Multiple Stationary Solutions to the Extremum Seeking Control Problem

Olle Trollberg and Elling W. Jacobsen

***Abstract*— Extremum seeking control was originally proposed for adaptive optimization of static systems and later extended to Hammerstein and Wiener systems. More recently, stability and convergence results were presented also for general type dynamic systems with a focus on the local behavior around the optimum and under assumptions of relatively slow gradient estimation and control. In this paper we derive properties characterizing any stationary solution of the extremum seeking control scheme, i.e., we do not restrict ourselves to solutions close to optimum and allow for any frequency in the sinusoidal perturbation based gradient estimation scheme. By considering the linear properties around a stationary solution of the system, we show that stationary solutions are characterized by either a zero gradient or a phase lag condition. The former condition is satisfied at the optimum only for systems in which the zero gradient at the optimum is due to a static nonlinearity. The phase lag condition is shown to be satisfied close to the optimum for low frequency excitations, but can also be satisfied at solutions arbitrarily far from the optimum. The results imply that the extremum seeking control scheme applied to general type dynamic systems can have multiple stable stationary solutions of which some are sub-optimal and potentially far removed from the optimum. For illustration we consider extremum seeking control of a tubular bioreactor, displaying a maximum yield, and show that the closed-loop has two saddle-node bifurcations resulting in a total of three possible stationary solutions for some perturbation frequencies. A stable sub-optimal solution, with a yield less than 10% of the optimal yield, exists even with relatively slow gradient estimation.**

I. Introduction

Extremum seeking control (ESC) is a classic adaptive control technique used to achieve and maintain optimal operating conditions even for complex processes with unknown input-output mappings. The classic approach to ESC is to employ a perturbation based metod for estimating the gradient and then combine this with feedback to force the gradient to zero. The use of feedback increases the robustness of the scheme by suppressing the effects of uncertainty and disturbances. Even some of the earliest descriptions of ESC were in principle based on this combination of perturbation based estimation and feedback, e.g., Leblanc (1922)[1]. Initially the method was derived for purely static systems, but in the 50s and 60s linear dynamics were added to yield models of Hammerstein or Wiener type. See e.g., [2], [3] for reviews. However, stability issues were largely neglected in the early works on ESC.

It was only at the beginning of this century that a rigorous local stability analysis of ESC applied to general dynamic systems was presented. Krstić and Wang [4] employ averaging and singular perturbation analysis of the system in the vicinity of the optimum to show that the ESC will converge to a stationary solution close to optimum under certain well defined conditions. The result is only local, and furthermore, employment of averaging combined with singular perturbations implies that it is necessary to assume that the feedback is slow relative to the perturbation based estimation, and that the perturbations again be slow relative to the open-loop dynamics of the plant. Tan et al. [5] extend the results to allow for semi-global stability analysis, still relying upon averaging and singular perturbations. Recently, Moase and Manzie [6] consider the problem of convergence from arbitrary initial conditions using arbitrarily fast gradient estimation and feedback. However, their method is only applicable to Hammerstein systems.

Krstić and Wang [4] also find that, for the case of general dynamic models, the ESC will locally converge to a solution deviating somewhat from the optimum, with the error being proportional to the square of the amplitude of the perturbation signal. Chioua et al. [7] consider the impact of the perturbation frequency and show that the error also will be proportional to the square of the frequency. Similar to the results of Krstić and Wang, their result is based on a local analysis around the optimum.

In this work we consider ESC of general dynamic systems, but rather than focus on the local properties around the optimum our aim is to characterize the properties of *any* stationary solution of the ESC scheme. Furthermore, we allow any frequency in the perturbation signal and put no restrictions on the bandwidth of the feedback. Our motivation for this work comes from a simulation study on ESC of the CANON process, a complex biological process for ammonium removal in wastewater [8]. This process has a sharp optimum in terms of removed ammonium with dissolved oxygen as the input, but the optimal conditions are unknown and furthermore varying with the quality of the incoming water. Thus, ESC is an obvious choice for this process. However, the CANON process involves biofilm transport and growth, something which makes the open-loop dynamics exceedingly slow and hence employing gradient estimation significantly slower than the process time-constant is not practical. Based on simulation studies we found that the ESC could move the CANON process close to optimum even with high perturbation frequencies and relatively fast feedback. But, we also found that for some initial condition the ESC could convergence to stationary solutions far from the optimum. Thus, we detected sub-optimal solutions as well as existence of multiple stable stationary points. The results presented in this paper serve to explain these observations.

