Extremal Problems
for Time Lag Parabolic Systems

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Abstract—Extremal problems for time lag parabolic systems are presented. An optimal boundary control problem for distributed parabolic systems in which constant time lags appear in the state equations and in the boundary conditions simultaneously is solved. Such equations constitute in a linear approximation a universal mathematical model for many processes of optimal heating. The time horizon is fixed. Making use of the Dubovicki-Milutin scheme, necessary and sufficient conditions of optimality for the Neumann problem with the quadratic performance functionals and constrained control are derived.

Keywords – Boundary control, parabolic systems; time lags.

I. INTRODUCTION

Extremal problems are now playing an ever-increasing role in applications of mathematical control theory. It has been discovered that notwithstanding the great diversity of these problems, they can be approached by a unified functional-analytic approach, first suggested by Dubovicki and Milutin. The general theory of extremal problems has been developed so intensely recently that its basic concepts may now be considered finalized.

Extremal problems were the object of mathematical research at the very earliest stages of the development of mathematics. The first results were then systematized and brought together under the heading of the calculus of variations with its innumerable applications to physics, control theory and mechanics.

The attention was devoted principally to the analysis of smooth functions and functionals defined over the entire space or restricted to some smooth manifold. The extremum conditions in this case are the Euler equations (with Lagrange multipliers in the case of constraints).

Independently of the calculus of variations, the theory of approximations was developed: the methods figuring in this theory, especially in the theory of Chebyshev approximations had a specialized nature.

Technological progress presented the calculus of variations with a new type of problem - the control of objects whose control parameters are varied in some closed set with boundary. Quite varied problems of this type were investigated by Pontryagin, Boltyanskii, Gamkrelidze and Mishchenko, who established a necessary condition for an extremum – the so-called Pontryagin maximum principle.

The nature of this condition and the form of the optimal solutions were so different from the classical theorems of the calculus of variations that popular – science writers began to speak of the advent of a “new” calculus of variations.

Something similar happened in the realm of extremal problems for functions of a finite number of variables. Economic necessity gave rise to the appearance of special methods for determining the extrema of smooth functions on closed domains with piecewise – smooth boundaries. The first results in this direction were obtained in 1939 by Kantorovich. This field of mathematics is now known as mathematical (nonlinear) programming.

These results and methods of mathematical programming were similar to the Pontryagin theory with obvious, but the subtle and elegant geometric technique of Boltyanskii and Pontryagin somewhat obscured the analytical content of the problem.

Finally, at the end of 1962 Dubovicki and Milutin found a necessary condition for an extremum, in the form of an equation set down in the language of functional analysis. They were able to derive, as special cases of this condition, almost all previously known necessary extremum conditions and thus to recover the lost theoretical unity of the calculus of variations.

The purpose of this paper is to show the use of the Dubovicki-Milutin method in solving optimal control problems for distributed parabolic systems.

As an example, an optimal boundary control problem for the system described by a linear partial differential equation of parabolic type in which constant time lags appear both in the state equation and in the boundary condition is considered. Such
The Dubovicki-Milutin method arises from the geometric form of the Hahn-Banach theorem (a theorem about the separation of convex sets).

We shall show it on the example.

Let us assume that

\[ E \] 

is a linear topological space, locally convex

\[ I(x) \] - a functional defined on \( E \)

\[ A_i, \ i = 1, 2, ..., n \] - sets in \( E \) with inner points which represent inequality constraints

\[ B \] - a set in \( E \) without inner points representing equality constraint.

We must find some conditions necessary for a local minimum of the functional \( I(x) \) on the set \( Q = \bigcap_{i=1}^{n} A_i \cap B, \) or find a point \( x_0 \in E \), so that \( I(x_0) = \min I(x) \), where \( U \) means a certain environment of the point \( x_0 \). We define the set

\[ A_0 = \{ x : I(x) < I(x_0) \}. \]

Then, we formulate the necessary condition of optimality as follows: in the environment of the local minimum point, the intersection of the system of sets (the set on which the functional attains smaller values than \( I(x_0) \) and the sets representing constraints) is empty, or \( \bigcap_{i=1}^{n} A_i \cap B = \emptyset \).

