ROBUST CONTROL OF DISTILLATION COLUMNS

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Abstract. Ill-conditioned plants are generally believed to be difficult to control. Using a high-purity distillation column as an example, the physical reason for the poor conditioning and its implications on control system design and performance are explained. It is shown that an acceptable performance/robustness trade-off cannot be obtained by simple loop-shaping techniques (via singular values) and that a good understanding of the model uncertainty is essential for robust control system design. Physically motivated uncertainty desciptions (actuator uncertainty) are translated into the H_∞/Structured Singular Value framework, which is demonstrated to be a powerful tool to analyze and understand the complex phenomena.

Keywords. Robust Control; Process Control; Linear systems; Robust Performance; Uncertainty; Distillation; Structured Singular Value.

I. INTRODUCTION

It is well known that ill-conditioned plants cause control problems (Morari and Doyle,1986, Skogestad and Morari,1985). By ill-conditioned we mean that the plant gain is strongly dependent on the input direction, or equivalently that the plant has a high condition number

$$\gamma(G(j\omega)) = \bar{\sigma}(G(j\omega))/\underline{\sigma}(G(j\omega)) \tag{1}$$

Here $\bar{\sigma}(G)$ and $\underline{\sigma}(G)$ denote the maximum and ninimum singular values of the plant

$$\bar{\sigma}(G) = \max_{u\neq 0} \frac{||Gu||_2}{||u||_2},\ \underline{\sigma}(G) = \min_{u\neq 0} \frac{||Gu||_2}{||u||_2}, \tag{2}$$

$||\cdot||_2$ denotes the usual Eucledian norm. We also say that an ill-conditioned plant is characterized by strong "directionality" because inputs in directions corresponding to high plant gains are strongly amplified by the plant, while inputs in directions corresponding to low plant gains are not.

The main reason for the control problems associated with ill conditioned plants is "uncertainty". Uncertainty in the plant model may have several origins:

1. There are always parameters in the linear model which are only known approximately.
2. Measurement devices have imperfections. This may give rise to uncertainty om the manipulated <u>inputs</u> in

a distillation columns, since they are usually measured and adjusted in a cascade manner. In other cases limited valve resolution may cause input uncertainty.

3. At high frequenvies even the structure and the model order is unknown, and the uncertainty will exceed 100% at some frequency.
4. The parameters in the linear model may vary due to nonlinearities or changes in the operating conditions.

For "tight control" of ill-conditioned plants the controller should compensate for the strong directionality by applying large input signals in the directions where the plant gain is low, that is, a controller similar to G^{-1} in directionality is desirable. However, because of uncertainty, the direction of the large input may not correspond exactly to the low plant-gain direction, and the amplification of these large input signals may be much larger than expected from the model. This will result in large values of the controlled variables y (Fig.1), leading to poor performance or even instability.

The concept of directionality is clearly unique to multivariable systems, and extensions of design methods developed for SISO systems are likely to fail for multivariable plants with a high degree of directionality. Furthermore, since the problems with ill-conditioned plants are closely related to how uncertainty affects the particular plant, it is very important to model the uncertainty as precicely as possible. Most multivariable design methods (LQG, LQG/LTR, DNA/INA, IMC, etc.) do not explicity take uncertainty into account, and these methods will in general not yield acceptable designs for ill-conditioned plants.

A distillation column will be used as an example of an ill-conditioned plant. Here the product compositions are very sensitive to changes in the external flows (high gain in this direction), but quite insensitive to changes in the internal flows (low gain in this direction). In this paper the main emphasis is on the general properties of ill-conditioned plants, rather than the control system design for a real distillation columns.

II. DISTILLATION COLUMN EXAMPLE

The objective of the distillation column (Fig.2) is to split the feed, F, which is a mixture a light and a heavy component, into a distillate product, D, which contains most of the light component, and a bottom product, B, which contains most of the heavy component. The compositions z_F, y_D and x_B of these streams refer to the mole fractions of light component. The distillation column in Fig.2 has five controlled variables

- Vapor holdup (expressed by the pressure p)
- Liquid holdup in the accumulator (M_D)

- Liquid holdup in the column base (M_B)
- Top composition (y_D)
- Bottom composition (x_B)

and five manipulated inputs

- Distillate flow (D)
- Bottom flow (B)
- Reflux (L)
- Boilup (V) (controlled indirectly by the reboiler duty)
- Overhead vapor (V_T) (controlled indirectly by the condenser duty)

Because the composition dynamics are usually much slower than the flow dynamics, we will make the simplifying assumption of perfect control of holdup (i.e., p, M_D, M_B constant) and instantaneous flow responses. Different control *configurations* are obtained by choosing different input pairs (e.g., L and V) for composition control; the remaining three manipulated inputs are then determined by the requirement of keeping p, M_D and M_B under perfect control. In this paper we will first consider the LV configuration and then the DV-configuration.