KTH Royal Institute of Technology, School of Electrical Engineering, Automatic Control Lab, Stockholm 100 44, Sweden (Email: olletr@kth.se, jacobsen@kth.se)

We start the paper by briefly describing the ESC algorithm with periodic excitation for gradient estimation. Expressions for the stationary solutions of the ESC in terms of the local linear frequency responses are then derived. Based on this we characterize the stationary solutions in terms of amplitude and phase lag of the open-loop frequency response. The results show that there may be two distinct properties that characterize the stationary solutions of ESC, one related to the gradient and one related to the phase-lag of the system at the excitation frequency. To shed some light on the relationship between the derived characteristics and properties of a general dynamic system around an extremum point, we present some results on the dynamics of systems with steady-state input multiplicity. We also present a simple stability analysis of the ESC to show that both type of solutions, satisfying either the gradient or the phase lag condition, can be locally stable. Finally, we illustrate the results with a simple example involving maximization of the yield in an tubular isothermal bioreactor with plug flow. As shown, the reactor with ESC displays multiple stable stationary solutions even for relatively low excitation frequencies in the gradient estimation. We finally discuss some possible remedies to avoid sub-optimal solutions to the ESC.

II. Extremum Seeking Control with Periodic Excitation

The principal idea behind extremum seeking control is to use gradient feedback to bring a process to the maximum or minimum, corresponding to the zero gradient point, of the input-output map in which the output represents the objective function and the input is the main control variable. There exists several approaches to ESC, e.g., based on sliding mode [9] or numerical optimization methods [10]. In this paper we consider the classical and much studied variant based on sinusoidal perturbations [2], [11], [3]. The corresponding ESC loop is outlined in Fig. 1, which also defines the various signals of the scheme. An important motivation for choosing this particular scheme is that it is model independent and also relatively simple to implement.

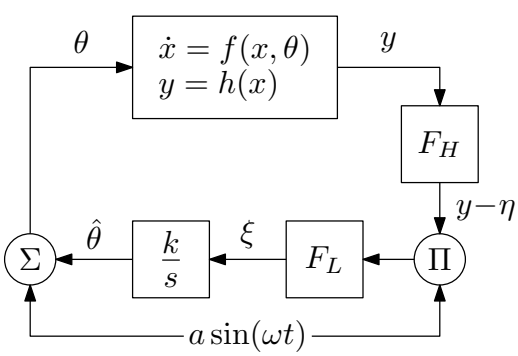

Fig. 1. Structure of the ESC system.

We consider the process to be described by a general set of nonlinear differential equations combined with a nonlinear state-to-output map. It is assumed that the system is asymptotically stable for all inputs θ and can be described by a state space model of the form

$$\begin{aligned} \dot{x} &= f(x,\theta) \\ y &= h(x) \end{aligned} \tag{1}$$

The assumption of asymptotic stability can easily be relaxed by introducing a stabilizing feedback control law. The functions $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are assumed to be sufficiently smooth such that all necessary derivatives exist. Furthermore, we assume that there exist a sufficiently smooth function $l : \mathbb{R} \to \mathbb{R}^n$ such that

$$0 = f(x,\theta)$$

if and only if

$$x = l(\theta)$$

The assumptions above imply that the stationary solutions of (1) are parametrized by θ and that the composite function

$$h \circ l : \mathbb{R} \to \mathbb{R} \tag{2}$$

exists and is sufficiently smooth. The function (2) is the steady-state map between θ and y, and as such, it is the function we want to optimize by employing ESC. We will assume that (2) has an extremum which is either a maximum or a minimum.

The addition of the sinusoid in Fig. 1 is motivated by the fact that the product of the sinusoid itself and the system response to the sinusoid will have a DC component which is proportional to the local gradient of the input-output map $h \circ l$, provided the system acts as a static map. The purpose of the high-pass filter F_H is to remove the DC component from the process response, while the low-pass filter F_L serves to retain only the DC component of the predicted gradient.

We next consider deriving the characteristics of the stationary solutions to the ESC as outlined above.

