Inverse Optimal Robust Control of Singularly Impulsive Dynamical Systems

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Abstract—In this paper for the class of nonlinear uncertain singularly impulsive dynamical systems we present optimal robust control and inverse robust optimal control results. We consider a control problem for nonlinear uncertain singularly impulsive dynamical systems involving a notion of optimality with respect to an auxiliary cost which guarantees a bound on the worst-case value of a nonlinear-nonquadratic hybrid cost criterion over a prescribed uncertainty set. Further we specialize result to affine uncertain systems to obtain controllers predicated on an inverse optimal hybrid control problem. In particular, to avoid the complexity in solving the steady-state hybrid Hamilton-Jacobi-Bellman equation we parameterize a family of stabilizing hybrid controllers that minimize some derived hybrid cost functional that provides flexibility in specifying the control law. The performance integrand is shown to explicitly depend on the fact that the Hermitian matrix is nonnegative (resp., positive) definite and the identity matrix. Furthermore, let $\dot{c}(t) \in C^r$ denote the set of continuous functions and $C^r$ denote the set of functions with $r$ continuous derivatives.

I. INTRODUCTION

For the class of nonlinear uncertain singularly impulsive dynamical systems presented in [2], we have developed robust stability results in [7]. In this paper we give optimal robust control and inverse robust optimal control results. For that purpose, we generalize results developed in [3]. We consider a control problem for nonlinear uncertain singularly impulsive dynamical systems involving a notion of optimality with respect to an auxiliary cost which guarantees a bound on the worst-case value of a nonlinear-nonquadratic hybrid cost criterion over a prescribed uncertainty set. Further we specialize result to affine uncertain systems to obtain controllers predicated on an inverse optimal hybrid control problem. In particular, to avoid the complexity in solving the steady-state hybrid Hamilton-Jacobi-Bellman equation we parameterize a family of stabilizing hybrid controllers that minimize some derived hybrid cost functional that provides flexibility in specifying the control law. Obtained results for nonlinear case are further specialized to linear singularly impulsive dynamical systems with polynomial and multilinear performance functional.

II. OPTIMAL ROBUST CONTROL FOR NONLINEAR UNCERTAIN SINGULARLY IMPULSIVE DYNAMICAL SYSTEMS

In this section we consider a control problem for nonlinear uncertain singularly impulsive dynamical systems involving a notion of optimality with respect to an auxiliary cost which guarantees a bound on the worst-case value of a nonlinear-nonquadratic hybrid cost criterion over a prescribed uncertainty set. The optimal robust hybrid time-invariant feedback controllers are derived as a direct consequence of Theorem 2.1 given in [7] and provide a generalization of the Hamilton-Jacobi-Bellman conditions for state-dependent singularly impulsive dynamical systems with optimality notions over the infinite horizon with an infinite number of resetting times, for addressing robust feedback controllers of nonlinear uncertain singularly impulsive dynamical systems. To address robust optimal control problem let $D \subset \mathbb{R}^n$ be an open set with $0 \in D$, and let $C_0 \subset \mathbb{R}^{m_0}$, $C_1 \subset \mathbb{R}^{m_1}$, where $0 \in C_0$ and $0 \in C_1$. Furthermore, let $\mathcal{F}_c \subset \{ F_c : D \times C_0 \to \mathbb{R}^n : F_c(0,0) = 0 \}$, and $\mathcal{F}_d \subset \{ F_d : D \times C_1 \to \mathbb{R}^n : F_d(0,0) = 0 \}$. For simplicity of exposition, we also define $(F_c(\cdot, \cdot), F_d(\cdot, \cdot)) \in \mathcal{F}_c \times \mathcal{F}_d \triangleq \mathcal{F}$. Next, consider the nonlinear uncertain singularly impulsive controlled dynamical system

$$E_c\dot{x}(t) = F_c(x(t), u_c(t)), \quad x(0) = 0, \quad x(t) \notin Z_x, \quad u_c(t) \in U_c, \quad (I.1)$$

$$E_d\Delta x(t) = F_d(x(t), u_d(t)), \quad x(t) \in Z_x, \quad u_d(t) \in U_d, \quad (I.2)$$

where $t \geq 0$, $x(t) \in D$ is the state vector, $(u_c(t), u_d(t_k)) \in U_c \times U_d \subset C_0 \times C_1$, $k \in \mathcal{N}$, is the hybrid control input,
where the control constraint sets \( U_c, U_d \) are given. We assume \((0, 0) \in U_c \times U_d, F_c : \mathcal{D} \times U_c \to \mathbb{R}^n \) is Lipschitz continuous and satisfies \( F_c(0, 0) = 0, F_d : \mathcal{D} \times U_d \to \mathbb{R}^n \) is continuous and satisfies \( F_d(0, 0) = 0 \), and \( Z_x \subset \mathbb{R}^n \). To address the robust optimal nonlinear hybrid feedback control problem let \( \phi_c : \mathcal{D} \to U_c \) be such that \( \phi_c(0) = 0 \) and let \( \phi_d : \mathcal{D} \to U_d \) be such that \( \phi_d(0) = 0 \). If \((u_c(t), u_d(t)) = (\phi_c(E_c(x(t)), \phi_d(E_d(x(t)))) \), where \( x(t), t \geq 0 \), satisfies (II.1), (II.2), then \((u_c(\cdot), u_d(\cdot))\) is a **hybrid feedback control**. Given the hybrid feedback control 

\[ u_c(t), u_d(t) = (\phi_c(E_c(x(t))), \phi_d(E_d(x(t)))) \]

for the closed-loop state-dependent singularly impulsive dynamical system has the form

\[
E_c \dot{x}(t) = F_c(x(t), \phi_c(E_c(x(t))), x(0) = x_0, \quad t \geq 0, \quad x(t) \notin Z_x, \quad (I.3)
\]

\[
E_d \Delta x(t) = F_d(x(t), \phi_d(E_d(x(t))), x(t) \in Z_x, \quad (I.4)
\]

for all \((F_c(\cdot, \cdot), F_d(\cdot, \cdot)) \in F\).