Model of the Distillation Column

The distillation column descibed in Table 1 will be used as an example. The overhead composition is to be controlled at $y_D = 0.99$ and the bottom composition at $x_B = 0.01$. Consider first using reflux L and boilup V as manipulated inputs for composition control, i.e.,

$$y = \begin{pmatrix} \Delta y_D \\ \Delta x_B \end{pmatrix}, \quad u = \begin{pmatrix} \Delta L \\ \Delta V \end{pmatrix}$$

This choice is often made since L and V have an immediate effect on the product compositions. By linearizing the steady-state model and assuming that the dynamics may be approximated by first order response with time constant τ = 75 min, we derive the following linear model

$$\begin{pmatrix} dy_D \\ dx_B \end{pmatrix} = G_{LV} \begin{pmatrix} dL \\ dV \end{pmatrix}, \; G_{LV} = \frac{1}{\tau s + 1} \begin{pmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{pmatrix} \tag{3}$$

This is admittedly a very crude model of this strongly nonlinear plant, but the model is simple and displays important features of the distillation column behavior.

Singular Value Analysis of the Model

The condition number of the plant (3) is $\gamma(G_{LV}) = 141.7$ which shows a high degree of directionality in the plant. More specific information about this directionality is obtained from the Singular Value Decomposition (SVD) of the steady-state gain matrix

$$G = U\Sigma V^H$$

or equivalently since $V^H = V^{-1}$

$$G\bar{v} = \bar{\sigma}(G)\bar{u}, \quad G\underline{v} = \underline{\sigma}(G)\underline{u}$$

where

$$\Sigma = diag\{\bar{\sigma}, \underline{\sigma}\} = diag\{1.972, 0.0139\}$$

$$V = (\bar{v}\ \underline{v}) = \begin{pmatrix} 0.707 & 0.708 \\ -0.708 & 0.707 \end{pmatrix}$$

$$U = (\bar{u}\ \underline{u}) = \begin{pmatrix} 0.625 & 0.781 \\ 0.781 & -0.625 \end{pmatrix}$$

The large plant gain, $\bar{\sigma}(G) = 1.972$, is obtained when the inputs are in the direction $\binom{dL}{dV} = \bar{v} = \binom{0.707}{-0.708}$. Since

$$dB = -dD = dL - dV \tag{4}$$

this physically corresponds to the largest possible change in the *external* flows, D and B. From the direction of the output vector $\bar{u} = \binom{0.625}{0.781}$, we see that it causes the outputs to move in the same direction, that is, mainly affects the average composition $\frac{y_D + x_B}{2}$.

The low plant gain, $\underline{\sigma}(G) = 0.0139$, is obtained for inputs in the direction $\binom{dL}{dV} = \underline{v} = \binom{0.708}{0.707}$. From (4) we observe that this physically corresponds to changing the *internal* flows only ($dB = -dD \approx 0$), and from the output vector $\underline{u} = \binom{0.781}{-0.625}$ we see that the effect is to move the outputs in different directions, that is, to change $y_D - x_B$. Thus, it takes a large control action to move the compositions in different directions and to make both products purer simultaneously.