The condition \( \bigcap_{i=0}^{n} A_i \cap B = \emptyset \) is also equivalent to the one in which there are approximations of the sets \( A_i \) \( (i = 1, 2, ..., n) \) and \( B \) instead of \( A_i \), or \( B \) ones. These approximations are cones with the vertex in a point \( \{ 0 \} \).

We shall approximate the inequality constraints by the regular admissible cones \( RAC(A_i, x_0) \) \( (i = 1, 2, ..., n) \), the equality constraint by the regular tangent cone \( RTC(B, x_0) \) and for the performance functional we shall construct the regular improvement cone \( RFC(I, x_0) \).

Then we have the necessary condition of the optimality \( I(x) \) on the set \( Q = \bigcap_{i=1}^{n} A_i \cap B \) in the form of Euler-Lagrange’s equation

\[ \sum_{i=1}^{n} f_i = 0 \]

where

\[ f_i \] \( (i = 0, 1, ..., n + 1) \) - are linear, continuous functionals, all of them are not equal to zero at the same time and they belong to the adjoint cones

\[ f_i \in [RAC(A_i, x_0)]^*, i = 1, 2, ..., n \]

\[ f_{n+1} \in [RTC(B, x_0)]^*, f_0 \in [RFC(I, x_0)]^* \]

\[ \{ f_i \in [RAC(A_i, x_0)]^* \Rightarrow f_1(x) \geq 0 \ \forall \ x \in RAC(A_i, x_0) \}. \]

For convex problems, i.e. problems in which the constraints are convex sets and the performance functional is convex, the Euler-Lagrange equation is the necessary and sufficient condition of optimality, if only certain additional assumptions are fulfilled (so-called Slater’s condition).

Using the Dubovicki-Milutin theorem we shall derive the necessary and sufficient conditions of optimality for time lag parabolic systems with the quadratic performance functionals and the constrained control.

II. THE DUBOVICKI-MILUTIN METHOD [7]

The Dubovicki-Milutin method arises from the geometric form of the Hahn-Banach theorem (a theorem about the separation of convex sets).

We shall show it on the example.

Let us assume that

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\[ \{ f_i \in [RAC(A_i, x_0)]^* \Rightarrow f_1(x) \geq 0 \ \forall \ x \in RAC(A_i, x_0) \}. \]

For convex problems, i.e. problems in which the constraints are convex sets and the performance functional is convex, the Euler-Lagrange equation is the necessary and sufficient condition of optimality, if only certain additional assumptions are fulfilled (so-called Slater’s condition).

Using the Dubovicki-Milutin theorem we shall derive the necessary and sufficient conditions of optimality for time lag parabolic systems with the quadratic performance functionals and the constrained control.

III. PROBLEM FORMULATION. OPTIMALITY CONDITIONS

Now we formulate the control problem for the system described by the following parabolic equation:

\[ \frac{\partial y}{\partial t} + A(t)y + y(x, t - h) = u \quad (x, t) \in \Omega \times (0, T) \] (1)

\[ y(x, t') = \Phi_0(x, t') \quad (x, t') \in \Omega \times [-h, 0) \] (2)

\[ y(x, 0) = y_0(x) \quad x \in \Omega \] (3)

\[ \frac{\partial y}{\partial x_1} = y(x, t - h) + v \quad (x, t) \in \Gamma \times (0, T) \] (4)

\[ y(x, t') = \Psi_0(x, t') \quad (x, t') \in \Gamma \times [-h, 0) \] (5)

where \( \Omega \subset \mathbb{R}^n \) – a bounded, open set with boundary \( \Gamma \), which is a \( C^\infty \) manifold of dimension \( (n-1) \). Locally, \( \Omega \) is totally on one side of \( \Gamma \).

\[ y = y(x, t; v), u = u(x(t), v) = v(x, t), \quad Q = \Omega \times (0, T), \quad \bar{Q} = \Omega \times [0, T], \quad Q_0 = \Omega \times [-h, 0), \quad \Sigma = \Gamma \times (0, T), \quad \Sigma_0 = \Gamma \times [-h, 0) \]

\( h \) is a specified positive number representing a time lag,

\( \Phi_0 \) is an initial function defined on \( Q_0 \).