Next we present sufficient conditions for characterizing robust nonlinear hybrid feedback controllers that guarantee robust stability over a class of nonlinear uncertain singularly impulsive dynamical systems and minimize an auxiliary hybrid performance functional. For the statement of this result let \( L_c : \mathcal{D} \times U_c \to \mathbb{R}, L_d : \mathcal{D} \times U_d \to \mathbb{R} \) and define the set of asymptotically stabilizing controllers for the nominal nonlinear singularly impulsive dynamical system \((F_{c0}(\cdot, \cdot), F_{d0}(\cdot, \cdot)) \) by

\[ C(x_0) \triangleq \{(u_c(\cdot), u_d(\cdot)) : (u_c(\cdot), u_d(\cdot)) \text{ is admissible and the zero solution } x(t) \equiv 0 \}
\]

Consider the nonlinear uncertain singularly impulsive dynamical system (II.1), (II.2) with hybrid performance functional

\[
J(E_c x_0, u_c(\cdot), u_d(\cdot)) = \int_0^\infty L_c(E_c x(t), u(t))dt + \sum_{k \in \mathbb{N}[0, \infty)} L_d(E_d x(t_k), u_d(t_k))(I.5)
\]

where \((F_{c0}(\cdot, \cdot), F_{d0}(\cdot, \cdot)) \in F \) and \((u_c(\cdot), u_d(\cdot)) \) is an admissible control. Assume there exist functions \( V : \mathcal{D} \to \mathbb{R}, G_c : \mathcal{D} \times U_c \to \mathbb{R}, G_d : \mathcal{D} \times U_d \to \mathbb{R} \), and a hybrid control law \( \phi_c : \mathcal{D} \to U_c \) and \( \phi_d : \mathcal{D} \to U_d \), where \( V(\cdot) \) is a C^1 function, such that

\[
V(0) = 0, \quad (I.6)
\]

\[
V(E_c x) \geq 0, \quad x \in \mathcal{D}, \quad x \neq 0, \quad (I.7)
\]

\[
\phi_c(0) = 0, \quad (I.8)
\]

\[
\phi_d(0) = 0, \quad (I.9)
\]

\[
V'(E_c x) F_c(x, \phi_c(x)) \leq V'(E_c x) F_{c0}(x, \phi_c(x)) + \Gamma_c(x, \phi_c(x)), \quad x \notin Z_x, \quad F_c(\cdot, \cdot) \in F_c, \quad (I.10)
\]

\[
V'(E_c x) F_{c0}(x, \phi_c(x)) + \Gamma_c(x, \phi_c(x)) < 0, \quad x \notin Z_x, \quad x \neq 0, \quad (I.11)
\]

\[
V(E_d x + F_d(x, \phi_d(x))) - V(E_d x) \leq (I.12)
\]

\[
\Gamma_d(x, \phi_d(x)), \quad x \in Z_x, \quad F_d(\cdot, \cdot) \in F_d, \quad (I.12)
\]

\[
V(E_d x + F_{d0}(x, \phi_d(x))) - V(E_d x) + \Gamma_d(x, \phi_d(x)) \leq 0, \quad x \in Z_x, \quad (I.13)
\]

\[
H_c(E_c x, \phi_c(x)) = 0, \quad x \notin Z_x, \quad (I.14)
\]

\[
H_c(E_c x, u_c(x)) \geq 0, \quad x \notin Z_x, \quad u_c \in U_c \quad (I.15)
\]

\[
H_d(E_d x, \phi_d(E_d x)) = 0, \quad x \in Z_x, \quad (I.16)
\]

\[
H_d(E_d x, u_d(x)) \geq 0, \quad x \in Z_x, \quad u_d \in U_d \quad (I.17)
\]

where \((F_{c0}(\cdot), F_{d0}(\cdot)) \in F \) defines the nominal singularly impulsive dynamical system and

\[
H_c(E_c x, u_c) \triangleq L_c(E_c x, u_c) + V'(E_c x) F_{c0}(x, u_c) + \Gamma_c(x, u_c), \quad (I.18)
\]

\[
H_d(E_d x, u_d) \triangleq L_d(E_d x, u_d) + V(E_d x + F_{d0}(x, u_d)) - V(x E_d x) + \Gamma_d(x, u_d). \quad (I.19)
\]

Then, with the hybrid feedback control \((u_c(\cdot), u_d(\cdot)) = (\phi_c(E_c x(\cdot)), \phi_d(E_d x(\cdot)))\), there exists a neighborhood of the origin \( D_0 \subset D \) such that if \( x_0 \in D_0 \), the zero solution \( x(t) \equiv 0 \) of the closed-loop system (II.3), (II.4) is locally asymptotically stable for all \((F_c(\cdot, \cdot), F_d(\cdot, \cdot)) \in F\). Furthermore,

\[
\sup_{(F_c(\cdot, \cdot), F_d(\cdot, \cdot)) \in F} J(E_c x_0, \phi_c(E_c x(\cdot)), \phi_d(E_d x(\cdot))) \leq J(E_c x_0, \phi_c(\cdot), \phi_d(\cdot)) = V(E_c x_0), \quad x_0 \in D_0, \quad (I.20)
\]

where

\[
\int_0^\infty [L_c(E_c x(t), u_c(t)) + \Gamma_c(x(t), u_c(t))]dt + \sum_{k \in \mathbb{N}(0, \infty)} [L_d(E_d x(t_k), u_d(t_k)) + \Gamma_d(x(t_k), u_d(t_k))], \quad (I.21)
\]

and where \((u_c(\cdot), u_d(\cdot)) \) is an admissible control and \( x(t), t \geq 0 \), is a solution of (II.1), (II.2) with \((F_c(x(t), u_c(t)), F_d(x(t), u_d(t))) = (F_{c0}(x(t), u_c(t)), F_{d0}(x(t), u_d(t)))\). In addition, if \( x_0 \in D_0 \) then the hybrid feedback control \((u_c(\cdot), u_d(\cdot)) = (\phi_c(E_c x(\cdot)), \phi_d(E_d x(\cdot)))\) minimizes \( J(E_c x_0, \phi_c(\cdot), \phi_d(\cdot)) \) in the sense that

\[
J(E_c x_0, \phi_c(E_c x(\cdot)), \phi_d(E_d x(\cdot))) = \min_{(u_c(\cdot), u_d(\cdot)) \in C(x_0)} J(E_c x_0, u_c(\cdot), u_d(\cdot)) \quad (I.22)
\]

Finally, if \( \mathcal{D} = \mathbb{R}^n \), and

\[
V(E_{c/d} x) \to \infty \quad \text{as} \quad ||x|| \to \infty, \quad (I.23)
\]

then the zero solution \( x(t) \equiv 0 \) of the closed-loop system (II.3), (II.4) is globally asymptotically stable for all \((F_c(\cdot, \cdot), F_d(\cdot, \cdot)) \in F\) [3] and [7].