The notion that some changes are more "difficult" than others is important, since it implies that some disturbances may be "easier" to reject than others. Let d represent the *effect* of the disturbance on the outputs (Fig.1), or let d represent a setpoint change. A disturbance d which has a direction close to $\bar{u}$, is expected to be easy to reject since it corresponds to the high plant gain. Similarly, a disturbance close to $\underline{u}$ in direction is expected to be more difficult. The disturbance condition number, $\gamma_d(G)$, gives a more precise measure of how the disturbance is "aligned" with the plant directions (Skogestad and Morari, 1986):

$$\gamma_d(G) = \frac{||G^{-1}d||_2}{||d||_2}\bar{\sigma}(G) \tag{5}$$

$\gamma_d(G)$ ranges in magnitude between 1 and $\gamma(G)$. A value close to 1 indicates that the disturbance is in the "good" direction ($\bar{u}$) corresponding to the high plant gain, $\bar{\sigma}(G)$. A value close to $\gamma(G)$ indicates that the disturbance is in the "bad" direction ($\underline{u}$) corresponding to the low plant gain, $\underline{\sigma}(G)$. We will consider the following two disturbances (actually setpoint changes) in the simulations

$$y_{s_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ with } \gamma_{d_1}(G) = 110.5$$

$$y_{s_2} = \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} \text{ with } \gamma_{d_2}(G) = 12.3$$

Linear Closed-Loop Simulations

Linear simulations of the distillation model (3) will now be used to support the following three claims regarding ill-conditioned plants:

1. **Inverse-based controllers are potensially very sensitive to uncertainty on the inputs.**

The inverse-based controller

$$C_1(s) = \frac{k_1}{s}G_{LV}^{-1}(s) = \frac{k_1(1+75s)}{s}\begin{pmatrix} 39.942 & -31.487 \\ 39.432 & -31.997 \end{pmatrix}$$

with $k_1 = 0.7 \text{ min}^{-1}$ is obtained using a steady-state decoupler plus a PI-controller. This controller should in theory remove all the directionality of the plant an give rise to a first-order response with time constant 1.43 min. This is indeed confirmed by the simulations in Fig.3 for the case with no uncertainty. In practice, the plant is different from the model and we also show the response when there is 20% error (uncertainty) in the change of each manipulated input:

$$dL = 1.2dL_c, \quad dV = 0.8dV_c \tag{7}$$

(dL and dV are the actual changes in the manipulated flow rates, while dL_c and dV_c are the desired values as specified by the controller). It is important to stress that this diagonal input uncertainty, which stems from our inability to know the exact values of the manipulated inputs, is *always* present. The simulated response with uncertainty to

y_{s1} differs drastically from the one presicted by the model, and the response is clearly not acceptable; the response is no longer decoupled, and Δy_D and Δx_B reach a value of about 6 before settling at their desired values of 1 and 0.

There is a simple physical reason for the observed poor response to the setpoint change in y_D. To accomplish this change, which occurs mostly in the "bad" direction corresponding to the low plant gains, the inverse-based controller generates a large change in the internal flows $(dL + dV)$, while trying to keep the changes in the external flows $(dB = \Leftrightarrow dD = dL \Leftrightarrow dV)$ very small. However, uncertainty with respect to the values of dL and dV makes it impossible to keep their sum large while keeping $dL \Leftrightarrow dV$ small - the result is a undesired large change in the external flows, which subsequently results in large changes in the product compositions because of the large plant gain in this direction. This sensitivity to input uncertainty may be avoided by controlling D or B dirictly as shown below.

2. **Low condition-number controllers are less sensitive to uncertainty, but the response is strongly dependent on the disturbance direction.**

The poor response for the case with uncertainty in the above example was caused by the high condition-number controller which generates large input signals in the direction corresponding to the low plant gain. The simplest way to make the closed-loop system insensitive to input uncertainty is to use a low condition-number controller which does not have large gains in any particular direction. The problem with such a controller is that little or no correction is made for the strong directionality of the plant. This results in a closed-loop response which depends strongly on the disturbance direction. To illustrate this consider the diagonal controller

$$C_2(s) = \frac{k_2(75s+1)}{s}\begin{pmatrix} 1 & 0 \\ 0 & \Leftrightarrow 1 \end{pmatrix}, \; k_2 = 2.4min^{-1} \qquad (8)$$

which consists of two equal single-loop PI controllers and has a condition number of one. As seen from the simulations in Fig.4 the quality of the closed-loop response depends strongly on the disturbance direction, but is only weakly influenced by uncertainty. The reponse to y_{s_1} is very sluggish, while the response to y_{s_2} is fast initially, but approaches the final steady state sluggishly. Note that a disturbance entirely in the "good" direction ($y_s = \bar{u}$) would give a first-order response with time constant $1/2.4 \cdot \bar{\sigma}(G) =$ 0.21 min. On the other hand, a disturbance in the "bad" direction ($y_s = \underline{u}$) generates a first-order response with time constant $1/2.4 \cdot \underline{\sigma}(G) = 30$ min. All other responses are linear combinations of these two extremes (Fig.4).