III. Stationary solutions of the ESC scheme

Consider the system given in (1) controlled by an ESC-loop as shown in Fig. 1. We are interested in determining the stationary solutions of this loop, here taken to be the solutions for which the control $\hat{\theta}$ is a constant. Clearly, this implies that $\xi(t) = 0$ for the solutions considered. With a constant $\hat{\theta}(t) = \bar{\theta}$, the input to the process becomes

$$\theta(t) = \bar{\theta} + a\sin(\omega t)$$

This input will in turn yield a stationary response in the process output $y(t)$ which is composed of a DC component, resulting from $\bar{\theta}$, combined with the frequency response for $a\sin(\omega t)$. If we assume that the amplitude of the sinusoid a is small, then the frequency response can be described by the transfer-function $G(s)$ obtained by linearizing the process around the steady-state corresponding to $\theta = \bar{\theta}$, i.e.,

$$y(t) = h \circ l(\bar{\theta}) + |G(i\omega)| a \sin(\omega t + \arg(G(i\omega)))$$

The presence of the high-pass filter F_H will effectively remove the DC component of y, resulting in the response

$$y(t) - \eta(t) = |G(i\omega)||F_H(i\omega)| a \sin(\omega t + \varphi)$$

where $\varphi = \arg(G(i\omega)) + \arg(F_H(i\omega))$ is the combined phase lag of the system and the high-pass filter. The signal $y - \eta$ is "demodulated" by multiplication with $a\sin(\omega t)$ to yield

$$(y(t) - \eta(t))a\sin(\omega t) = |G(i\omega)||F_H(i\omega)|a^2\sin(\omega t + \varphi)\sin(\omega t).$$

The trigonometric identity

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left(\cos(\alpha - \beta) - \cos(\alpha + \beta)\right)$$

yields

$$(y(t) - \eta(t))a\sin(\omega t) = \frac{a^2}{2}|G(i\omega)||F_H(i\omega)|(\cos(\varphi) - \cos(2\omega t + \varphi))$$

Note that the demodulated signal consists of a DC component and a sinusoidal component with twice the excitation frequency. Low-pass filtering the demodulated signal yields

$$\xi = \frac{a^2}{2}|F_L(0)||G(i\omega)||F_H(i\omega)|\cos(\varphi) - \frac{a^2}{2}|F_L(i2\omega)||G(i\omega)||F_H(i\omega)|\cos(2\omega t + \varphi + \arg(F_L(i2\omega)))$$

The low pass filter is assumed to effectively filter out the high frequency component, i.e.,

$$|F_L(i2\omega)| = 0, \tag{3}$$

which yields

$$\xi = \frac{a^2}{2}|F_L(0)||G(i\omega)||F_H(i\omega)|\cos(\varphi). \tag{4}$$

Since $\xi = 0$ is required to yield a constant $\hat{\theta}$, it follows that we for stationarity require

$$\frac{a^2}{2}|F_L(0)||G(i\omega)||F_H(i\omega)|\cos(\varphi) = 0 \tag{5}$$

Clearly, $\frac{a^2}{2}|F_L(0)||F_H(i\omega)| > 0$, so the only possibility for (5) to be true is if either

$$|G(i\omega)| = 0 \tag{6}$$

or

$$\cos(\varphi) = 0 \Rightarrow \varphi = \frac{\pi}{2} + n\pi,\ n = 0, 1, 2, \ldots \tag{7}$$

From the analysis above we draw the conclusion that the stationary solutions are characterized either by the system being output invariant with $|G(i\omega)| = 0\ \forall\omega$ or the phase lag fulfilling (7). Since these criteria can be fulfilled irrespective of the optimality conditions, there may exist stationary solutions completely unrelated to the optimum. Furthermore, the phase lag φ at the frequency ω can in principle vary with the input θ in such a way that (7) can be fulfilled for several different stationary points. Some systems could therefore have multiple stationary solutions for a single excitation frequency.