\( \Psi_0 \) is an initial function defined on \( \Sigma_0 \).

The parabolic operator \( D_t + A(t) \) in the state equation (1) satisfies the hypothesis of Lions and Magenes [9] and \( A(t) \) is given by

\[ A(t)y = - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x, t) \frac{\partial y(x, t)}{\partial x_j} \right) \] (6)

and the coefficients \( a_{ij} \) are real \( C^\infty \) functions on \( \bar{Q} \) (closure of \( Q \) satisfying the ellipticity condition

\[ \sum_{i,j=1}^{n} a_{ij}(x, t) \phi_i \phi_j \geq \sum_{i=1}^{n} \phi_i^2, \] (7)

\( v > 0, \forall (x, t) \in \bar{Q}, \forall \phi_i \in \mathbb{R} \).

The equation (1)–(5) constitutes a Neumann problem. Then the left-hand side of (4) is written in the form

\[ \frac{\partial y}{\partial y_A} = - \sum_{i,j=1}^{n} a_{ij}(x, t) \cos(n, x_i) \frac{\partial y(x, t)}{\partial x_j} = q(x, t) \] (8)
Then for $v \in L^2(\Sigma)$ it follows that for any $\epsilon > 0$ and $\nu \in L^2(\Sigma)$, where:

$$q(x, t) = y(x, t - h) + v(x, t)$$

(9)

The existence of a unique solution of the parabolic equation (1)-(5) for $v \in L^2(\Sigma)$ can be proved using a constructive method.

Then the following result is fulfilled [4]:

**Theorem 1:** Let $y_0, \Phi_0, \Psi_0, v,$ and $u$ be given with $y_0 \in H^1(\Omega), \Phi_0 \in H^2(\Gamma), \Psi_0 \in L^2(\Sigma), v \in L^2(\Sigma)$ and $u \in H^{1, 1}(Q)$. Then, there exists a unique solution $y \in H^2(\Sigma)$ for the mixed initial-boundary value problem (1)-(5). Moreover, $y(j, \cdot) \in H^2(\Omega)$ for $j = 1, ..., K$.

In this paper we shall consider the optimal boundary control problem i.e. $v \in L^2(\Sigma)$.

Let us denote by $Y = H^2(\Sigma)$ the space of states and by $U = L^2(\Sigma)$ the space of controls. The time horizon $T$ is fixed in our problem.

The performance functional is given by

$$I(y, v) = \lambda_1 \int_\Omega |y(x, t; v) - z_d|^2 \, dx + \lambda_2 \int_0^T (Nv) \, dt$$

(10)

where: $\lambda_1 \geq 0$ and $\lambda_1 + \lambda_2 > 0$; $z_d$ is a given element in $L^2(Q)$ and $N$ is a strictly positive linear operator on $L^2(\Sigma)$ into $L^2(\Sigma)$.

From the Theorem 1 [4] it follows that for any $v \in U_{ad}$ the cost function (10) is well-defined since $y(v) \in H^2(\Sigma) \subset L^2(\Sigma)$. We assume the following constraints on control: $v \in U_{ad}$ is a closed, convex set with non-empty interior, a subset of $U$. (11)

The optimal control problem (1)-(5), (10), (11) will be solved as the optimization one in which the function $v$ is the unknown function.

Making use of the Dubovicki-Milutin theorem [7] we shall derive the necessary and sufficient conditions of optimality for the optimization problem (1)-(5), (10), (11).