**Proof**: Local and global asymptotic stability is a direct consequence of (I.6)–(I.13) by applying Theorem 2.1 of [7] to the closed-loop system (II.3), (II.4). Next, let
(u_c(\cdot), u_d(\cdot)) \in C(x_0) and let \( x(\cdot) \) be the solution of (II.1), (II.2) with \((F_c(\cdot, \cdot), F_d(\cdot, \cdot)) = (F_{c0}(\cdot, \cdot), F_{d0}(\cdot, \cdot))\).

Then it follows that
\[
0 = -\dot{V}(E_c x(t)) + V'(E_c x(t)) F_c(x(t), u_c(t)), x(t) \not\in Z_x, 
\]
(II.24)
\[
0 = -\Delta V(E_d x(t)) + V(E_d x + F_d(x(t), u_d(t))) - V(E_d x(t)), \quad x(t) \in Z_x. 
\]
(II.25)

Hence,
\[
L_c(E_c x(t), u_c(t)) + \Gamma_c(E_c \tilde{x}(t), u_c(t)) = -\dot{V}(E_c x(t)) + L_c(E_c x(t), u_c(t)) + V'(E_c x(t)) F_{c0}(x(t), u_c(t)) + \Gamma_c(E_c \tilde{x}(t), u_c(t)) = -\dot{V}(E_c x(t)) + H_c(E_c x(t), u_c(t)), \quad x(t) \not\in Z_x. 
\]
(II.26)

Similarly,
\[
L_d(E_d x(t), u_d(t)) + \Gamma_d(x(t), u_d(t)) = -\dot{V}(E_d x(t)) + L_d(E_d x(t), u_d(t)) + \Delta V(E_d x(t)) + \Gamma_d(x(t), u_d(t)) = -\dot{V}(E_d x(t)) + H_d(E_d x(t), u_d(t)), \quad x(t) \in Z_x. 
\]
(II.27)

Now, over the interval \([0, t]\) yields
\[
\int_0^t [L_c(E_c x(t), u_c(t)) + \Gamma_c(\tilde{x}(t), u_c(t))])dt + \sum_{k \in N_{\{0, t\}}} [L_d(E_d x(t_k), u_d(t_k)) + \Gamma_d(x(t_k), u_d(t_k))] 
- \dot{V}(E_c x(t)) + H_c(x(t), u_c(t))dt 
\geq V(E_c x_0) 
= J(E_c x_0, \phi_c(x(\cdot)), \phi_d(x(\cdot))). 
\]
(II.28)

Letting \( t \to \infty \) and noting that \( V(E_c x(t)) \to 0 \) for all \( x_0 \in D_0 \) yields (II.22).

Next, we specialize Theorem II to linear uncertain singularly impulsive dynamical systems. Specifically, in this case we consider \( \mathcal{F} \triangleq F_c \times F_d \) to be the set of uncertain linear singularly impulsive dynamical systems, where
\[
\mathcal{F}_c = \{ (A_c + \Delta A_c)x + B_c u_c : x \in \mathbb{R}^n, A_c \in \mathbb{R}^{n \times n}, B_c \in \mathbb{R}^{n \times m_c}, \Delta A_c \in \Delta A_c \}, 
\]
\[
\mathcal{F}_d = \{ (A_d + \Delta A_d - E_d)x + B_d u_d : x \in \mathbb{R}^n, A_d \in \mathbb{R}^{n \times n}, B_d \in \mathbb{R}^{n \times m_d}, \Delta A_d \in \Delta A_d \}, 
\]
where \( \Delta A_c, \Delta A_d \subset \mathbb{R}^{n \times n} \), are given bounded uncertainty sets of uncertain perturbations \( \Delta A_c, \Delta A_d \) of the nominal system matrices \( A_c, A_d \), such that \( 0 \in \Delta A_c \) and \( 0 \in \Delta A_d \).

For simplicity of exposition, we also define \((\Delta A_c, \Delta A_d) \in \Delta A_c \times \Delta A_d \triangleq \Delta \). For the following result let \( R_{c1} \in \mathbb{P}^n \), \( R_{c2} \in \mathbb{P}^{m_c} \), \( R_{d1} \in \mathbb{N}^n \), \( R_{d2} \in \mathbb{N}^{m_d} \) be given.

Consider the linear state-dependent uncertain singularly impulsive controlled dynamical system
\[
E_c \dot{x}(t) = (A_c + \Delta A_c)x(t) + B_c u_c(t), \quad x(0) = x_0, 
\]
(II.29)
\[
E_d \dot{x}(t) = (A_d + \Delta A_d - E_d)x(t) + B_d u_d(t), \quad x(t) \in Z, 
\]
(II.30)

with performance functional
\[
J_{\Delta A_c, \Delta A_d}(E_c x_0, u_c(\cdot), u_d(\cdot)) \triangleq \int_0^\infty \left[ x^T(t) E_c^T R_{c1} E_c x(t) + u_c^T(t) R_{c2} u_c(t) \right] dt + \sum_{k \in N_{\{0, \infty\}}} [x^T(k) E_d^T R_{d1} E_d x(k) + u_d^T(k) R_{d2} u_d(k)],
\]
(II.31)

where \((u_c(\cdot), u_d(\cdot))\) is admissible, \((\Delta A_c, \Delta A_d) \in \Delta \). Furthermore, assume there exist \( P \in \mathbb{P}^n \), \( \Omega_c : \mathbb{P}^n \to \mathbb{N} \), \( \Omega_{dxx} : \mathbb{P}^n \to \mathbb{N} \), \( \Omega_{dxxu} : \mathbb{N}^{m_d} \to \mathbb{N} \), and \( \Omega_{dxxuu} : \mathbb{N}^{m_d} \to \mathbb{N}^{m_d} \), such that
\[
x^T(A_c^T E_c^T P + PA_c E_c)x \leq \eta_c^T E_c^T \Omega_c(P) E_c x, 
\]
(II.32)
\[
x^T(A_d^T P A_d + P A_d E_d - \Delta A_d P B_d (R_{d2})^{-1} + B_d^T P B_d + \Omega_{dxxu_d} (P)) x \leq \eta_d^T P A_d + \Omega_{dxxu_d} (P), 
\]
(II.33)