The derivation above is not strict since it depends on (3), a condition that no filters actually fulfill. However, if we consider the full ESC-loop, it is clear that high-frequency components will be attenuated not only by the low-pass filter but also by the integrator and typically by the process itself as well. Furthermore, if we consider the average of ξ over one period $T = 2\pi/\omega$ we get

$$\frac{1}{T}\int_0^T \xi dt = \frac{1}{T}\frac{a^2}{2}|G(i\omega)||F_H(i\omega)|\int_0^T (|F_L(0)|\cos(\varphi) - |F_L(i2\omega)|\cos(2\omega t + \varphi + \arg(F_L(i2\omega))))dt =$$

$$\frac{1}{T}\frac{a^2}{2}|G(i\omega)||F_H(i\omega)|\left(\int_0^T |F_L(0)|\cos(\varphi)dt - \underbrace{\int_0^T |F_L(i2\omega)|\cos(2\omega t + \varphi + \arg(F_L(i2\omega)))dt}_{=0}\right) =$$

$$\frac{a^2}{2}|F_L(0)||G(i\omega)||F_H(i\omega)|\cos(\varphi).$$

This corresponds exactly to what is assumed in (3), i.e., the assumption is equivalent to studying the average behaviour of the system which makes sense since we are interested in stationary solutions. If the assumption (3) is not made, then $\hat{\theta}$ would not be constant for stationary solutions and we would instead have to consider limit-cycles of small amplitude.

A. Relation of stationary solutions to optimality

From the above we find that stationary solutions of the ESC scheme either satisfy the amplitude condition

$$|G(i\omega)| = 0$$

or the phase lag condition

$$\varphi = \frac{\pi}{2} + n\pi$$

To better understand how these conditions relate to properties of a general dynamic system at the optimum, it is interesting to consider the dynamic properties of systems with an extremum in the input-output mapping, i.e., systems with steady-state input multiplicity. Steady state input multiplicity is a property that all systems viable for ESC exhibits, i.e., there are multiple inputs yielding the same output at steady-state due to the existence of a maximum or minimum. Such systems have previously been shown to possess certain dynamical properties [12] that are relevant for the stationary solutions of the ESC as derived above.

Let

$$G(s) = K\frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n}, \quad n \geq m$$

be the transfer-function from input u to output y of (1) linearized about a steady-state solution $\bar{x} = l(\bar{\theta})$. Then the stationary gain is given by

$$G(0) = K\frac{b_0}{a_0} = (h \circ l)'(\bar{\theta})$$

Let θ^* be the value for which $h \circ l$ achieves its extremum. Then it follows that $(h \circ l)'$ switches sign through zero at $\theta = \theta^*$, i.e. the sign of $(h \circ l)'(\theta^* + \varepsilon)$ is opposite of $(h \circ l)'(\theta^* -$

ε) for small values of ε. This implies that the stationary gain $G(0)$ also will switch sign through zero when linearized about $x^* = l(\theta^*)$. This can only happen if either of

$$K = \pm 0, \text{ or } b_0 = \pm 0 \tag{8}$$

are true at the optimum ($a_0 = \pm\infty$ is not possible for proper systems). If $K = 0$, then all dynamics disappear at the extremum since $G(s) = 0\ \forall s$, i.e., the output is invariant at the optimum. If instead $b_0 = 0$ and some $b_k \neq 0$, then a real zero will cross between the LHP and the RHP as $G(0)$ changes sign and hence there exist s such that $G(s) \neq 0$. In this case the system will have a dynamic response to small changes in θ even at the extremum point and, furthermore, the linearized system will be non-minimum phase, at least locally, on one side of the extremum. This is an interesting observation as it severely limits the ability to stabilize or speed up the dynamics of a system prior to applying ESC. However, we will here merely focus on the implications of the zero crossing for the existence of stationary solutions to the ESC itself and will leave the implications for inner loop feedback to future work.

Consider now the case of a Hammerstein/Wiener model, as considered in most previous studies on ESC. For such models it is clear that the optimum corresponds to $K = 0$, and hence the optimum is a stationary solution of ESC for sufficiently small a. Furthermore, the phase lag of such models does not vary with $\bar{\theta}$ and hence there will only be singular frequencies for which the ESC can lock on to a solution with $\varphi = \frac{\pi}{2} + n\pi$. Thus, for essentially any choice of the excitation frequency ω there will be a unique stationary solution of the ESC which is the optimal solution for small a. Furthermore, the deviation from optimality for larger a will only depend on the degree of non-symmetry of the mapping $h \circ l$ around the optimum. As shown in [4], the solution will also be stable for an appropriate choice of controller parameters, including a sufficiently small excitation frequency ω.