The solution of the stated optimal control problem is equivalent to seeking a pair $(y^*, v^*) \in E = H^2(\Sigma) \times L^2(\Sigma)$ which satisfies the equation (1) – (5) and minimizing the performance functional (10) with the constraints on control (11).

We formulate the necessary and sufficient conditions of the optimality in the form of Theorem 2.

**Theorem 2:** The solution of the optimization problem (1)-(5), (10), (11) exists and it is unique with the assumptions mentioned above; the necessary and sufficient conditions of the optimality are characterized by the following system of partial differential equations and inequalities:

State equation

$$\frac{\partial y^*}{\partial t} + A(t)y^* + y^0(x, t - h) = u \quad (x, t) \in \Omega \times (0, T)$$

$$y^0(x, t') = \Phi_0(x, t') \quad (x, t') \in \Omega \times [-h, 0)$$

$$y^0(x, 0) = \Phi_0(x) \quad x \in \Omega$$

$$\frac{\partial y^*}{\partial x_i} = y^0(x, t - h) + v^0 \quad (x, y) \in \Gamma \times (0, T)$$

$$y^0(x, t') = \Psi_0(x, t') \quad (x, t') \in \Gamma \times [-h, 0)$$

(12)

Adjoint equations

$$- \frac{\partial p}{\partial t} + A^*(t)p + p(x, t + h) = \lambda_1(y^* - z_d) \quad (x, t) \in \Omega \times (0, T - h)$$

$$- \frac{\partial p}{\partial t} + A^*(t)p = \lambda_1(y^* - z_d) \quad (x, t) \in \Omega \times (T - h, T)$$

$$\frac{\partial p}{\partial y_A} = p(x, t + h) \quad (x, y) \in \Gamma \times (0, T - h)$$

$$\frac{\partial p}{\partial y_A} = 0 \quad (x, t) \in \Gamma \times (T - h, T)$$

$$p(x, T) = 0 \quad x \in \Omega$$

(13)

Maximum condition

$$\int_0^T \int_\Omega (p + \lambda_2 Nv^*)(v - v^0) \, dx \, dt \geq 0 \quad \forall v \in U_{ad}$$

(14)

We can also notice that

$$\frac{\partial p}{\partial y_A} = \sum_{i,j=1}^n a_{ij}(x, t) \cos(n_j, x_i) \frac{\partial p}{\partial x_j}$$

(15)

Outline of the proof

According to the Dubovicki-Milutin theorem we approximate the set representing the inequality constraints by the regular admissible cone, the equality constraint by the regular tangent cone and the performance functional by the regular improvement cone.

a) Analysis of the equality constraint

The set $Q$, representing the equality constraint has the form
\[ Q_1 = \begin{cases} 
\frac{dy}{dt} + A(t)y + y(x, t - h) = u \quad (x, t) \in \Omega \times (0, T) \\
y(x, t') = \Phi_0(x, t') \quad (x, t') \in \Omega \times [-h, 0) \\
y(x, 0) = y_p(x) \quad x \in \Omega \\
\frac{\partial y}{\partial y_A} = y(x, t - h) + v \quad (x, y) \in \Gamma \times (0, T) \\
y(x, t') = \Psi_0(x, t') \quad (x, t') \in \Gamma \times [-h, 0) \\
y(x, u) \in E 
\end{cases} \] 

We construct the regular tangent cone of the set \( Q_1 \) using the Ljustenik theorem (Theorem 9.1 [2]). For this purpose we define the operator \( P \) in the form 
\[ P(y, v) = \left( \frac{\partial y}{\partial t} + Ay + y(x, t - h) - u, y(x, t') - \Phi_0(x, t'), 
\quad y(x, 0) - y_o(x), \frac{\partial y}{\partial y_A} - y(x, t - h), 
\quad -v, y(x, t') - \Psi_0(x, t') \right) \] 

The operator \( P \) is the mapping from the space \( H^{33}(Q) \times L^2(\Sigma) \) into the space \( H^{32}(\Omega) \times H^{32}(Q_0) \times H^{21}(\Omega) \times L^2(\Sigma) \times L^2(\Sigma_0) \). 