Furthermore, suppose there exists \( P \in \mathbb{P}^n \) satisfying
\[
0 = x^T(A_d^T PA_d + E_d^T P E_d + E_d^T R_{d1} E_d + \Omega_{dxx}(P) - (B_d^T P A_d + \Omega_{dxxu_d} (P))^T R_{d2} + B_d^T P B_d + \Omega_{dxxu_d} (P)) x, 
\]
(II.34)
\[
0 < R_{d2} + B_d^T P B_d + \Omega_{dxxu_d} (P), 
\]
(II.35)
\[
0 = x^T(A_c^T PA_c + E_c^T P E_c + E_c^T R_{c1} E_c + \Omega_c(P) - (B_c^T P A_d + \Omega_{dxxuu_d} (P))^T R_{c2} + B_c^T P B_c + \Omega_{dxxuu_d} (P)) x, 
\]
(II.36)
Then, with hybrid feedback control \((u_c, u_d) = (\phi_c(x), \phi_d(x))\) = \((-R_{e1}^{-1}B_c^TPcex, -(R_{d1} + B_d^TP_0) + \Omega_{duu_1}(P))^{-1}(B_d^TP_0A_d + \Omega_{duu_1}(P))x\) the zero solution \(x(t) \equiv 0\) to (II.29), (II.30) is globally asymptotically stable for all \(x_0 \in \mathbb{R}^n\), \((\Delta A_c, \Delta A_d) \in \mathcal{D}_{ac} \times \mathcal{D}_{ad}\)

\[
\sup_{(\Delta A_c, \Delta A_d) \in \mathcal{D}} J_{(\Delta A_c, \Delta A_d)}(E_0x_0) \leq J(E_cx_0, \phi_c(\cdot), \phi_d(\cdot)) = x_0^T E_c^T P E_cx_0, \quad x_0 \in \mathbb{R}^n, \tag{III.37}
\]

where

\[
J(E_cx_0, \phi_c(\cdot), \phi_d(\cdot)) \triangleq \int_0^\infty [x^T(t)E_c^T P c1 x(t) + u_c^T(t)Rc2 u_c(t) + x^T(t)\Omega_c(P)x(t)]dt + \sum_{k \in \mathbb{N}_0} [x^T(t_k)E_d^T R_{d1} E_d x(t_k) + u_d^T(t_k)R_{d2} u_d(t_k) + x^T(t_k)\Omega_{duu_1}(P)u_d(t_k) + u_d^T(t_k)\Omega_{duu_1}(P)u_d(t_k)], \tag{III.38}
\]

and where \((u_c, u_d)\) is admissible and \(x(t), t \geq 0\), is a solution to (II.29), (II.30) with \((\Delta A_c, \Delta A_d) = (0, 0)\). Furthermore,

\[
J(E_cx_0, \phi_c(x(\cdot)), \phi_d(x(\cdot))) = \min_{(u_c(\cdot), u_d(\cdot)) \in C(x_0)} J(E_cx_0, u_c(\cdot), u_d(\cdot)), \tag{III.40}
\]

where \(C(x_0)\) is the set of asymptotically stabilizing hybrid controllers for the nominal singularly impulsive dynamical system and \(x_0 \in \mathbb{R}^n\), [3] and [7].

**Proof:** The detailed proof is given in [7]. □

III. INVERSE OPTIMAL ROBUST CONTROL FOR NONLINEAR AFFINE UNCERTAIN SINGULARLY IMPULSIVE DYNAMICAL SYSTEMS

In this section we specialize Theorem II to affine uncertain systems. The controllers obtained are predicated on an inverse optimal hybrid control problem. In particular, to avoid the complexity in solving the steady-state hybrid Hamilton-Jacobi-Bellman equation we do not attempt to minimize a given hybrid cost functional, but rather, we parametrize a family of stabilizing hybrid controllers that minimize some derived hybrid cost functional that provides flexibility in specifying the control law. The performance integrand is shown to explicitly depend on the nonlinear singularly impulsive system dynamics, the Lyapunov function of the closed-loop system, and the stabilizing hybrid feedback control law wherein the coupling is introduced via the hybrid Hamilton-Jacobi-Bellman equation. Hence, by varying the parameters in the Lyapunov function and the performance integrand, the proposed framework can be used to characterize a class of globally stabilizing hybrid controllers that can meet the closed-loop system response constraints.

Consider the state-dependent affine (in the control) uncertain singularly impulsive dynamical system

\[
E_c\dot{x}(t) = f_c(x(t)) + \Delta f_c(x(t)) + G_c(x(t))u_c(t), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}_e, \tag{III.41}
\]

\[
E_d\Delta x(t) = f_d(x(t)) + \Delta f_d(x(t)) + G_d(x(t))u_d(t), \quad x(t) \in \mathcal{Z}_d, \tag{III.42}
\]

where \(t \geq 0\), \(f_{c0}, f_{d0} : \mathcal{D} \to \mathbb{R}^n\) and satisfies \(f_{c0}(0) = 0, f_{d0}(0) = 0\), \(\mathcal{D} = \mathbb{R}^n, \mathcal{U}_c = C_c = \mathbb{R}^{m_c}, \mathcal{U}_d = C_d = \mathbb{R}^{m_d}\), and \((\Delta f_c, \Delta f_d) \in \mathcal{F}_c \times \mathcal{F}_d \triangleq \mathcal{F}, \) where

\[
\Delta f_c(\cdot) \in \mathcal{F}_c \subset \{ \Delta f_c : \mathbb{R}^n \to \mathbb{R}^n : \Delta f_c(0) = 0 \}, \quad \Delta f_d(\cdot) \in \mathcal{F}_d \subset \{ \Delta f_d : \mathbb{R}^n \to \mathbb{R}^n : \Delta f_d(0) = 0 \}.
\]

In this section no explicit structure is assumed for the elements of \(\mathcal{F}\). Furthermore, we consider performance integrands \(L_c(E_c x, u_c)\) and \(L_d(E_d x, u_d)\) of the form

\[
L_c(E_c x, u_c) = L_{c1}(E_c x) + u_c^T R_{c2} u_c, \quad x \notin \mathcal{Z}_c, \tag{III.43}
\]

\[
L_d(E_d x, u_d) = L_{d1}(E_d x) + u_d^T R_{d2} u_d, \quad x \in \mathcal{Z}_d, \tag{III.44}
\]

where \(L_{c1} : \mathbb{R}^n \to \mathbb{R}\) and satisfies \(L_{c1}(E_c x) \geq 0\), \(x \in \mathbb{R}^n\), \(R_{c2} : \mathbb{R}^n \to \mathbb{R}^{m_c}, L_{d1} : \mathbb{R}^n \to \mathbb{R}\) and satisfies \(L_{d1}(E_d x) \geq 0, x \in \mathbb{R}^n\), and \(R_{d2} : \mathbb{R}^n \to \mathbb{R}^{m_d}\) so that (II.5) becomes

\[
J(E_c x_0, u_c(\cdot), u_d(\cdot)) = \int_0^\infty \left[ L_{c1}(E_c x(t)) + u_c^T(t)R_{c2} u_c(t) + \sum_{k \in \mathbb{N}_0} [L_{d1}(E_d x(t_k)) + u_d^T(t_k)R_{d2} u_d(t_k)] \right]dt, \tag{III.45}
\]

