteration: Reduction of Computational Effort

The above results can be used to reduce the computational effort involved in the K step of the DK-iteration procedure in two ways. First, instead of solving one large H_∞-synthesis problem, one may solve n smaller H_∞-synthesis subproblems. Second, some of these n subproblems may be repeated (identical), for example, this occurs for the important case when both the plant and weights are symmetric circulant (or parallel). In general, the computational effort is *not* reduced in the D step where the upper bound to μ is computed, since for the case of full block uncertainty we have $D = dI$ so d should be the same for all subproblems. This restriction is difficult to incorporate unless a simultaneous approach is used. However, all *repeated* subproblems need only be considered once in finding the D_i (see item 3 in Theorem 4). Thus repeated subproblems can be deleted before starting the DK-iteration design procedure, and for a large number of subsystems the size of the DK-iteration and μ-analysis problems can be reduced dramatically.

When all uncertainty blocks are diagonal except possibly one full block, and the weights for the diagonal blocks satisfy Definition 2, the subproblems can be considered independently for the D step, since the D_i corresponding to the full block can be normalized to be the identity matrix.

Below we summarize the general DK-iteration procedure for designing SVD controllers for SVD problems.

Algorithm for μ-optimal SVD Controllers using DK-iteration

1. Test whether the problem is a Robust SVD-problem as given by Definitions 1 and 2. If the structure of an uncertainty Δ_i and its corresponding weights $W_i(s)$ do not satisfy Definition 2, then an SVD controller may not be optimal. To use the design procedure, treat the uncertainty as a full block, realizing that this is potentially conservative.

2. Form $\tilde{N}(s)$ as given by Eq.13 and rearrange it such that it is block-diagonal.

3. Delete all identical subproblems in $\tilde{N}$.

4. **K step:** Design an H_∞-optimal controller for each independent unique subproblem, and collect the optimal $\tilde{K}_i(s)$ (without repetitions) into a diagonal matrix.

5. **D step:** Calculate the tight upper bound on μ in (21) and obtain $D(s)$. Return to step 4 until DK-iteration converges.

6. Collect the optimal $\tilde{K}_i(s)$ (including repetitions for identical subproblems) into a diagonal matrix $\Sigma_K(s)$. Form $K(s) = V\Sigma_K(s)U^H$.

7. If the DK-iteration procedure converged to the global minimum, then this would be the μ-optimal controller under the assumptions of Theorem 4, for the uncertainty assumed in Step 1 of this algorithm.

Performing DK-iteration on the transformed system will converge faster and is numerically better conditioned than on the original system. This is both because the H_∞ subproblems are smaller than the original problem, and because the algorithm will be initialized with a controller which has the correct (optimal) directionality. This will be illustrated in the examples in Section 7.

6 One Source of Uncertainty

In the previous two sections we have shown that an SVD controller is optimal for classes of problems of engineering interest. Here we consider a class of problems for which an SVD controller may not be optimal, but in *choosing* the controller to on SVD form we get a substantial simplification in system analysis and controller synthesis.

In particular, we consider control problems with one source of uncertainty. When this one source of uncertainty is full, then at each frequency μ equals its upper bound and $\mu(M) = \overline{\sigma}(M)$, which corresponds to minimizing the H_∞-norm and was studied in Section 4. When this one source of uncertainty is not full, then an SVD controller may *not* be optimal for SVD problems (the optimality depends on the structure of the uncertainty and weights). Still, an SVD controller may be used, and we next state an interesting result for cases when $M(s)$ is a *normal* matrix.