3. **Changing the plant may make even an ill-conditioned plant insensitive to input uncertainty.**

We already argued physically that the plant might be made less sensitive to uncertainty by controlling the external flows directly. Consider the case of distillate flow D and boilup V as manipulated variables ("direct material balance control"). Assuming perfect level and pressure control, i.e., $dL = dV dD$, we have

$$\begin{pmatrix} dL \\ dV \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dD \\ dV \end{pmatrix} \tag{9}$$

and the following linear model is derived from (3)

$$\begin{pmatrix} dy_D \\ dx_B \end{pmatrix} = G_{DV} \begin{pmatrix} dD \\ dV \end{pmatrix}$$

$$G_{DV} = G_{LV} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \frac{1}{1+75s} \begin{pmatrix} -0.878 & 0.014 \\ -1.082 & -0.014 \end{pmatrix} \tag{10}$$

In practice, the condenser level loop introduces a lag between the change in distillate flow, dD, and the reflux flow, dL (which is the input which actually affects the compositions), but this is neglected here. To confirm that the system is much less sensitive to uncertainty in this case, consider the following inverse-based controller

$$c_3(s) = \frac{k_3}{s} G_{DV}^{-1}(s) = \frac{k_3(1+75s)}{s} \begin{pmatrix} -0.5102 & -0.5102 \\ 39.43 & -32.00 \end{pmatrix}$$

with $k_3 = 0.7$ min^{-1}. Without uncertainty this controller gives the *same* response as with $C_1(s)$ applied to the LV-configuration. However, for the DV-configuration the decoupled first-order response with time constant 1.43 min is also maintained when there is 20% uncertainty on the manipulated inputs (Fig.5). From this example we see that ill-conditioned plants by themselves may not give performance problems provided the uncertainty is appropriately "aligned" with the plant.

III. ROBUSTNESS ANALYSIS WITH μ

It is quite evident from the linear simulations above that multivariable systems exhibit a type of "directionality" which make the closed-loop response strongly dependent on the particular disturbance and model error assumed. One of the major weaknesses with the simulation approach is that it may be very difficult and time-consuming to find the particular input signal and model error which causes control problems. Therefore there is a need for a tool which solves the following robust performance problem in a systematic manner:

Given a nominal plant model, an uncertainty description, a set of possible external input signals (disturbances, setpoints), a desired performance objective, and a controller: Will the "worst case" response satisfy the desired performance objective?

Achieving robust performance is clearly the ultimate goal for the controller design. However, it may be easier to solve this problem by first considering some subobjectives which have to be satisfied in order to achieve this:

Nominal Stability (NS): The closed-loop system with the controller applied to the (nominal) plant model must be stable.

Nominal Performance (NP): In addition to stability, the quality of the response should satisfy some minimum requirement. We will define performance in terms of the H_∞-norm of the weightes sensitivity operator

$$NP \quad \Leftrightarrow \quad \bar{\sigma}(W_{1P} S W_{2P}) \leq 1 \quad \forall \omega \tag{11a}$$

The input weight W_{2P} is often equal to the disturbance model. The output weight W_{1P} is used to specify the frequency range over which the errors are to be small and (if W_{1P} is not equal to $w_P I$) which outputs are more important.

Robust Stability (RS): The closed-loop system must remain stable for all possible plants (G_p)as defined by the uncertainty description.

Robust Performance (RP): The closed-loop system must satisfy the performance specifications for all possible plants. As an example we may require (16a) to be satisfied when G is replaced by any of the possible G_p:

$$RP \quad \Leftrightarrow \quad \bar{\sigma}(W_{1P}(I + G_p C)^{-1} W_{2P}) \leq 1 \quad \forall \omega, \quad \forall G_p \tag{11b}$$