Consider the nonlinear uncertain controlled affine singularly impulsive system (III.41), (III.42) with performance functional (III.45). Assume there exists a \(C^1\) function \(V : \mathbb{R}^n \to \mathbb{R}\), and functions \(P_{12} : \mathbb{R}^n \to \mathbb{R}^{1 \times m_c}, P_1 : \mathbb{R}^n \to \mathbb{R}^{m_c}, P_{1 f_1} : \mathbb{R}^n \to \mathbb{R}^{1 \times n}, P_{2} : \mathbb{R}^n \to \mathbb{R}^{m_d}, P_{2 f_1} : \mathbb{R}^n \to \mathbb{R}^{m_d \times n}, \Gamma_c : \mathbb{R}^n \to \mathbb{R}, \Gamma_{dx} : \mathbb{R}^n \to \mathbb{R}^n, \Gamma_{dxx} : \mathbb{R}^n \to \mathbb{R}^{1 \times m_d}\) and \(\Gamma_{duu_1} : \mathbb{R}^n \to \mathbb{R}^{m_d}\) such that

\[
P_{12}(0) = 0, \tag{III.46}
\]

\[
P_{1 f_1}(0) = 0, \tag{III.47}
\]

\[
\gamma_{duu_1}(0) = 0, \tag{III.48}
\]

\[
V(0) = 0, \tag{III.49}
\]

\[
V(E_c x) \geq 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{III.50}
\]

\[
V'(E_c x) \Delta f_c(\cdot) \leq \Gamma_c(x), \quad x \notin \mathcal{Z}_c, \tag{III.51}
\]

\[
V'(E_c x)[f_{c0}(x) - \frac{1}{2}G_c(x)R_{c2}G_c^T(x)V''(E_c x)] \geq \Gamma_c(x) \leq 0, \quad x \notin \mathcal{Z}_c, \quad x \neq 0, \tag{III.52}
\]

\[
P_{1 f_1}(x) \Delta f_d(\cdot) + \Delta f_d^T(\cdot)P_{1 f_1}(x) \geq \Gamma_{dx}(x) + \Gamma_{dxx}(x) \tag{III.53}
\]

\[
\leq \Gamma_{dx}(x) + \Gamma_{duu_1}(x) \phi_d(x) + \phi_d^T(\cdot) \Gamma_{duu_1}(x) \phi_d(x), \quad x \in \mathcal{Z}_d, \Delta f_d(\cdot) \in \mathcal{F}_d, \tag{III.53}
\]
\[ V(E_q x + f_{0 q}(x)) - V(E_q x) + P_{1 q}(x) \phi_0(x) + \phi_0(x)^T P_{2 q}(x) \phi_0(x) + \Gamma_{d,xx}(x) + \Gamma_{d,ux}(x) \phi_0(x) + \phi_0^T(x) \Gamma_{d,u,a_0}(x) \phi_0(x) \leq 0, \quad x \in \mathbb{Z}_x, \]

\[ V(E_q x + f_{0 q}(x) + G_q(x) u_q) = V(E_q x + f_{0 q}(x)) + P_{1 q}(x) u_q + u_q^T P_{2 q}(x) u_q, \]

\[ V(E_q x + f_{0 q}(x) + \Delta f_d(x) + G_q(x) u_q) - V(E_q x) = V(E_q x + f_{0 q}(x) + G_q(x) u_q) - V(E_q x) + P_{1 q}(x) \Delta f_d(x) + \Delta f_d^T(x) P_{1 q}^{-1}(x) \Delta f_d(x) \]

\[ \cdot P_{2 q}(x) \Delta f_d(x) + u_q^T P_{u,a_q}(x) \Delta f_d(x) + \Delta f_d^T(x) P_{u,a_q}(x) u_q, \]

\[ x \in \mathbb{Z}_x, \quad u_q \in \mathbb{R}^{m_q}, \Delta f_d(x) \in \mathcal{F}, \]

and

\[ V(E_q x) \to \infty \text{ as } \|x\| \to \infty. \quad (\text{III.61}) \]

Then the zero solution \( x(t) \equiv 0 \) to the closed-loop system

\[ E_q \dot{x}(t) = f_c(x(t)) + \Delta f_c(x(t)) + G_c(x(t)) \phi_c(x(t)), \quad x(0) = x_0, \quad x(t) \not\in \mathcal{Z}_x, \quad (\text{III.62}) \]

\[ E_q \Delta x(t) = f_d(x(t)) + \Delta f_d(x(t)) + G_d(x(t)) \phi_d(x(t)), \quad x(t) \in \mathcal{Z}_x, \quad (\text{III.63}) \]

is globally asymptotically stable for all \( (\Delta f_c, \Delta f_d) \in \mathcal{F} \) with the hybrid feedback control law

\[ \phi_c(x) = -\frac{1}{2} R_c^{-1}(x) G_c^T(x) V^{\prime}(E_q x), \quad x \not\in \mathcal{Z}_x, \quad \text{(III.64)} \]

\[ \phi_d(x) = -\frac{1}{2} (R_d(x) + P_d(x) + \Gamma_{d,u,a_d}(x))^{-1} \cdot (P_{1 q} + \Gamma_{d,xx}(x))^T(x), \quad x \in \mathcal{Z}_x, \quad \text{(III.65)} \]

and performance functional \( (\text{III.45}) \), satisfies

\[ J(E_q x_0, \phi_c(x(\cdot)), \phi_d(x(\cdot))) = \min_{(u_c(\cdot), u_d(\cdot)) \in \mathcal{C}(x_0)} J(E_q x_0, u_c(\cdot), u_d(\cdot)), x_0 \in \mathbb{R}^n, \quad \text{(III.66)} \]

where

\[ J(E_q x_0, u_c(\cdot), u_d(\cdot)) \triangleq \int_0^\infty \left[ L_c(E_q x(t), u_c(t)) + \Gamma_c(x(t), u_c(t)) dt + \sum_{k \in \mathbb{N}} [L_d(x(t_k), u_d(t_k)) + \Gamma_d(x(t_k), u_d(t_k))] \right] \quad \text{(III.67)} \]

and

\[ \Gamma_c(x, u_c) = \Gamma_{c,xx}(x), \quad x \not\in \mathcal{Z}_x, \quad \text{(III.68)} \]

\[ \Gamma_d(x, u_d) = \Gamma_{d,xx}(x) + \Gamma_{d,ux}(x) u_d + u_d^T \Gamma_{d,u,a_d}(x) u_d, \quad x \in \mathcal{Z}_x, \quad \text{(III.69)} \]

and where \((u_c(\cdot), u_d(\cdot))\) is an admissible control and \( x(t), t \geq 0 \), is a solution of \( (\text{III.41}), (\text{III.42}) \) with \((\Delta f_c, \Delta f_d) = (0, 0)\).