Theorem 5 (Structure of the Uncertainty Block) *Consider an SVD problem where the objective is to minimize* $\sup_\omega \mu_\Delta(M)$. *Make the additional assumption that the weights are of the form* $W_i(s) = U\Sigma_{W_i}(s)U^H$, $U^H\Sigma_{W_i}(s)U$, $V\Sigma_{W_i}(s)V^H$, *or* $V^H\Sigma_{W_i}(s)V$. *Assume that the system has only one multiplicative or inverse multiplicative uncertainty block (e.g.,* $\Delta = \Delta_I$*), and an SVD controller is used. Then the robust stability for the system is independent of the structure of the uncertainty block.*

Proof: Trivial algebra shows that M is a normal matrix for multiplicative or inverse multiplicative uncertainty blocks. For example, consider multiplicative input uncertainty for which $M = W_1KG(I+KG)^{-1}W_2$. Substituting in $G = U\Sigma_G(s)V^H, K = V\Sigma_K(s)U^H, W_1 = V\Sigma_{W_1}(s)V^H, W_2 = V\Sigma_{W_2}(s)V^H$ gives $M(s) = V\Sigma_M(s)V^H$, where $\Sigma_M(s)$ is a diagonal matrix and we see that $M(s)$ is a normal matrix. The result then follows directly from (20) and the fact that $\rho(M) = \overline{\sigma}(M)$ for normal matrices. □

Remark 1. The weights can always be written on the form required by Thm. 5 when the uncertainty weight is scalar times identity $W_i = w_iI$. This is a reasonable assumption in many cases.

Remark 2. The matrix $M(s)$ may not be normal for additive uncertainty and *general* plants. However, if the *plant* is also a normal transfer function matrix, that is, $G(s) = U\Sigma_G(s)U^H$, then $M(s)$ will be a normal matrix for any robust stability problem with a single source of uncertainty (including additive or inverse additive uncertainty). Symmetric circulant plants are normal, for example. Thus, in this case the robust stability of the system will be independent of the structure of the uncertainty block whenever there is only one source of uncertainty in the system.

The significance of Thm. 5 is that, under the conditions of the theorem, the robust stability problem can be replaced by an H_∞ problem, thus substantially simplifying system analysis and controller synthesis. This simplification holds regardless of whether the uncertainty is described as linear time invariant [10], arbitrarily-slow linear time-varying [29], arbitrary linear time-varying [30], or arbitrary nonlinear operators [30] (see references for descriptions of these other uncertainty types).

7 Examples

The following examples illustrate the computational usefulness of the results of this paper.

7.1 Example 1: Distillation Column

Consider the robust controller design problem for the simplified distillation column example studied by Skogestad et al. [34], which under certain assumptions regarding the structure of the uncertainty can be shown to be a Robust SVD problem according to Definition 2. The nominal plant for this problem is of the form $G(s) = k(s)A$ given in Eq. (3):

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{bmatrix} \tag{25}$$

The plant $G(s)$ can be decomposed into $G(s) = U\Sigma_G(s)V^H$ with

$$\Sigma_G(s) = \begin{bmatrix} \frac{1.9721}{75s+1} & 0 \\ 0 & \frac{0.0139}{75s+1} \end{bmatrix};\ U = \begin{bmatrix} 0.6246 & -0.7809 \\ 0.7809 & 0.6246 \end{bmatrix};\ V = \begin{bmatrix} 0.7066 & -0.7077 \\ -0.7077 & -0.7066 \end{bmatrix} \tag{26}$$

The plant has a condition number of 141.7 and an RGA-value of 35.5 at all frequencies [34]. Although not a good model of a real distillation column, this model is an excellent example for demonstrating the problems with ill-conditioned plants and has been studied by many other researchers. For example, in a somewhat altered form this robust controller design problem has been considered by Yaniv and Barlev [37], and was used as a benchmark for the 1991 CDC [7].

For this problem, the relative magnitude of the uncertainty in each of the manipulated variables is given by $w_1(s) = 0.2(5s+1)/(0.5s+1)$. The robust performance specification is that $\|w_2 S_p\|_\infty < 1$ where $w_2(s) = 0.5(10s+1)/10s$ and S_p is the worst sensitivity function possible with the given bounds on the uncertainty in the manipulated variables. This robust controller design problem is easily captured in the framework of the structured singular value, μ. The resulting μ condition for Robust Performance (RP) becomes:

$$\text{RP} \quad \Longleftrightarrow \quad \mu(M) < 1 \quad \forall\omega \tag{27}$$

$$M = \begin{bmatrix} -W_1KSG & W_1KS \\ W_2SG & -W_2S \end{bmatrix}; \quad \Delta = \text{diag}\{\Delta_1, \Delta_2\} \tag{28}$$

where Δ_1 is a *diagonal* 2×2 perturbation block, Δ_2 is a full 2×2 perturbation block, and

$$W_1 = w_1 I_2 \quad \text{and} \quad W_2 = w_2 I_2$$

Note that in this case with only three perturbation blocks the upper bound in terms of the scaled singular value is equal to the structured singular value.