Consider next the case in which the optimum corresponds to $b_0 = \pm 0$, i.e., a transmission zero crosses the imaginary axis through zero as $\bar{\theta}$ passes the extremum. In this case, the system has a zero at $s = 0$ and hence a phase lag of $\pi/2$ at $\omega = 0$ at the optimum. We here make the usual assumption that the cut-off frequency $\omega_h \leq \omega$ in the high-pass filter. Thus, the stationary solution will asymptotically approach the optimum as $\omega \to 0$. For solutions close to the extremum there will be a zero close to 0, either in the LHP or RHP, and hence a small non-zero frequency for which the phase lag $\varphi = \pi/2$. Thus, for small non-zero excitation frequencies ω the ESC will converge to a solution in the vicinity of the optimum. Since the zero moves away from the origin as $\bar{\theta}$ moves away from the optimum, it implies that the distance to the optimum will increase with increasing frequency. This also corresponds well with the results based on local approximations around the optimum in [7]. Note that the slower the zero moves with changes in $\bar{\theta}$, the larger the distance to the optimum will in general be for a given excitation frequency. Also, note that as the zero has moved some distance from the imaginary axis the impact of other poles and zeros are likely to interfere with its phase contribution and we may not get a phase lag of $\varphi = \frac{\pi}{2} + n\pi$ at any solution in the vicinity of the optimum. Thus, for sufficiently large frequencies there will probably not be any stationary solutions related to the existence of a process optimum.

Finally, it is clear that a process may have frequencies where the phase lag $\varphi = \frac{\pi}{2} + n\pi$ without any relation whatsoever to the optimality conditions discussed above. If this frequency varies with $\bar{\theta}$, then we will have a continuous range of excitation frequencies for which a sub-optimal stationary solution will exist. In such cases it is also possible that multiple solutions will exist, of which one is related to the optimality of the process while the others are not. However, there may also exist situations where all stationary solutions are sub-optimal in the sense that they are not related to the optimality conditions of the process. This is shown for the example bio-reactor below. First, we derive a simple stability condition for the stationary solutions of the ESC as derived above.

B. Stability of the stationary solutions

As shown above, depending on the dynamic properties of the process and the excitation frequency ω, the ESC may lock on to different types of stationary solutions. In practice, one will of course only observe stable stationary solutions and hence it is of interest to determine if all the various types of stationary solutions can be stable, at least for some choices of controller parameters.

To simplify the stability analysis we will assume that the control is so slow that the process in combination with the low-pass and high-pass filters acts as a static map from θ to ξ. Note that the control can be slow even if the excitation frequency is relatively high since the response time also depends on other parameters such as excitation amplitude a and integrator gain k. Also note that the purpose of the stability analysis presented here simply is to show that in principle all types of stationary solutions discussed above can be asymptotically stable.

Consider the ESC closed loop in Fig. 1. The block diagram can be simplified into the one shown in Fig. 2, in which all blocks but the integrator block have been included in the block labelled L.

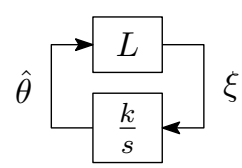

Fig. 2. Simplified representation of ESC scheme in Fig.1

To investigate the stability of the simplified loop we seek to find an algebraic expression for the relation

$$\xi = L(\hat{\theta}).$$

If $\hat{\theta}$ is varying slowly, then the local response of the system (1) to small and relatively fast perturbations can

be approximated by the system linearized about the current $\hat{\theta}$. Thus, we approximate the system by a linear parameter varying (LPV) system with $\hat{\theta}$ as a parameter. We form

$$\begin{aligned}\dot{x} =& A(\hat{\theta})x + B(\hat{\theta})\theta \\ \tilde{y} =& C(\hat{\theta})x\end{aligned}$$

which yields the $\hat{\theta}$-parametrized transfer-function

$$G(s,\hat{\theta}) = C(\hat{\theta})(sI - A(\hat{\theta}))^{-1}B(\hat{\theta}).$$

If ω is fast compared to the variations in $\hat{\theta}$, we can consider the problem using separate time scales. For the fast time scale, we approximate $\hat{\theta}$ as a constant and follow the same steps as in deriving (4) to characterize the stationary solutions. We get

$$\xi = \frac{a^2}{2}|F_L(0)||G(i\omega,\hat{\theta})||F_H(i\omega)|\cos(\varphi(\hat{\theta})) = L(\hat{\theta}).$$