[3] and [7].

**Proof:** The result is a direct consequence of Theorem II with \( D = \mathbb{R}^n, \quad \mathcal{U}_c = \mathbb{R}^{m_c}, \quad \mathcal{U}_d = \mathbb{R}^{m_d}, \quad F_c(x, u_c) = f_{0 q}(x) + \Delta f_c(x) + G_c(x) u_c, \quad F_d(x, u_d) = f_{0 q}(x) + \Delta f_d(x) + G_d(x) u_d, \quad (\text{III.72}) \) given by \( (\text{III.43}), \quad (\text{III.68}) \) by \( (\text{III.63}) \), \( \Gamma_c(x, u_c) \) given by \( (\text{III.69}) \), for \( x \in \mathcal{Z}_x \). Specifically, with \( (\text{III.41})-(\text{III.44}), (\text{III.68}), \) and \( (\text{III.69}) \), the Hamiltonian have the form

\[ H_c(E_q x, u_c) = L_{c,1}(E_q x) + u_c^T R_c x \phi_c(x) + V'(E_q x)(f_{0 q}(x) + G_c(x) u_c) + \Gamma_{c,xx}(x), \quad x \not\in \mathcal{Z}_x, \quad u_c \in \mathcal{U}_c, \quad (\text{III.73}) \]

\[ H_d(E_q x, u_d) = L_{d,1}(E_q x) + u_d^T R_d x \phi_d(x) + V'(E_q x)(f_{0 q}(x) + G_d(x) u_d) - V(E_q x) + \Gamma_{d,xx}(x) + \Gamma_{d,ux}(x) u_d + u_d^T \Gamma_{d,u,a_d}(x) u_d, \quad x \in \mathcal{Z}_x, \quad u_d \in \mathcal{U}_d, \quad (\text{III.74}) \]

Now, the hybrid feedback control law \( (\text{III.64}), (\text{III.65}) \) is obtained by setting \( \frac{\partial H_c}{\partial u_c} = 0 \) and \( \frac{\partial H_d}{\partial u_d} = 0 \). With \( (\text{III.64}) \) and \( (\text{III.65}) \) it follows that \( (\text{III.51})-(\text{III.60}) \) imply \( (\text{II.10})-(\text{II.13}) \). Next, since \( V(\cdot) \) is \( C^1 \) and \( x = 0 \) is a local minimum of \( V(\cdot) \), it follows that \( V'(0) = 0 \) and, hence, since by assumption \( P_{1 q}(0) = 0 \) and \( \Gamma_{d,xx}(0) = 0 \), it follows that \( \phi_c(0) = 0 \) and \( \phi_d(0) = 0 \) which proves \( (\text{II.8}), (\text{II.9}) \). Next, with \( L_{c,1}(E_q x) \) and \( L_{d,1}(E_q x) \) given by \( (\text{III.70}) \) and \( (\text{III.71}) \), respectively, and \( \phi_c(x), \phi_d(x) \) given by \( (\text{III.64}) \) and \( (\text{III.65}) \), \( (\text{II.14}) \) and \( (\text{II.16}) \) hold. Finally, since

\[ H_c(E_q x, u_c) = H_c(E_q x, u_c) - H_c(E_q x, \phi_c(x)) \]

\[ = [u_c - \phi_c(x)]^T R_c(x) [u_c - \phi_c(x)], \quad x \not\in \mathcal{Z}_x, \quad u_c \in \mathcal{U}_c, \quad (\text{III.75}) \]

\[ H_d(E_q x, u_d) = H_d(E_q x, u_d) - H_d(E_q x, \phi_d(x)) \]

\[ = [u_d - \phi_d(x)]^T R_d(x) [u_d - \phi_d(x)], \quad x \in \mathcal{Z}_x, \quad u_d \in \mathcal{U}_d, \quad (\text{III.76}) \]

where \( R_c(x) > 0, \quad x \not\in \mathcal{Z}_x \), and \( R_d(x) + P_d(x) + \Gamma_{d,u,a_d}(x) > 0, \quad x \in \mathcal{Z}_x \), conditions \( (\text{II.15}) \) and \( (\text{II.17}) \) hold. The result now follows as a direct consequence of Theorem II. \( \square \)
IV. ROBUST NONLINEAR HYBRID CONTROL WITH POLYNOMIAL PERFORMANCE FUNCTIONAL

In this section we specialize the results of Section IV to linear singularly impulsive systems controlled by inverse optimal nonlinear hybrid controllers that minimize a derived polynomial cost functional. Specifically, assume \( \mathcal{F} = \mathcal{F}_c \times \mathcal{F}_d \) to be the set of uncertain systems, where

\[
\mathcal{F}_c = \{ (A_c + \Delta A_c)x + B_cu_c : x \in \mathbb{R}^n, A_c \in \mathbb{R}^{n \times n}, B_c \in \mathbb{R}^{n \times m_c}, \Delta A_c \in \Delta A_c \},
\]

\[
\mathcal{F}_d = \{ (A_d + \Delta A_d)x : x \in \mathbb{R}^n, A_d \in \mathbb{R}^{n \times n}, \Delta A_d \in \Delta A_d \},
\]

where \( \Delta A_c, \Delta A_d \subset \mathbb{R}^{n \times n} \) are given bounded uncertainty sets of uncertain perturbations \( \Delta A_c, \Delta A_d \) of the nominal asymptotically stable system matrices \( A_c, A_d \) such that 0 \( \notin \) \( \Delta A_c \) and 0 \( \notin \) \( \Delta A_d \). For simplicity of exposition, we also define \( \Delta A_c \times \Delta A_d \equiv \Delta \). For the results in this section we assume \( u_d(t) \equiv 0 \). Furthermore, let \( R_{1c} \in \mathbb{R}^n, R_{1d} \in \mathbb{N}^n, R_{2c} \in \mathbb{R}^{m_c}, \hat{R}_q, \hat{R}_q \in \mathbb{R}^n, q = 2, \ldots, r, \) be given, where \( r \) is a positive integer, and define \( S_c \equiv B_cR_{2c}^{-1}B_c^T \).