As stated this is not a Robust SVD Problem according to Definition 2. However, if we allow unstructured (full block) input uncertainty, i.w. Δ_I is a full rather than diagonal matrix, then this is a Robust SVD Problem, and we know from Theorem 4 that an SVD controller $K(s) = V\Sigma_K(s)U^H$ will be μ-optimal.

Controller Design

Skogestad et al. [34] used DK-iteration with some early H_∞-software to design a controller with 6 states giving a value of $\mu = 1.067$. Lundström et al. [25] assumed full block input uncertainty (for numerical convenience) and used the latest state-space H_∞ software [2] to design a μ-optimal controller with 22 states and with $\mu = 0.978$. As just noted we know that the μ-optimal controller for this case with full block input uncertainty should be an SVD controller. Indeed, Engstad [13] found for Lundström's [25] controller that the diagonal elements in $\tilde{K}(s) = V^H K(s)U$ were more than 10^7 times larger than the off-diagonal elements, and removing these off-diagonal elements did not affect the value of μ, which suggests that Lundström's controller is nearly μ-optimal. We have made attempts to improve on the design which gave $\mu = 0.978$ by considering diagonal rather than full block input uncertainty. Somewhat surprisingly, this has not proved successful. Actually, the value of μ with Lundström's [25] controller is not reduced by restricting the input uncertainty to be diagonal. Thus, it seems that in this special case the worst-case uncertainty Δ_i occurs when Δ_i is diagonal. Though we have no proof of this, it does seem reasonable since the input singular vector matrix V in Eq. (26) has large off-diagonal terms, which allows independent input uncertainty to cause strong interactions between the nominal subplants $\Sigma_{Gi}(s)$. A similar conjecture has been made earlier by Chen and Freudenberg [8].

Design of SVD controller. The optimal SVD controller may be obtained by designing two SISO-controllers, $\sigma_{K1}(s)$ and $\sigma_{K2}(s)$, using DK-iteration which involves solving two independent 2×2 H_∞-problems in the K-step and considering the full 4×4 μ-problem in the D-step to obtain the scaling $D(s) = \text{diag}\{d(s)I_2, I_2\}$. The order of this controller will depend on the order selected for $d(s)$ when fitting the D-scales.

Alternatively, one may design directly a low-order SVD-controller using "μ-K iteration", that is, by optimizing the parameters in a given controller to minimize μ. This approach only requires software to *compute* the structured singular value, as the DK-iteration involving H_∞-norm minimization is not used. Freudenberg [14] used this approach. He *assumed* the controller to be on the SVD form and obtained two SISO controller with 2+3=5 states giving $\mu = 1.054$, and he also used this problem as an example in [9]. Lin [24] used the same approach and obtained two SISO controllers with 7+4=11 states giving $\mu = 1.038$ (observed from plot).

Engstad [13] also used the same approach, but he restricted the input uncertainty to be diagonal rather than full, and used PID controllers of the form

$$\sigma_{Kj} = K_j \frac{1+\tau_{Ij}s}{\tau_{Ij}s} \frac{1+\tau_{Dj}s}{1+0.1\tau_{Dj}s} \tag{29}$$

Each controller has two states and three adjustable parameters. By numerical optimization[2] he obtained a value of $\mu = 1.036$ which is only slightly higher than the optimal value of 0.978, in spite of the fact that the overall controller only has 4 states. The optimal PID parameters for the SVD controller were:

$$K_1 = 38.3, \quad \tau_{I1} = 3.21; \quad \tau_{D1} = 0.50 \tag{30}$$

$$K_2 = 5.65, \quad \tau_{I2} = 1.24; \quad \tau_{D2} = 79.2 \tag{31}$$

Note that the second controller is not really a PID controller since the derivative time is larger than the integral time.

