"Model Reduction applied on Natural Gas Pipeline Systems"

Hans Aalto
Main pipeline system components: Compressor stations, pipeline segments and offtakes

Compressor station discharge pressures are usually used to operate the pipeline

50-100 km
Start with the PDE for a (=each!) pipeline segment

\[
\frac{\partial P}{\partial t} + b \frac{\partial q}{A \partial z} = 0
\]

\[
\frac{\partial q}{\partial t} + A \frac{\partial P}{\partial z} + f \frac{b}{DA} \frac{q^2}{P} = 0
\]

This is the simplest isothermal PDE model for pipelines in the horizontal plane only and with small gas velocities!
Discretize w.r.t. to the space co-ordinate \( z \), using \( N \) elements (nodes) for each segment (\(!\)) \( i=1,2,\ldots,N \)

\[
\frac{dP_i}{dt} = \frac{b_i}{A_i \Delta z_i} (q_{i-1} - q_i)
\]

\[
\frac{dq_i}{dt} = \frac{A_i}{\Delta z_i} (P_i - P_{i+1}) - f_i \frac{b_i}{D_i A_i} \frac{q_i^2}{P_i}
\]

Compressor station between node “k-1” and “k” : PI-controller of discharge pressure manipulating gas flow:

\[
\frac{dq_{k-1}}{dt} = -K \beta_k (q_{k-1} - q_k) + \frac{K}{T_i} (P_{k,SET} - P_k)
\]
Nonlinearity measure (example)

1 bar perturbation: @ 48 bar Gain=3.08 , Timeconst. 119 min.  
@ 64 bar Gain=1.53 , Timeconst. 59 min.
Transfer functions from reduced models:

How would we obtain the transfer function between 2 variables of a given pipeline?

- Identify from true pipeline system data
- Identify from dynamic simulator data

“Direct method”: from design data to transfer functions!
... Linearize this \textbf{large} ODE model in a given steady state operating point

\[
\frac{d \Delta P_i}{dt} = \alpha_i (\Delta q_{i-1} - \Delta q_i)
\]

\[
\frac{d \Delta q_i}{dt} = \beta_i (\Delta P_i - \Delta P_{i+1}) - 2\gamma_i \frac{\Delta q_i}{P_{i,SS}} + \gamma_i \frac{q_{i,SS}^2 \Delta P_i}{P_{i,SS}^2}
\]

or:

\[
\frac{dx(t)}{dt} = Ax(t) + Bu(t),
\]

\[y(t) = Cx(t)\]

where \(x \land [\Delta P_1 \Delta q_1 \Delta P_2 \Delta q_2 \ldots \Delta P_N \Delta q_N]^{\top}\)
Matrices $A$ (2Nx2N) and $B$ (2Nxm) depend on the geometry, physical parameters, node partition and steady state data = design (engineering) information

$C$ (1x2N) is needed just to select which state variable is of interest

The rest is easy, obtain the transfer function from $(A,B,C)$ using standard methods

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t),$$

$$y(t) = Cx(t)$$

$$G(s) = \frac{K(T_a s + 1)(T_b s + 1)\ldots}{(T_1 s + 1)(T_2 s + 1)(T_3 s + 1)\ldots}$$

eg. ss2tf of Matlab
**NO!** Transfer function from large system is difficult, even if dominating time constants may be obtained. In our case, numerator dynamics has relevance!
Use Linear Model Reduction techniques!

Truncation: Solve \( P \) and \( Q \) from

\[
\begin{align*}
AP + PA^T + BB^T &= 0 \\
A^TQ + QA + C^TC &= 0
\end{align*}
\]

Compute Hankel singular values

\[ \sigma_i = \sqrt{\lambda_i(PQ)} \]

Arrange eigenvectors of \( PQ \) into: a transformation matrix

\[ T = [v_1 \ v_2 \ ... \ v_{2N}] \]

The upper \( N_r << 2N \) submatrices of the transformed matrices = a reduced linear state space system

\[
\begin{align*}
\tilde{A} &= T^{-1}AT \triangleq W^TAV \\
\tilde{B} &= T^{-1}B \triangleq W^TB \\
\tilde{C} &= CT \triangleq CV
\end{align*}
\]
Balanced truncation: \( P \) and \( Q \) required to be diagonal

\[ \ldots \]

Transfer function from reduced model with \( N_r = 3 \ldots 4 \) is easily obtained with standard methods!
Pipeline system w. 6 segments, 8 offtakes, 4 compressor stations and 70 nodes => 2N=140
Transfer function from CS2 discharge pressure to “Pa”, far downstream CS2 [time constant]=minutes!:

\[
\frac{1.44(116.9s + 1)}{(117.7s + 1)(190.6s + 1)} \approx \frac{1.44}{(190.6s + 1)}
\]

Dito for “Pb”, close to CS2:

\[
\frac{1.16(83.5s + 1)}{(11.2s + 1)(183.6s + 1)}
\]
**Empirical calculation of the Gramians P and Q**