Consider the linear uncertain controlled singularly impulsive system

\[
E_c\dot{x}(t) = (A_c + \Delta A_c)x(t) + B_cu_c(t), \quad x(0) = x_0,
\]

\[
E_d\Delta x(t) = (A_d + \Delta A_d - E_d)x(t), \quad x(t) \in \mathbb{Z}_x,
\]

where \( u_c \) is admissible and \( (\Delta A_c, \Delta A_d) \in \Delta \). Let \( \Omega_c, \Omega_d : N_p \subseteq S^n \rightarrow \mathbb{N}^n, P \in N_p, \) be such that

\[
x^T(\Delta A^T_c PE_c + E^T_c P \Delta A_c)x \leq x^T \Omega_c(P)x,
\]

\[
x \notin \mathbb{Z}_x, \quad \Delta A_c \in \Delta A_c,
\]

\[
x^T(\Delta A^T_d PA_d + \Delta A^T_d PA_d + \Delta A^T_d PA_d)x \leq x^T \Omega_d(P)x, \quad x \in \mathbb{Z}_x, \quad \Delta A_d \in \Delta A_d.
\]

Assume there exist \( P \in \mathbb{P}^n \) and \( M_q \in \mathbb{N}^n, q = 2, \ldots, r, \) such that

\[
0 = x^T(A^T_c PE_c + E^T_c PA_c + E^T_c R_{1c}E_c + \Omega_c(P) - PS_cP)x, \quad x \notin \mathbb{Z}_x,
\]

\[
0 = x^T[(A_c - S_c P)^T M_q E_c + E^T_c M_q (A_c - S_c P) + \hat{R}_q x], \quad x \notin \mathbb{Z}_x, q = 2, \ldots, r,
\]

\[
0 = x^T(A^T_d PA_d - E^T_d PE_d + E^T_d R_{1d}E_d + \Omega_d(P))x, \quad x \in \mathbb{Z}_x,
\]

\[
0 = x^T(A^T_d M_q A_d - E^T_d M_q E_d + \hat{R}_q)x, \quad x \in \mathbb{Z}_x, q = 2, \ldots, r.
\]

Then the zero solution \( x(t) \equiv 0 \) of the uncertain closed-loop

system

\[
E_c\dot{x}(t) = (A_c + \Delta A_c)x(t) + B_c\phi_c(x(t)), \quad x(0) = x_0,
\]

\[
E_d\Delta x(t) = (A_d + \Delta A_d - E_d)x(t), \quad x(t) \in \mathbb{Z}_x,
\]

is globally asymptotically stable with the feedback control law

\[
\phi_c(x) = -R_{2c}^{-1}B_c^T(P + \sum_{q=2}^r (x^T E^T_c M_q E_c x)^{q-1} M_q) E_c x, \quad x \notin \mathbb{Z}_x
\]

and the performance functional (III.45) satisfies

\[
\sup_{(\Delta A_c, \Delta A_d) \in \Delta} J_{\Delta A_c, \Delta A_d}(E_c x_0, \phi_c(x_0)) \leq J(E_c x_0, \phi_c(x_0))
\]

\[
= x^T_0 E^T_c P E_c x_0 + \sum_{q=2}^r \frac{1}{q} (x^T_0 E^T_c M_q E_c x_0)^{q-1}, \quad x_0 \in \mathbb{R}^n,
\]

where

\[
\mathcal{J}(E_c x_0, \phi_c(\cdot)) \leq \int_0^\infty \left[ L_c(E_c x(t), u_c(t)) + \Gamma_c(\dot{x}(t), u_c(t)) \right] dt
\]

\[
+ \sum_{k \in N(0, \infty)} [L_d(E_d x(tk)) + \Gamma_d(x(tk))] dt
\]

and where \( u_c \) is admissible, and \( x(t), t \geq 0 \), is a solution to (IV.79), (IV.80) with \( (\Delta A_c, \Delta A_d) = (0, 0) \), and

\[
\Gamma_c(x, u_c) = x^T(\Omega_c(P) + \sum_{q=2}^r (x^T E^T_c M_q E_c x)^{q-1} \Omega_c(M_q)) E_c x,
\]

\[
x \notin \mathbb{Z}_x
\]

\[
\Gamma_d(x) = x^T \Omega_d(P) x + \sum_{q=2}^r \frac{1}{q} [(x^T \hat{R}_q x) \sum_{j=1}^q (x^T E^T_d M_q E_d x)^{j-1}]
\]

\[
\cdot [x^T(A^T_d M_q A_d + \Omega_d(x))x)^{q-j} - (x^T A^T_d M_q A_d x)^{q-j}], \quad x \in \mathbb{Z}_x,
\]

where \( u_c \) is admissible and \( (\Delta A_c, \Delta A_d) \in \Delta \). In addition, the performance functional (III.45), with \( R_{2c}(x) = R_{2c} \) and

\[
L_{1c}(E_c x) = x^T(E_c^T R_{1c} E_c + \sum_{q=2}^r (x^T E^T_c M_q E_c x)^{q-1} \hat{R}_q)
\]

\[
+ \sum_{q=2}^r (x^T E^T_c M_q E_c x)^{q-1} M_q) T S_c
\]

\[
\cdot [(x^T E^T_c M_q E_c x)^{q-1} M_q) x,
\]

\[
L_{1d}(E_d x) = x^T E^T_d R_{1d} E_d x + \sum_{q=2}^r \frac{1}{q} [(x^T \hat{R}_q x)
\]

\[
\cdot \sum_{j=1}^q (x^T E^T_d M_q E_d x)^{j-1}
\]

(IV.90)
is minimized in the sense that \( I(E_c, x_0, \phi_c(x(\cdot))) = \min_{u_c(\cdot) \in C(x_0)} J(E_c x_0, u_c(\cdot)), \quad x_0 \in \mathbb{R}^n \) \((IV.95)\) where \( C(x_0) \) is the set of asymptotically stabilizing controllers for the nominal system and \( x_0 \in \mathbb{R}^n \), \([3]\) and \([7]\).

Proof: The result is a direct consequence of Corollary III.