7.2 Example 2: Parallel Reactors With Combined Precooling

A simplified model $G(s)$ of four parallel reactors with combined precooling [33] is

$$G(s) = \frac{1}{100s+1} \begin{bmatrix} 1 & 0.7 & 0.7 & 0.7 \\ 0.7 & 1 & 0.7 & 0.7 \\ 0.7 & 0.7 & 1 & 0.7 \\ 0.7 & 0.7 & 0.7 & 1 \end{bmatrix} \tag{32}$$

[2]Standard optimization software in Matlab was used. Numerical problems with local minima were reduced by switching the optimization objective between minimizing the peak of μ (i.e., $\|\mu(j\omega)\|_\infty$) and minimizing the integral square deviation of μ from 1 (i.e., $\|\mu(j\omega) - 1\|_2$).

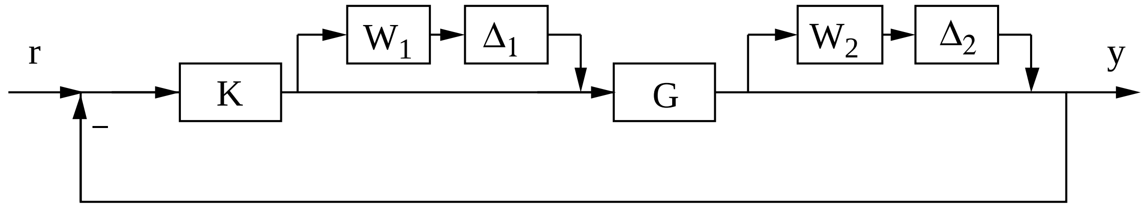

Figure 5: Block diagram for plant with uncertainties in Example 2.

The Fourier matrix diagonalizes the plant, i.e. $G(s) = F^H \Sigma_G(s) F$ where the plant singular values are

$$\sigma_{G1}(s) = \frac{3.1}{100s+1}; \qquad \sigma_{G2}(s) = \sigma_{G3}(s) = \sigma_{G4}(s) = \frac{0.3}{100s+1}$$

Consider the process with input and output uncertainty as shown in Fig. 5. The input uncertainty Δ_1 and output uncertainty Δ_2 are both assumed to be independent diagonal, with uncertainty weights $W_1(s) = \mathrm{diag}\{0.2\frac{5s+1}{0.5s+1}\}$ and $W_2(s) = \mathrm{diag}\{0.2\frac{2.5s+1}{0.25s+1}\}$. To reject disturbances at the plant output, we include the performance specification $\bar{\sigma}(W_3 S_p) < 1, \forall\omega$, with $W_3(s) = \mathrm{diag}\{0.5\frac{10s+1}{10s}\}$. The overall problem (before SVD reduction) has two diagonal 4×4 uncertainty blocks and one full 4×4 performance block, and we get a 12×12 μ interconnection matrix:

$$M = \begin{bmatrix} -W_1KG(I+KG)^{-1} & -W_1K(I+GK)^{-1} & W_1K(I+GK)^{-1} \\ W_2G(I+KG)^{-1} & -W_2GK(I+GK)^{-1} & W_2GK(I+GK)^{-1} \\ -W_3G(I+KG)^{-1} & -W_3(I+GK)^{-1} & W_3(I+GK)^{-1} \end{bmatrix} \tag{33}$$

In order to make this a Robust SVD problem (see Definition 2) we need to assume that all the uncertainty blocks are full. We then have three full blocks and μ is equal to its upper bound such that Theorem 4 applies. Thus, for this potentially-conservative case we know that an SVD controller is μ-optimal. In addition, when we follow the Algorithm for μ-optimal SVD Controllers, we find that three of the four subproblems in $\tilde{N}$ are identical. This means that with DK-iteration we may solve two 3×3 independent H_∞ problems in the K Step, and obtain the scalings $d_1(s)$ and $d_2(s)$ from a 6×6 μ matrix M in the D Step. Using this procedure we were able find a controller resulting in a μ-value of 0.93. The state space representation of the eigenvalues of this controller are given in Tables 1 and 2.