This static map is the relation between $\hat{\theta}$ and ξ in the slow time scale. Again, we are interested in stationary solutions corresponding to $\xi = 0$, or $\hat{\theta} = \bar{\theta}$ constant. To determine the stability of such solutions we consider a linearization of L around the stationary solutions for $L(\bar{\theta}) = 0$

$$L(\hat{\theta}) \approx \frac{dL(\bar{\theta})}{d\hat{\theta}}\tilde{\theta}, \quad \tilde{\theta} = \hat{\theta} - \bar{\theta}.$$

Now if we replace L by its linear approximation in the closed loop in Fig. 2, it should be clear the closed loop will have a single pole at

$$k\frac{dL(\bar{\theta})}{d\hat{\theta}}.$$

The stability of the loop is determined by the sign of the pole and the stability criterion thus becomes

$$k\frac{dL(\bar{\theta})}{d\hat{\theta}} < 0 \tag{9}$$

Condition (9) can in principle be satisfied for any type of stationary solution discussed above, and hence all types of solutions can in principle be stable. Also, note that any stationary solution can be made stable by simply choosing the appropriate sign of the controller gain k.

IV. Example: Control of a Continuous Tubular Reactor

As stated in the introduction, our motivation behind this work was observations made when applying ESC to a bioreactor used for ammonium removal from waste water. The model for this process is highly complex and we therefore consider a simpler example here to make the results more transparent as well as reproducible by the reader. The qualitative results obtained are similar to those observed in the bioreactor for wastewater treatment.

The system considered here is a simple isothermal tubular bioreactor with plug-flow for converting species A into species B, but in which there also is a side reaction producing species C from B [13]. The side reaction implies that there is a maximum in the yield of B with respect to the residence time in the reactor. The residence time can be controlled using the total flow into the reactor as the input. The reactor model is

$$\begin{aligned}\frac{\partial\alpha}{\partial t} + \frac{1}{q}\frac{\partial\alpha}{\partial z} &= -\alpha\beta \\ \frac{\partial\beta}{\partial t} + \frac{1}{q}\frac{\partial\beta}{\partial z} &= \alpha\beta - \frac{\beta}{\phi(1+\rho\beta)}\end{aligned}$$

Here α and β are dimensionless concentrations of A and B, respectively, q is the total flow and ϕ and ρ are parameters describing the reaction kinetics. The dimensionless length of the reactor is 1, i.e., $z \in [0,1]$. We consider the same nominal parameter values as in [13], i.e., $\phi = 20$, $\rho = 3$ and $\alpha(t,0) = 0.8$ and $\beta(t,0) = 0.2$. We use the method of lines for simulation and employ simple backward Euler with $N = 10$ elements for the spatial discretization.

The static map from the flow $u = q$ to the product concentration of B, $y = \beta(1)$, is shown in Fig. 3. As can be seen, there is a maximum concentration of B, $y = 0.915$, for a flow $u = 6.01$. From linearization we find that there is a zero at $s = 0$ in the transfer-function of the linearised system at the optimum, and this zero moves into the RHP for higher values of the input flow q. Thus, the system does not have any steady-states for which the amplitude $|G(i\omega)| = 0$ for any non-zero frequency ω. According to the analysis presented above, this implies that that any stationary solution to the ESC problem for the bioreactor must satisfy the phase-lag condition $\varphi = \frac{\pi}{2} + n\pi$.

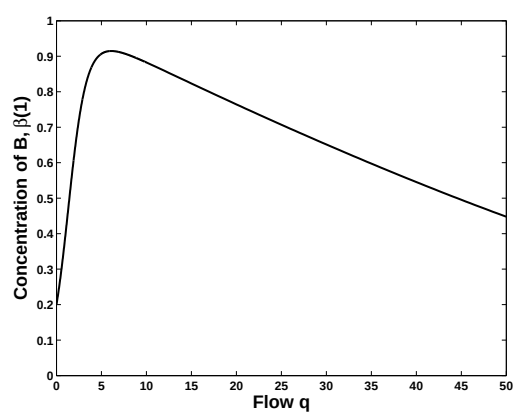

Fig. 3. Steady-state input-output map for bioreactor in Example.