V. ROBUST NONLINEAR HYBRID CONTROL WITH MULTILINEAR PERFORMANCE FUNCTIONAL

Finally, we specialize the results of Section VI to linear singularly impulsive systems controlled by inverse optimal hybrid controllers that minimize a derived multilinear functional. First, however, we give several definitions involving multilinear forms. A scalar function \( \psi : \mathbb{R}^n \to \mathbb{R} \) is \( q \)-multilinear if \( q \) is a positive integer and \( \psi(x) \) is a linear combination of terms of the form \( x_{i_1}^{q_1}x_{i_2}^{q_2} \cdots x_{i_q}^{q_q} \), where \( i_j \) is a nonnegative integer for \( j = 1, \ldots, n \), and \( i_1 + i_2 + \cdots + i_n = q \). Furthermore, a \( q \)-multilinear function \( \psi(x) \) is nonnegative definite (resp., positive definite) if \( \psi(x) \geq 0 \) for all \( x \in \mathbb{R}^n \) (resp., \( \psi(x) > 0 \) for all nonzero \( x \in \mathbb{R}^n \)). Note that if \( q \) is odd then \( \psi(x) \) cannot be positive definite. If \( \psi(x) \) is a \( q \)-multilinear function then \( \psi(x) \) can be represented by means of Kronecker products, that is, \( \psi(x) \) is given by \( \psi(x) = \Psi x[x] \), where \( \Psi \in \mathbb{R}^{1 \times n^q} \) and \( x[x] \equiv x \otimes x \otimes \cdots \otimes x \) \((q \text{ times})\), where \( \otimes \) denotes Kronecker product. For the next result recall the definition of \( S_c \), let \( R_{1c} \in \mathbb{R}^n \), \( R_{1d} \in \mathbb{R}^n \), \( R_{2c} \in \mathbb{R}^m \), \( R_{2d} \in \mathbb{R}^m \), \( q = 2, \ldots, r \), be given, where \( N^{(2q,n)} \equiv \{ \Psi \in \mathbb{R}^{1 \times n^q} \mid \Psi x[x] \geq 0, \ x \in \mathbb{R}^n \} \), and define the repeated \((q \text{ times})\) Kronecker sum as \( \oplus A \equiv A \oplus A \oplus \cdots \oplus A \).

Consider the linear controlled singularly impulsive system \((IV.79), (IV.80)\). Assume there exist \( P \in \mathbb{R}^n \) and \( \tilde{P}_q \in N^{(2q,n)} \), \( q = 2, \ldots, r \), such that

\[
0 = x^T (A_c^T P A_c + E_c^T P A_c + E_c^T R_{1c} E_c - P B_c R_{2c}^{-1} B_c^T P) x, \quad x \notin Z_x, \quad q = 2, \ldots, r, \quad (V.96)
\]

\[
0 = x^T (P_q[\tilde{\Psi} (E_c^T A_x - S_c P)] + \tilde{R}_{2q}) x, \quad x \notin Z_x, \quad q = 2, \ldots, r, \quad (V.97)
\]

\[
0 = x^T (A_c^T P A_q - E_c^T P E_q + E_q^T R_{1d} E_q) x, \quad x \in Z_x, \quad q = 2, \ldots, r, \quad (V.98)
\]

\[
0 = x^T (P_q[\tilde{\Psi} (E_c^T A_q - E_c^T E_q)] + \tilde{R}_{2q}) x, \quad x \in Z_x, \quad q = 2, \ldots, r, \quad (V.99)
\]

Then the zero solution \( x(t) = 0 \) of the closed-loop system \((IV.79), (IV.80)\) is globally asymptotically stable with the feedback control law

\[
\phi_c(x) = -R_{2c}^{-1} B_c^T (P E_c x + \frac{1}{2} g^T (E_c x)), \quad x \notin Z_x, \quad (V.100)
\]

where \( g(x) \equiv \sum_{q=2}^{r} \tilde{P}_q E_c x[x]^{[2q]} \), and the performance functional \((III.45)\), with \( R_{2c}^{-1} = R_{2c}^{-1} \) and

\[
L_{1c}(E_c x) = x^T E_c R_{1c} E_c x + \sum_{q=2}^{r} \tilde{R}_{2q} E_c x[x]^{[2q]} + \frac{1}{4} g^T (E_c x) S_c g^T (E_c x), \quad (V.101)
\]

\[
L_{1d}(x) = x^T E_d^T R_{1d} E_d x + \sum_{q=2}^{r} \tilde{R}_{2q} E_d x[x]^{[2q]} \quad (V.102)
\]

is minimized in the sense that

\[
J(E_c x_0, \phi_c(x(\cdot))) = \min_{u_c(\cdot) \in C(x_0)} J(E_c x_0, u_c(\cdot)), \quad x_0 \in \mathbb{R}^n. \quad (V.103)
\]

Finally,

\[
J(E_c x_0, \phi_c(x(\cdot))) = x_0^T E_c^T P E_c x_0 + \sum_{q=2}^{r} \tilde{P}_q E_c x[x]^{[2q]}, \quad x_0 \in \mathbb{R}^n. \quad (V.104)
\]

[3] and [7].

Proof: The result is a direct consequence of Theorem II with \( f_c(x) = A_c x, \ f_d(x) = (A_d - E_d) x, \ G_c(x) = B_c, \ G_d(x) = 0, \ u_d = 0, \ R_{2c}(x) = R_{2c}, \ R_{2d}(x) = I, \) and

\[
V(E_c x, x) = x^T E_c^T P E_c x + \sum_{q=2}^{r} \tilde{P}_q E_c x[x]^{[2q]}, \quad (V.81), (V.97), \text{ and } (V.100)
\]

\[
V'(E_c x,x) = -\frac{1}{2} G_c(x) R_{2c}^{-1} G_c^T (x) V^T (E_c x) = -x^T E_c^T R_{1c} E_c x + \sum_{q=2}^{r} \tilde{R}_{2q} E_c x[x]^{[2q]} - \phi_c^T (x) R_{2c} \phi_c(x) - \frac{1}{4} g^T (E_c x) S_c g^T (E_c x), \quad (V.105)
\]

which implies (2.2.13). For \( x \in Z_x \) it follows from (IV.81), (IV.97), and (IV.100) that

\[
\Delta V(E_d x) = V(E_d x, f_d(x)) - V(E_d x) = -x^T E_d^T R_{1d} E_d x + \sum_{q=2}^{r} \tilde{R}_{2q} E_d x[x]^{[2q]}, \quad (V.99)
\]

which implies (2.2.14) with \( G_d(x) = 0 \). Finally, with \( u_d = 0 \), condition is automatically satisfied so that all the conditions of Corollary V are satisfied.

VI. CONCLUSION

In this paper we have developed optimal robust control and inverse optimal robust control results for the class of nonlinear uncertain singularly impulsive dynamical systems [5]. Results are based on Lyapunov and asymptotic stability theorems developed in [6], and results presented in [7].

VII. FUTURE WORK

Further work will specialize results of this paper to time-delay systems.

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