Thereafter we attempted to use DK-iteration to improve the controller design by using the true diagonal structure for the uncertainties Δ_1 and Δ_2, the original 12×12 M-matrix (33), and the above controller as a starting point. However, we found that this increased the complexity of the controller synthesis problem so much that we were unable to improve the design using DK-iteration. The best controller the software was able to obtain had a μ-value of 0.96, which is larger than the μ-value for the controller the algorithm was initialized with. This result shows that there are numerical inaccuracies with the off-the-shelf software. It also demonstrates the important advantage of reduced problem size which results from applying our method.

8 Discussion

Here we discuss additional uses of the SVD control structure.

8.1 Generalizations of the Results

The results of this paper are easily generalized to cases with multivariable, possibly non-square subplants. Synthesis problems similar to class B in Section 2 arise naturally whenever identical multivariable plants are arranged in parallel or in a symmetric manner, respectively.

8.2 General Use of the SVD Controller Structure

The structure of an SVD controller may be useful also for problems that do not fit into the problem definition in this paper. The reason is that we convert a multivariable design problem into designing n single-loop controllers. The results of this paper (see above) imply that at a fixed frequency the SVD structure is optimal (with some restrictions on the structures of the perturbation blocks given in Definition 2). This provides a theoretical justification for a design method based on obtaining an SVD of the plant at some important frequency, for example, the closed loop bandwidth, and use this as a basis for design a realizable controller. Indeed this has been suggested by several authors [21, 19]. One problem is that we need to obtain real approximations of the singular vector matrices U and V. The ALIGN algorithm of MacFarlane [19] deals with this particular issue. This design procedure should perform well for process control problems which do not have U and V varying rapidly as a function frequency, but may perform poorly for processes such as flexible structures.

8.3 Design of Low Order Controllers

DK-iteration is known for resulting in controllers with many states. We have shown that the SVD controller is the optimal structure for a certain class of problems, and this may be used for designing controllers with a low number of states. Using V as a pre-compensator and U^H as a post-compensator, we are left with n SISO controllers to design for a plant of dimension $n \times n$. This design problem is similar to the conventional decentralized control problem (e.g., [18]), and may be solved by sequential design, independent design, or simultaneous design (parameter optimization). The last approach was used to obtain the SVD-PID controllers for the distillation example. However, unlike the regular decentralized control problem, there is also the possibility to make use of the DK-iteration, such that each SISO controller is obtained by minimizing the H_∞-norm (i.e., peak value) of one scalar transfer function. This simplifies significantly the parameter optimization, but requires several iterations between designing K and obtaining D.

8.4 Use as a Controllability Measure

The SVD controller structure can be used for obtaining a simple lower bound on the achievable value for the upper bound to μ. The frequency response of the μ interconnection matrix can be decomposed frequency-by-frequency. At each frequency the plant can be decomposed into its singular value decomposition $G = U\Sigma_G V^H$ where in this cases all matrices may be complex. The Algorithm for robust optimal SVD controllers can then be used at each frequency. When the perturbation blocks are full and/or multiplicative/inverse multiplicative repeated diagonal, DK-iteration will result in a controller of the SVD form. Because each design subproblem at a fixed frequency only involves finding one complex scalar, the synthesis part is very simple (the state-space algorithm need not be used). This frequency-by-frequency approach will not yield a realizable controller, since issues such as causality and phase-gain relationships are ignored. Instead, the resulting value for the upper bound for μ will be a lower bound on the upper bound for μ obtainable by any realizable controller, and may therefore (since μ is usually close in magnitude to its upper bound) can be used as a controllability measure

Lee et al. [23] also suggest to use the upper bound on μ on a frequency-by-frequency basis as a controllability measure. Use of the SVD structure in the calculation of the upper bound will simplify the calculations involved.

9 Conclusions

For plants where the directionality is independent of frequency, the singular value decomposition (SVD) is used to decouple the system into nominally independent subsystems of lower dimension. In H_2- and H_∞-optimal control, the controller synthesis can thereafter be performed for each of these subsystems independently, and the resulting overall SVD controller will be optimal (the same will hold for any norm which is invariant under unitary transformations). In H_∞-optimal control the resulting controller is also *super-optimal*, as a controller of dimension $n \times n$ will minimize the norm in n directions.