Practical Conditions for RS and RP

The definition (11b) of robust performance is of no value without a test which tells whether (11b) is satisfied for all possible plants G_p. Below we state computationally useful conditions for RS and RP using the Structured Singular Valuse μ. To use μ the uncertainty(i.e., the set of possible plants) must be modelled in terms norm-bounded perturbations on the nominal system. Bye use of weights each perturbation is normalized to have magnitude one: $\bar{\sigma}(\Delta_i) \leq 1,\ \forall\omega$. The perturbations, which may occur at different locations in the system, are collected in the *diagonal* matrix $\Delta = \text{diag}\{\Delta_1, \ldots, \Delta_n\}$ and the system is rearranged to match the structure of Fig.6. The signals $\hat{d}$ in Fig.6 represent the external inputs (weighted disturbances and setpoints) to the system. The signals $\hat{e}$ represents the weighted errors, or more generally all signals we want to keep "small" (e.g., $y \Leftrightarrow y_s$, manipulated inputs u). The interconnection matrix N in Fig.6 is a function of the nominal plant model G, the controller C and the uncertainty weights. Performance weights are also absorbed into N in order to normalize the performance specifications involving $\hat{d}$ and $\hat{e}$:

$$RP \quad \Leftrightarrow \quad \bar{\sigma}(E) \leq 1 \quad \forall\omega, \quad \forall\Delta \tag{12}$$

$$\text{where} \quad \hat{e} = E\hat{d},\ E = N_{22} + N_{21}\Delta(IN_{11}\Delta)^{-1}N_{12}$$

An example of such a performance specification is (11b). With these assumptions for the uncertainty and the performance we have the following results (Doyle et al., 1982):

$$NS \quad \Leftrightarrow \quad N \text{ stable (internally)} \tag{13}$$

$$NP \quad \Leftrightarrow \quad \mu_{NP} = \sup_{\omega} \bar{\sigma}(N_{22}) < 1 \tag{14}$$

$$RS \quad \Leftrightarrow \quad \mu_{RS} = \sup_{\omega} \mu_{\Delta}(N_{11}) < 1 \tag{15}$$

$$RP \quad \Leftrightarrow \quad \mu_{RP} = \sup_{\omega} \mu_{\hat{\Delta}}(N) < 1 \tag{16}$$

where $\hat{\Delta} = \text{diag}\{\Delta, \Delta_P\}$. The quantities μ_{NP}, μ_{RS} and μ_{RP} represent the "μ-norms" and are introduced as a convenient notation. The conditions for NP and RS are necessary in order to satisfy the RP-condition. Note that μ of a matrix depends on both the matrix and of the *structure* of the perubations. The RP-condition (16) is computed with respect to the structure $\text{diag}\{\Delta, \Delta_P\}$, where Δ_P is a full matrix of the same size as N_{22}.

IV. μ-ANALYSIS OF THE COLUMN

Problem definition

The same uncertainty and performance specifications will be assumed for the LV-configuration (3) and the DV-configuration (10).

Uncertainty: The uncertainty with respect to the manipulated inputs which was used in the simulations in Section II may be represented as multiplicative input uncertainty (Fig.7)

$$G_p = G(1 + w_I(s)\Delta_I), \quad \bar{\sigma}(\Delta_i) < 1 \quad \forall\omega \tag{17}$$

where $w_I(s)$ gives the magnitude of the realitive uncertainty on each manipulated input. We choose the following weight

$$w_I(s) = 0.2\frac{5s+1}{0.5s+1} \tag{18}$$

This implies an input error of up to 20% in the low frequency range as was used in the simulations. The uncertainty increases at high frequency; reaching a value one (100%) at about ω=1min^{-1}. This increase may take care of neglected flow dynamics: It allows for a time delay of about 1 min in the responses between L and V and the outputs y_D and x_B. At first the uncertainty will be assumed to be unstructured, that is, the perturbation matrix Δ_I is a *full* 2×2 matrix rather than a diagonal matrix. This does not make much sense from a physical point of view, but it is done for mathematical convenience. It will turn out that this assumption does not make any difference for the LV-configuration.

Performance: We use $\hat{d} = y_s$ as external inputs and $\hat{e} = w_P(y \Leftrightarrow y_s)$ as weighted errors (Fig.7).

$$w_P(s) = 0.5\frac{10s+1}{10s} \tag{19}$$

The RP-specification (12) then becomes

$$RP \quad \Leftrightarrow \quad \bar{\sigma}(S_p) < 1/|w_P|, \quad \forall\omega,\ \forall G_p \tag{20}$$

The performance weight $w_P(s)$ (19) implies that we require integral action at low frequency ($w_P(0) = \infty$) and allow an amplification of disturbances at high frequencies of at most a factor of two ($|w_P(j\infty)|^{-1} = 2$). A particular sensitivity function which exactly matches the performance bound (20) at low frequencies and satisfies it easily at high frequencies is $S = \frac{20s}{20s+1}I$. This corresponds to a first-order response with time constant 20 min.