- Step (impulse) perturbations on the system or on a full-scale simulation model
- After obtaining $T$, define a transformed state and apply a Galerkin projection to get a reduced non-linear model:

$$\dot{x}(t) = f(x(t), u(t)) \ , \ y(t) = g(x(t), u(t))$$

$$\dot{x}_r(t) = P T f(T^{-1} \tilde{x}(t), u(t))$$

$$\dot{x}_{N-r}(t) = 0, \ x_{N-r}(t) = x_{N-r, SS}$$

$$y(t) = g(T^{-1} \tilde{x}(t), u(t))$$

$$\tilde{x}(t) \triangleq \begin{bmatrix} x_r \\ x_{N-r} \end{bmatrix} = T x(t) \ , \ P = \begin{bmatrix} I_r & 0 \end{bmatrix}$$
Response of original ODE model for 92 km pipeline with 58 states, reduced nonlinear model Nr= 8 (circles) and dito with Nr=4.
General problem (not necessarily natural gas pipeline): We need an EKF for a large non-linear system

\[ \dot{x}(t) = f(x(t), u(t)) \]
\[ y(t) = g(x(t)) \]

\[ \hat{x}(t) = f(\hat{x}(t), u(t)) + K[y_M(t) - g(\hat{x}(t))] \]

Steady state Riccati equation applies for linearized continuous time system, \( \text{dim}(K) = n \times ny \):

\[ K = PC^T R^{-1} \]
\[ AP + PA^T + Q - PC^T R^{-1} CP = 0 \]
All “mappings” in the sequel from a smaller matrix $K_r$ to a bigger $K$, actually mean:

$$K = f(K_r) \overset{\triangle}{=} f\left(\begin{bmatrix} K_r & 0 \\ 0 & \varepsilon I \end{bmatrix}\right), \quad \varepsilon \rightarrow 0$$

Kalman filter for the reduced model also obeys Riccati:

$$K_r = P_r C_r^T R^{-1}$$

$$A_r P_r + P_r A_r^T + Q_r - P_r C_r^T R^{-1} C_r P_r = 0$$

Then, if:

$$K = V K_r, \quad Q = W Q_r W^T$$

The Riccati equation for the full system holds, and:

$$P = W P_r W^T$$

True also for discrete-time system
Example: 90-km long true pipeline segment:

$$\Delta z = 1667 \text{ m} \Rightarrow 82 \text{ elements} = 164 \text{ states}$$

Design Kalman filter for $$n_r = 4$$ and then scale up $$K = V^* K_r$$ to obtain state estimator for 164 states
... Results:

Estimated upstream pressure

Estimated $P_a$

Measured $P_a$

Estimated gas flow after CS

Simulated gas flow
What if we want to do the EKF exercise but do not have access to the full scale linear n-dimensional system \((A,B,C)\) ?

Recall:
- Empirical Gramians would give us \(P,Q,V\) and \(W\)
- Low dimensional model \((A_r, B_r, C_r)\) could be obtained by identification

\(\Rightarrow\) Do the matrices match, can we do “scale up” \(K = V*K_r\) ?

Let us borrow some results from discrete-time subspace identification (The state space model realisation part of it)
Use the system impulse response to form a Hankel matrix:

\[
H = \begin{bmatrix}
    h_1 & h_2 & h_{N+1} \\
    h_2 & h_3 & h_{N+2} \\
    \vdots & \vdots & \vdots \\
    h_N & h_{N+1} & h_{2N+2}
\end{bmatrix}
\]

\[h_i = CA^{i-1}B\]

Svd of \(H\), which is actually an estimate from data, \(H^\hat{\}

Choose a model order "r" and partition:

\[
Q = \begin{bmatrix} Q_r & Q_{N-r} \end{bmatrix} \quad V = \begin{bmatrix} V_r & V_{N-r} \end{bmatrix}
\]

\[
H = QS\!V^T
\]
$S_r$ is a diagonal matrix with $r$ principal singular values. Observability and Controllability matrix estimates:

$$
\Gamma_N = \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{N-1}
\end{bmatrix} = Q_r S_r^{1/2} \\

\Omega_N = \begin{bmatrix}
B \\
AB \\
\vdots \\
AB^{N-1}
\end{bmatrix} = S_r^{1/2} V_r^T
$$

Read $C$, actually $C_r$ from $\Gamma_N$ and $B_r$ from $\Omega_N$

For $A$, actually $A_r$, apply the "shift invariance" $\Gamma_N = \Gamma_{N-1} A$

Solve $A$ using pseudo-inverse
Actually we have done a **balanced truncation**!

- Using $H$, make a full state dimension model with $r \rightarrow n < N$:
  - $P = \Omega_N \Omega_N^T$
  - $Q = \Gamma_N^T \Gamma_N$
- Calculate $A_n$, $B_n$ and $C_n$ as above = **linearization**!
- Calculate $T$ (as above), call $W = T^{-T}$ and $V = T$
  Use $W$ and $V$ for a lower-dimensional model $r<n$:

$$\tilde{A} = W^T A_n V, \quad A_r = \tilde{A}(1:n_r,1:n_r) \quad \text{etc.}$$

$r$:th order model can also be obtained by repeating the realisation procedure.

**NOTE**, that $V$ is needed to do the ”scale up” for the Kalman filter.
Impulse response data from a simulation model = no noise problem = should work, but non-linearity may harm

Fix: Discrete-time EKF innovations part to be combined with continuous-time non-linear model

Find weak points of "scale up" procedure, some horrible counter-example etc.

Subspace realisation for a large full order linear (=n) system may be tough; almost redundant states etc.
Thank You!