We employ the cut-off frequency $\omega_h = 0.01$ in the high-pass filter and compute solutions in terms of ω to $\varphi = \frac{\pi}{2} + n\pi$ for steady-states within the input range $q \in [0, 200]$. The results in terms of stationary flow and concentration of B as functions of the perturbation frequency ω is shown in Fig. 4. As can be seen, for perturbation frequencies up to $\omega = 0.36$ the ESC scheme has a stationary solution close to the optimal concentration 0.915. However, as can be seen from Fig. 4, there is a bifurcation at $\omega = 0.36$, resulting in multiple steady-states in the range $\omega \in [0.035,\ 0.36]$. From simulations as well as from the simplified stability analysis presented above, we find that the solutions at the upper and lower branches are stable while the solutions at

the intermediate branch are unstable. For instance, for the perturbation frequency $\omega = 0.05$ rad/min we find three solutions with concentrations $y = 0.913$ (stable), $y = 0.226$ (unstable) and $y = 0.033$ (stable), respectively. See also Fig. 4. Thus, depending on the initial conditions, the ESC may converge to a solution close to the optimal yield or to a solution with essentially no yield. Note that this is a perturbation frequency which is low relative to the main dynamics of the reactor with a time-constant around 5 min

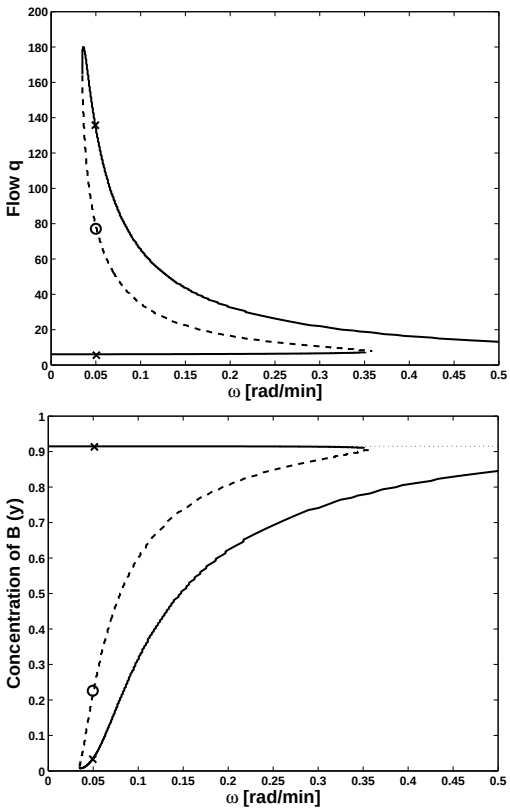

Fig. 4. Stationary solutions of bioreactor in terms of flow and product composition of B as a function of the perturbation frequency ω with ESC. The solutions for the perturbation frequency $\omega = 0.05$ are marked, x-stable, o-unstable. The dotted line is the maximum product concentration.

V. Summary and Conclusions

We have in this paper considered stationary solutions of the extremum seeking control scheme. An analysis of a general dynamic model revealed that the stationary solutions of the ESC are characterized either by the linearized dynamics having zero gain for all frequencies $|G(i\omega)| = 0\ \forall\omega$, or the phase lag at the excitation frequency being $\frac{\pi}{2} + n\pi$. This result was related to previous results on properties of dynamic systems close to extremum points in the input-output map [12], and based on this we could conclude that either condition for stationarity is likely to be fulfilled in some vicinity of the optimum for sufficiently low excitation frequencies. In particular, for systems in which the nonlinearity with an extremum point is purely static, e.g., as in Hammerstein systems, the zero gain condition is fulfilled at the optimum. For systems which display transient behavior even at the optimum, there is a transmission zero crossing the imaginary axis at the optimum and hence there will be a solution close to optimum fulfilling the phase lag condition for low perturbation frequencies. However, there may also exist stationary solutions fulfilling the phase lag condition at operating points with no relation to the optimality condition whatsoever. Such solutions may coexist with solutions close to the optimum, resulting in multiple stationary solutions to the ESC problem. For higher excitation frequencies, there may exist no stationary solutions related to the optimality of the process and only sub-optimal solutions exist.

We stress that the phase lag condition presented in this paper, if properly utilized, may represent an advantage for ESC as it represents a dynamic property reflecting nearness to optimum, and hence higher excitation frequencies allowing for faster convergence may be employed. However, the problem of avoiding convergence to sub-optimal solutions also fulfilling the phase lag condition is an open problem.

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