Performance and Stability Conditions

With the information given above the matrix N in the ΔN-structure (Fig.6) becomes

$$N = \begin{pmatrix} \Leftrightarrow w_I CSG & w_I CS \\ w_P SG & \Leftrightarrow w_P S \end{pmatrix}, \quad S = (I + GC)^{-1} \tag{21}$$

Conditions for NP and RS are derived from (21) by using (14) and (15)

$$NP \quad \Leftrightarrow \bar{\sigma}(S) \leq 1/|w_P|\ \forall\omega$$

$$RS \Leftrightarrow \bar{\sigma}(H_I) \leq 1/|w_I|, \ \forall\omega, H_I = CG(I+CG)^{-1} = CSG$$

The condition for RS is expressed in terms of the singular value $\bar{\sigma}$ since Δ_I is assumed to be a full matrix. The RP-specification (20) is tested by computing μ of the whole matrix N (Eq. 21):

$$RP \quad \Leftrightarrow \quad \mu_{\hat{\Delta}}(N) \leq 1, \ \forall\omega \tag{22}$$

Analysis of the LV-configuration

We will analyze the LV-configuration for the inverse-based and the diagonal controller used in the simulations

$$C_1(s) = c_1(s) G_{LV}^{-1}(s), \quad c_1(s) = \frac{k_1}{s} \tag{23}$$

$$C_2(s) = c_2(s) \begin{pmatrix} 1 & 0 \\ 0 & \Leftrightarrow 1 \end{pmatrix}, \quad c_2(s) = \frac{k_2(1+75s)}{s} \tag{24}$$

We will first consider the choices $k_1 = 0.7$ and $k_2 = 2.4$ used in the simulations and then see if robust performance can be improved with other choices for k_1 and k_2. Finally, we will consider the "μ-optimal" controller, $C_\mu(s)$, which is obtained by minimizing μ_{RP}.

Nominal Performance and Robust Stability. One way of designing controllers to meet the NP and RS specifications is to use multivariable loop-shaping (Doyle and Stein. 1981): For NP, $\underline{\sigma}(GC)$ must be above $|w_P|$ at low frequencies. For RS with input uncertainty, $\bar{\sigma}(CG)$ must lie below $1/|w_I|$ at high frequencies (Fig.8). For the inverse-based controller (23) we get $\bar{\sigma}(C_1G) = \underline{\sigma}(GC_1) = |c_1|$ and it is trivial to choose a $c_1(s)$ which satisfies these conditions. The choice $c_1(s) = 0.7/s$ used in the simulations satisfied the NP- and RS-conditions easily (Fig. 9 and 10). For the diagonal controller (24) we find $\bar{\sigma}(C_2G) = 1.972|c_2|$ and $\underline{\sigma}(GC_2) = 0.0139|c_2|$, and the difference between these two singular values is so large that no choice of c_2 is able to satify both NP and RS (see Fig.9 and 10).

Robust Performance. In the case with input uncertainty, sufficients ("conservative") tests for RP in terms of singular values are easily derived:

$$RP \quad \Leftrightarrow \quad \gamma \cdot \bar{\sigma}(w_P S_I) + \bar{\sigma}(w_I H_I) \leq 1, \quad \forall\omega \tag{25a}$$

$$or\ RP \quad \Leftrightarrow \quad \bar{\sigma}(w_P S) + \gamma \cdot \bar{\sigma}(w_I H) \leq 1, \quad \forall\omega \tag{25b}$$

Here γ denotes the condition number of the plant *or* the controller (the smallest one should be used). These conditions indicate that the use of an ill-conditioned controller (e.g., $\gamma(C_1) = 141.7$) may give very poor robust performance even though both the nominal performance ($\bar{\sigma}(w_P S) < 1$) and robust stability conditons ($\bar{\sigma}(w_I H_I) < 1$) are individually satisfied. If a controller with a low condition number (e.g., $\gamma(C_2) = 1$) is used then we get RP for "free" provided we have satisfied NP and RS. This always the case for SISO systems and gives a partial explanation for why robust performance was never an important issue in classical control literature. Furthermore, for SISO systems (25) is necessary and sufficient for RP.