For robust control in terms of the structured singular value, μ, the SVD controller is optimal for most systems with full block complex uncertainty and repeated diagonal complex uncertainty. In this case computational savings can be achieved in the controller synthesis step of the DK-iteration scheme.

Nomenclature

D - Block diagonal scaling matrix
F - Fourier matrix
F_l - Lower linear fractional transformation (see Eq. (8))
$G(s) = U\Sigma_G(s)V^H$ - Transfer function matrix for the plant
$K(s)$ - Transfer function matrix for the controller
$M(s)$ - Matrix whose norm is to be minimized in the controller synthesis
$M_0(s)$ - The matrix $M(s)$ with the weights removed
n - Plant dimension $(n \times n)$
$N(s)$ - Interconnection matrix for the synthesis problem
R - "Real Fourier matrix"; real, unitary eigenvector matrix for symmetric circulant matrices
s - Laplace variable
U - Output singular vector matrix of the plant $G(s)$
U_{Oi} - Output singular vector matrix for output weight i
V - Input singular vector matrix of the plant $G(s)$
V_{Ii} - Input singular vector matrix for input weight i
$W_I(s) = \text{diag}\{W_{Ii}(s)\}$ - Block diagonal matrix of weights for the inputs to $M(s)$
$W_O(s) = \text{diag}\{W_{2i}(s)\}$ - Block diagonal matrix of weights for the outputs from $M(s)$
Δ - Block diagonal matrix of perturbations
Δ_i - i'th block on the diagonal of Δ (of the same size as $G(s)$)
μ - Structure singular value
ρ - Magnitude of largest eigenvalue
σ - Singular value
$\bar{\sigma}$ - Largest singular value
$\Sigma(s)$ - Matrix of singular values

ω - Frequency

Subscripts
I - Input to synthesis problem
O - Output from synthesis problem
i - Block i on the diagonal of a block diagonal matrix j - Singular value j

Superscripts
H - Hermitian (complex conjugate transpose)
$\tilde{}$ - Denotes that the matrix has been transformed by pre- and postmultiplication with unitary matrices.

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Appendix

The following is a more detailed proof of item 4 in the proof of Theorem 1. Essentially, it shows that a decentralized controller is optimal for a decentralized plant with decentralized weights (costs).

The optimal controller solve

$$\inf_{\tilde{K}\in\mathcal{K}_s} \left\| F_l(\tilde{N}, \tilde{K}) \right\| \tag{34}$$

where $\mathcal{K}_s$ represents the set of all stabilizing controllers.

The key to a rigorous proof that a diagonal controller $\tilde{K}$ can be chosen to be optimal is to reparametrize the above optimization over $\tilde{K}$ as an optimization over the Youla matrix Q, and then use matrix dilation theory to show that Q is diagonal. The set of all stabilizing $\tilde{K}$ is given by

$$\begin{aligned} \mathcal{K}_s &= \left\{K : K = (Y - TQ)(X - SQ)^{-1}, Q \in \mathcal{RH}_\infty\right\} && (35) \\ &= \left\{K : K = (\tilde{X} - Q\tilde{S})^{-1}(\tilde{Y} - Q\tilde{T}), Q \in \mathcal{RH}_\infty\right\} && (36) \end{aligned}$$

where (S, T) and $(\tilde{S}, \tilde{T})$ are right and left coprime factors of $\tilde{N}_{22}$ respectively (*i.e.*, $\tilde{N}_{22} = ST^{-1} = \tilde{T}^{-1}\tilde{S}$), and $(X, Y, \tilde{X}, \tilde{Y})$ is a solution to the following Bezout identity:

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{S} & \tilde{T} \end{bmatrix} \begin{bmatrix} T & Y \\ S & X \end{bmatrix} = I \tag{37}$$

Note that, since $\tilde{N}_{22}$ is diagonal we may choose $T, S, \tilde{X}, \tilde{Y}, X, Y, \tilde{T}, \tilde{S}$ to all be diagonal (to do this, first construct the right and left coprime factors of each subsystem and stack these on the diagonal to construct right and left coprime factors of the overall system).