The Fréchet differential of the operator \( P \) can be written in the following from: 
\[ P'(y^o, v^o)(\vec{y}, \vec{v}) = \left( \frac{\partial \vec{y}}{\partial t} + A\vec{y} + \vec{y}(x, t - h), \vec{y}|_{t=0}(x, t'), 
\quad \vec{y}(x, 0), \frac{\partial \vec{y}}{\partial y_A} - \vec{y}(x, t - h), 
\quad -\vec{v}, \vec{y}|_{t=0}(x, t') \right) \] 

Really, \( \frac{\partial}{\partial t} \) (Theorem 2.8 [10]), \( A(t) \) (Theorem 2.1 [8]) and \( \frac{\partial}{\partial y_A} \) (Theorem 2.3 [9]) are linear and bounded mappings. 

Using Theorem 1 [4] we can prove that \( P' \) is the operator “one to one” from the space \( H^{33}(Q) \times L^2(\Sigma) \) onto \( H^{32}(\Omega) \times H^{32}(Q_0) \times H^{21}(\Omega) \times L^2(\Sigma) \times L^2(\Sigma_0) \). 

Considering that the assumptions of Ljustenik’s theorem are fulfilled, we can write down the regular tangent cone for the set \( Q_1 \) in a point \( (y^o, v^o) \) in the form 
\[ \text{RTC}(Q_1, (y^o, v^o)) = \{(\vec{y}, \vec{v}) \in E; P'(y^o, v^o)(\vec{y}, \vec{v}) = 0\} \] 

It is easy to notice that it is a subspace. Therefore, using Theorem 10.1 [2] we know the form of the functional belonging to the adjoint cone 
\[ f_1(\vec{y}, \vec{v}) = 0 \quad \forall (\vec{y}, \vec{v}) \in \text{RTC}(Q_1, (y^o, v^o)) \] 

\[ b) \text{Analysis of the constraint on controls} \]

The set \( Q_2 = Y \times U_{ad} \) representing the inequality constraints is a closed and convex one with non-empty interior in the space \( E \). 

Using Theorem 10.5 [2] we find the functional belonging to the adjoint regular admissible cone, i.e. 
\[ f_2(\vec{y}, \vec{v}) \in [\text{RAC}(Q_2, (y^o, v^o))]^* \]

We can note if \( E_1, E_2 \) are two linear topological spaces, then the adjoint space to \( E = E_1 \times E_2 \) has the form 
\[ E^* = \{ f = (f_1, f_2); f_1 \in E_1, f_2 \in E_2 \} \] 
and 
\[ f(x) = f_1(x_1) + f_2(x_2) \]
So we note the functional \( f_2(\vec{y}, \vec{v}) \) as follows 
\[ f_2(\vec{y}, \vec{v}) = f_1(\vec{y}) + f_2(\vec{v}) \] 

where 
\[ f_2(\vec{y}) = 0 \quad \forall \vec{y} \in Y \quad (\text{Theorem 10.1}[2]) \]

\[ f_2(\vec{v}) \] is a support functional to the set \( U_{ad} \) in a point \( v_o \) (Theorem 10.5 [2]). 

\[ c) \text{Analysis of the performance functional} \]

Using Theorem 7.5 [2] we find the regular improvement cone of the performance functional (10) 
\[ R\text{FC}(I, (y^o, v^o)) = \{(\vec{y}, \vec{v}) \in E; I'(y^o, v^o)(\vec{y}, \vec{v}) < 0\} \]
where: \( I'(y^o, v^o)(\vec{y}, \vec{v}) \) is the Fréchet differential of the performance functional (10) and it can be written as 
\[ I'(y^o, v^o)(\vec{y}, \vec{v}) = 2\lambda_o\lambda_1 \int_0^T (y^o - z_d)\vec{y}dxdt + 2\lambda_o\lambda_2 \int_0^T (Nv^o)\vec{v}dl'dt \]