Conditions (26) are useful since they directly relate robust performance to NP,RS and the condition number. However, (25) may be very conservative and in order to get a "tight" condition for RP the μ-condition (22) has to be used. μ for RP is plotted in Fig. 11 and 12 for the two controllers used in the simulations. As expected, the inverse based controller, $C_1(s)$, is far from satisfying the RP requirememts (μ_{RP} is about 5.8), even though the controller was shown to achieve both NP and RP. On the other hand, the performance of the diagonal controllser, $C_2(s)$, is much less affected by the uncertainty and we find $\mu_{RP} = 1.71$ in this case.

Optimizing k_1 and k_2 with respect to RP. For the inverse-based controller the optimal value for k_1 is 0.14 corresponding to a value of μ_{RP} equal to 3.3 which still implies poor robust performance. For the diagonal controller, the optimal gain is $k_2 = 2.4$, which is the value already used. It is not clear how low μ_{RP} can be made if $C(s)$ is only resticted to be diagonal (decentralized control); we were able to get μ_{RP} down to 1.42 by repeatedly minimizing μ for one loop at the time with the controller for the other loop fixed.

μ-optimal Controller. μ for RP has a peak value of 1.06 (Fig.13), which means that this controller "almost" satisfies the robust performance condition. This value for μ_{RP} is significantly lower than that for the diagonal controller, and the time response is also better as seen from Fig. 14. In particular, the approach to steady state is much faster.

Structure of Δ_I. Note that Δ_I was assumed a full matrix in all the above calculations. It turns out that for this particular plant (3), the same values are found for μ_{RS} and μ_{RP} also when Δ_I is assumed diagonal which is a more reasonable assumption from physical considerations (there is no reason to expect the manipulated variables to influence each other). On the other hand, for the DV-configuration below it is cruical that Δ_I is modelled as a diagonal and not as a full matrix.

Analysis of the DV-configuration

We will consider the inverse-based controller

$$C_3(s) = \frac{k_3}{s} G_{DV}^{-1}(s)$$

In the simulations in Section II we studied the controller $C_3(s)$ with $k_3 = 0.7$. For this controller the NP- and RP-conditions are identical to those of the controller $C_1(s)$ with the LV-configuration. However, based on the simulations and other arguments presented before, we expect μ for robust performance to be much better for the DV-configuration. This is indeed confirmed by the μ-plots in Fig.15. μ_{RP} is 0.95 which means that the performance criterion is satisfied for all possible model errors. The uncertainty block Δ_I was assumed diagonal. If Δ_I were full (which is not the case) the value of μ_{RP} would be about 4.1. The reason for this high value is that the off-diagonal elements in Δ_I introduce errors in D when V is changed.

Even lower values for μ are obtained by reducing k_3 from 0.7 to 0.13 which yields $\mu_{RP} = 0.63$. In fact, this controller seems to be close to the μ-optimal as we were not able to reduce μ_{RP} below this value.

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References

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$\alpha = 1.5$
$N = 40$
$N_F = 21$
$z_F = 0.5$

$y_D =$	0.99
$x_B =$	0.01
$D/F =$	0.500
$L/F =$	2.706

$$\begin{pmatrix} dy_D \\ dx_B \end{pmatrix} \begin{pmatrix} 0.878 \Leftrightarrow 0.864 \\ 1.082 \Leftrightarrow 1.096 \end{pmatrix} \begin{pmatrix} dL/F \\ dV/F \end{pmatrix}$$

Figure Captions

$\Delta y_D \Delta x_B C_1(s), k_1 = 0.7$
$\Delta y_D \Delta x_B C_2(s), k_2 = 2.4$
$\Delta y_D \Delta x_B C_3(s), k_3 = 0.7$
$\Delta \hat{d} \hat{e}$
N
$\underline{\sigma}(GC) |w_P| \bar{\sigma}(CG) 1/|w_I|$

$\mu C_1(s), k_1 = 0.7$
$\mu C_2(s), k_2 = 2.4$
$\mu \mu C_\mu(s)$
$\Delta y_D \Delta x_B \mu C_\mu(s)$
$\mu C_3(s), k_3 = 0.7$