Using the parametrization (35)-(36), (34) becomes

$$\inf_{Q \in \mathcal{RH}_\infty} \|N_{11} + N_{12} Q N_{21}\| \tag{38}$$

where

$$\begin{aligned} N_{11} &= \tilde{N}_{11} + \tilde{N}_{12} T \tilde{Y} \tilde{N}_{21} && (39) \\ N_{12} &= \tilde{N}_{12} T && (40) \\ N_{21} &= \tilde{T} \tilde{N}_{21} && (41) \end{aligned}$$

The only restriction on Q is that it should be analytic in the closed RHP.

N_{11} consists of diagonal blocks because $\tilde{N}_{11}$, $\tilde{N}_{12}$, and $\tilde{N}_{21}$ consist of diagonal blocks and T and $\tilde{Y}$ are diagonal. Similarly, N_{12} and N_{21} also consist of diagonal blocks. Thus, each entry of $N_{11} + N_{12} Q N_{21}$ will have only one Q_{ij} in it, and the rows and columns of this matrix can be permuted so that the permuted matrix can be partitioned with only one Q_{ij} in each partition (permuting the rows and columns of a matrix does not change the value of its Unitary-invariant norm). Call this permuted matrix $P(Q)$ and let $P_{ij}(Q_{ij})$ be the partition containing Q_{ij}. Then

$$\inf_{Q \in \mathcal{RH}_\infty} \|N_{11} + N_{12} Q N_{21}\| = \inf_{Q \in \mathcal{RH}_\infty} \|P(Q)\| \tag{42}$$

Now we will specialize to the H_2- and H_∞-norms. For the H_∞ norm, recall from basic linear algebra that the maximum singular value of a matrix (in this case, $P(Q)$) is greater than the maximum singular values of each partition P_{ij} of $P(Q)$, that is,

$$\inf_{Q \in \mathcal{RH}_\infty} \|P(Q)\|_\infty = \inf_{Q \in \mathcal{RH}_\infty \text{ and } Q \text{ full}} \sup_\omega \bar{\sigma}\left(P(Q)|_{s=j\omega} \right) \tag{43}$$

$$\geq \inf_{Q_{ij} \in \mathcal{RH}_\infty} \max_{i,j} \left\{ \sup_\omega \bar{\sigma}\left(P_{ij}(Q_{ij})|_{s=j\omega} \right) \right\} \tag{44}$$

$$\geq \inf_{Q_{ii} \in \mathcal{RH}_\infty} \max_{i} \left\{ \sup_\omega \bar{\sigma}\left(P_{ii}(Q_{ii})|_{s=j\omega} \right) \right\} \tag{45}$$

$$= \inf_{Q \in \mathcal{RH}_\infty \text{ and } Q \text{ diagonal}} \sup_\omega \bar{\sigma}\left(P(Q)|_{s=j\omega} \right) \tag{46}$$

$$= \inf_{Q \in \mathcal{RH}_\infty \text{ and } Q \text{ diagonal}} \sup_\omega \bar{\sigma}\left(N_{11} + N_{12} Q N_{21}|_{s=j\omega} \right). \tag{47}$$

Thus minimizing over diagonal Q gives an H_∞-norm less than or equal to the value obtained by minimizing over full Q. Since Q being diagonal is more restrictive than allowing Q to be full, the above inequalities are equalities and the optimal Q can be taken to be diagonal. That diagonal Q corresponds to diagonal K can be seen from (35)-(36), that is

$$\inf_{Q \in \mathcal{RH}_\infty \text{ and } Q \text{ diagonal}} \sup_\omega \bar{\sigma}\left(N_{11} + N_{12} Q N_{21}|_{s=j\omega} \right) \tag{48}$$

$$= \inf_{\tilde{K} \in \mathcal{K}_s \text{ and } \tilde{K} \text{ diagonal}} \left\| F_l(\tilde{N}, \tilde{K}) \right\|_\infty. \tag{49}$$

QED (H_∞-norm case).