On the basis of Theorem 10.2 [2] we find the functional belonging to the adjoint regular improvement cone, which has the form 
\[ f_3(\vec{y}, \vec{v}) = -\lambda_o\lambda_1 \int_0^T (y^o - z_d)\vec{y}dxdt - \lambda_o\lambda_2 \int_0^T (Nv^o)\vec{v}dl'dt \]

where: \( \lambda_o > 0 \). 

\[ d) \text{Analysis of Euler-Lagrange’s equation} \]

The Euler-Lagrange equation for our optimization problem has the form
\[
\sum_{i=1}^{3} f_i = 0 \quad (24)
\]

Let \( p(x, t) \) be the solution of (13) for \( (y^o, v^o) \).
Let us denote by \( \tilde{y} \) the solution of \( P'(\tilde{y}, \tilde{v}) = 0 \) for any fixed \( \tilde{v} \).
Then taking into account (20), (21) and (23) we can express (24) in the form

\[
f_2'(\tilde{v}) = \lambda_0 \lambda_1 \int_{Q} (y^o - z_d)\tilde{y}dxdt
+ \lambda_0 \lambda_2 \int_{\Gamma} (Nv^o) \tilde{v}d\Gamma dt \quad (25)
\]

\[\forall (\tilde{y}, \tilde{v}) \in RTC(Q, (y^o, v^o)).\]

We transform the first component of the right-hand side of (25) introducing the adjoint variable by adjoint equations (13).

After transformations we get

\[
\lambda_0 \lambda_1 \int_{Q} (y^o - z_d)\tilde{y}dxdt = ... \lambda_0 \int_{\Gamma} p \tilde{v}d\Gamma dt \quad (26)
\]

Substituting (26) into (25) gives

\[
f_2'(\tilde{v}) = \lambda_0 \int_{\Gamma} \left(p + \lambda_2 Nv^o\right) \tilde{v}d\Gamma dt \quad (27)
\]

Using the definition of the support functional [2] and dividing both members of the obtained inequality by \( \lambda_0 \), we finally get

\[
\int_{\Gamma} \left(p + \lambda_2 Nv^o\right) (v - v^o) d\Gamma dt \geq 0 \quad \forall v \in U_{ad} \quad (28)
\]

The last inequality is equivalent to the maximum condition (14).

In order to prove the sufficiency of the derived conditions of the optimality we use the fact that constraints and the performance functional are convex and that the Slater’s condition is satisfied (Theorem 15.3 [2]). Then, there exists a point \((\tilde{y}, \tilde{v}) \in intQ_2\) such that \((\tilde{y}, \tilde{v}) \in Q_1\).

This fact follows immediately from existence of non-empty interior of the set \(Q_2\) and from the existence of the solution of the equation (1)-(5) as well.

The uniqueness of the optimal control follows from the strict convexity of the performance functional (10).

This last remark finishes the proof of Theorem 2.

One may also consider analogous optimal control problem with the performance functional

\[I(y, v) = \lambda_1 \int_{\mathcal{L}} |y(v)|_L - z_{zd}|^2 d\Gamma dt
+ \lambda_2 \int_{\Gamma} (Nv) v_d d\Gamma dt \quad (29)\]

where: \(z_{zd}\) is a given element in \(L^2(\Sigma)\).

From the Theorem 1 [4] and the trace theorem [9] for each \( v \in L^2(\Sigma) \), there exists a unique solution \( y \in H^{3,3}(Q) \) with \( y|_{\Gamma} \in L^2(\Sigma) \). Thus \( I(y, v) \) is well-defined. Then the solution of the formulated optimal control problem is equivalent to seeking a pair \((y^o, v^o) \in E = H^{3,3}(Q) \times L^2(\Sigma)\) which satisfies the equation (1)-(5) and minimizing the cost function (29) with the constraints on control (11).

We can prove the following theorem:

**Theorem 3:** The solution of the optimization problem (1)-(5), (29), (11) exists and it is unique with the assumptions mentioned above; the necessary and sufficient conditions of the optimality are characterized by the following system of partial differential equations and inequalities:

**State equation** (12)

**Adjoint equations**

\[
\begin{align*}
\frac{\partial p}{\partial t} + A'(t) p + p(x, t + h) &= 0 \\
&\quad (x, t) \in \Omega \times (0, T - h) \\
\frac{\partial p}{\partial t} + A'(t) p &= 0 \\
&\quad (x, t) \in \Omega \times (T - h, T) \\
\frac{\partial p}{\partial \gamma_{A^*}} &= p(x, t + h) + \lambda_1 (y^o - z_{zd}) \\
&\quad (x, y) \in \Gamma \times (0, T - h) \\
\frac{\partial p}{\partial \gamma_{A^*}} &= \lambda_1 (y^o - z_{zd}) \\
&\quad (x, t) \in \Gamma \times (T - h, T) \\
p(x, T) &= 0 \\
&\quad x \in \Omega
\end{align*}
\]

**Maximum condition**

\[
\int_{\Gamma} \left(p + \lambda_2 Nv^o\right) (v - v^o) d\Gamma dt \geq 0 \quad \forall v \in U_{ad} \quad (30)
\]

The idea of the proof of the Theorem 3 is the same as in the case of the Theorem 2.

We must notice that the conditions of optimality derived above (Theorems 2 and 3) allow us to obtain an analytical formula for the optimal control in particular cases only (e.g. there are no constraints on control). This results from the following: the determining of the function \(p(v^o)\) in the maximum condition from the adjoint equation is possible if and only if we know \(y^o\) which corresponds to the control \(v^o\). These mutual connections make the practical use of the derived optimization formulas difficult. Therefore we resign from the exact determining of the optimal control and we use approximation methods [4].
In the case of performance functionals (10) and (29) with \( \lambda_1 > 0 \) and \( \lambda_2 = 0 \), the optimal control problem reduces to the minimizing of the functional on a closed and convex subset in a Hilbert space. Then, the optimization problem is equivalent to a quadratic programming one (Refs. [4], [5], [6]) which can be solved by the use of the well-known algorithms, e.g. Gilbert’s algorithm to optimal control problem for a parabolic system with boundary condition involving a time lag is presented in [6]. Using the Gilbert’s algorithm, a one-dimensional numerical example of the plasma control process is solved [6].

**IV. CONCLUSIONS**

The derived conditions of the optimality (Theorems 2 and 3) are original from the point of view of application of the Dubovicki-Milutin theorem [7] in solving optimal control problems for time lag parabolic systems with the Neumann boundary conditions. The obtained optimization theorems (Theorems 2 and 3) demand the assumption dealing with the non-empty interior of the set \( \mathcal{Q}_2 \) representing the inequality constraints. Therefore we approximate the set \( \mathcal{Q}_2 \) by the regular admissible cone, (if \( \text{int} \ \mathcal{Q}_2 = \emptyset \) then this cone does not exist) [3].

It is worth mentioning that the obtained results can be reinforced by omitting the assumption concerning the non-empty interior of the set \( \mathcal{Q}_2 \) and utilizing the fact that the equality constraints in the form of the state equations are “decoupling”. The optimal control problem reduces to seeking \( v^0 \in \mathcal{Q}_2^1 \) and minimizing the performance index \( I(v) \). Then, we approximate the set \( \mathcal{Q}_2^1 \) representing the inequality constraints by the regular tangent cone and for the performance index \( I(v) \) we construct the regular improvement cone [3].

The proposed methodology based on the Dubovicki-Milutin scheme can be presented on a specific case study concerning plasma control process described by partial differential equations of the parabolic type in which time lags appear in the integral form both in the state equations and in the Neumann boundary conditions.

The same procedure (methodology) can be applied to solving optimal boundary control problems for time lag parabolic systems with the free final time.

An interesting possible future research direction may consist in formulation of extremal problems for advanced modern control strategies, for example, event-based control [11].

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