For the H_2-norm, recall from basic linear algebra that the square of the Frobenius norm of a partitioned matrix (in this case, $P(Q)$) is equal to the sum of the squares of the Frobenius norms of its partitions (in this case, P_{ij}). Thus,

$$\inf_{Q \in \mathcal{RH}_\infty \text{ and } Q \text{ full}} \|P(Q)\|_2 \tag{50}$$

$$= \inf_{Q \in \mathcal{RH}_\infty \text{ and } Q \text{ full}} \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}\left(P^*(Q) P(Q) \right) d\omega} \tag{51}$$

$$= \inf_{Q \in \mathcal{RH}_\infty \text{ and } Q \text{ full}} \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \|P(Q)\|_F^2 \; d\omega} \tag{52}$$

$$= \inf_{Q_{ij} \in \mathcal{RH}_\infty} \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{i,j} \|P_{ij}(Q_{ij})\|_F^2 \; d\omega} \tag{53}$$

$$\geq \inf_{Q_{ii} \in \mathcal{RH}_\infty} \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{i} \|P_{ii}(Q_{ii})\|_F^2 \; d\omega} \tag{54}$$

$$= \inf_{Q \in \mathcal{RH}_\infty \text{ and } Q \text{ diagonal}} \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \|P(Q)\|_F^2 \; d\omega} \tag{55}$$

$$= \inf_{Q \in \mathcal{RH}_\infty \text{ and } Q \text{ diagonal}} \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}\left(P^*(Q)P(Q)\right) \; d\omega} \tag{56}$$

$$= \inf_{Q \in \mathcal{RH}_\infty \text{ and } Q \text{ diagonal}} \|N_{11} + N_{12} Q N_{21}\|_2. \tag{57}$$

Thus minimizing over diagonal Q gives an H_2 norm less than or equal to the H_2 norm obtained by minimizing over a full Q. Because forcing Q to be diagonal is more restrictive than allowing Q to be full, the inequality is an equality and the optimal Q can be taken to be diagonal. That diagonal Q corresponds to diagonal K can be seen from (35)-(36), that is

$$\inf_{Q \in \mathcal{RH}_\infty \text{ and } Q \text{ diagonal}} \|N_{11} + N_{12} Q N_{21}\|_2 \tag{58}$$

$$= \inf_{\tilde{K} \in \mathcal{K}_s \text{ and } \tilde{K} \text{ diagonal}} \|F_l(\tilde{N}, \tilde{K})\|_2. \tag{59}$$

QED (H_2-norm case).

A		
Sub-diagonal	Main diagonal	Super-diagonal
	-9.692$E-$08	0
0	-1.509$E-$01	0
0	-1.021$E+$00	0
0	-9.600$E+$00	9.525$E+$00
-9.526$E+$00	-9.600$E+$00	0
0	-1.338$E+$02	0
0	-1.110$E+$04	
B	C^T	D
-1.944$E-$01	-5.222$E-$01	0
-7.293$E-$02	9.074$E-$00	
2.644$E-$01	1.218$E+$01	
-2.228$E+$00	-4.392$E+$01	
-1.747$E+$00	-2.743$E+$01	
2.460$E+$01	-2.472$E+$03	
-1.633$E+$01	-3.098$E+$05	

Table 1: State space description of σ_{K1} for the controller found in Example 2.

A		
Sub-diagonal	Main diagonal	Super-diagonal
	-9.975$E-$08	0
0	-1.535$E-$02	0
0	-3.325$E-$01	0
0	-4.099$E-$01	0
0	-2.192$E-$00	9.257$E-$01
-9.257$E-$01	-2.192$E+$01	0
0	-1.645$E+$01	0
0	-1.250$E+$02	0
0	-1.987$E+$02	
B	C^T	D
4.426$E-$01	6.897$E-$01	0
-8.813$E-$02	1.931$E-$01	
2.441$E-$01	-1.079$E-$00	
6.614$E-$01	1.413$E+$01	
5.825$E-$02	-5.937$E+$00	
5.437$E-$01	-5.180$E+$00	
-3.673$E-$01	-6.958$E+$01	
-1.435$E+$02	9.608$E+$02	
-1.437$E+$02	-1.541$E+$03	

Table 2: State space description of $\sigma_{K2} = \sigma_{K3} = \sigma_{K4}$ for the controller found in Example 2.