Design of multivariable LQ-optimal PID controllers based on convex optimization

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Problem formulation
Consider the following LTI system model:
\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + n^y
\end{align*}
\]
(1)
Here \(x \in \mathbb{R}^{n_x}\) are the states, \(u \in \mathbb{R}^{n_u}\) are the inputs, \(y \in \mathbb{R}^{n_y}\) are the outputs we want to control, and \(n^y \in \mathbb{R}^{n_y}\) is a vector of additive noise.

In this work we present a method for design of multivariable LQ-optimal PID controllers based on convex optimization for systems that can be described by (1).

Theory
A key result, which is the basis for this paper, is the nullspace theorem [Alstad et al., 2008]:

**Theorem 1.** (Loss by introducing linear constraint for nullspace theorem) Consider the un-constrained optimization problem
\[
\min_u J(u, d) = \left[ \begin{array}{c} u \\ d \end{array} \right]^T \left[ \begin{array}{cc} J_{uu} & J_{ud} \\ J_{ud}^T & J_{dd} \end{array} \right] \left[ \begin{array}{c} u \\ d \end{array} \right]
\]
(2)
and a set of noise measurements \(y_m = y + n^y\), where \(y = G^yu + G^wd\). Assume that \(n_c = n_u\) constraints \(c = H^y y_m = c_s\), with \(\text{rank}(H) = n_c\), are added to the problem, which will result in a non-optimal solution with a loss \(L = J(u, d) - J_{\text{opt}}(d)\). Consider disturbances \(d\) and noise \(n^y\) with magnitudes
\[
d = Wd' \quad n^y = W_{n^y}n^y' \quad \|\left[ \begin{array}{c} d' \\ n^y' \end{array} \right]\|_2 \leq 1.
\]
(3)
Then for a given \(H\), the worst-case loss introduced by adding the constraint \(c = H^y y\) is \(L_{wc} = \bar{\sigma}(M)/2\), where \(M\) is
\[
M \triangleq \begin{bmatrix} M_u & M_{n^y} \\ M_{n^y}^T & M_d \end{bmatrix}, \quad M_d = -J^{-1/2}_{uu}(HG^y)^{-1}HFW_d, \quad M_{n^y} = -J^{-1/2}_{uu}(HG^y)^{-1}HW_{n^y}\]
(4)
The optimal \(H\) that minimizes the loss can be found by solving the convex optimization problem
\[
\min_{H} \|HF\|_F \quad \text{subject to } HG^y = J^{-1/2}_{uu}
\]
(5)
Here \(\tilde{F} = \begin{bmatrix} FW_d & W_{n^y} \end{bmatrix}\) and \(F = -(G^y J^{-1}_{uu} J_{ud} - G^w_d)\).
The reason for using the Frobenius norm is that minimization of this norm also minimizes \(\bar{\sigma}(M)\) Kariwala et al. [2008].

Derivation of multivariable PID controller
Assuming that the available “measurements” in \(y\) include the present, integrated, and derivative value of the output, Theorem 1 can be used for design of multivariable PID controllers. The following procedure is proposed:

1. To include integral action in the LQ problem formulation, augment the plant with \(n_d = n_y\) disturbances such that offset-free tracking is guaranteed, i.e., by using the rank-conditions from [Pannocchia and Rawlings, 2003], and \(n_c = n_d\) integrators that belongs to the controller. The augmented plant becomes:
\[
\begin{align*}
\dot{x} &= \begin{bmatrix} A & 0 & B_d \\ C & 0 & C_d \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \\
\dot{y}^P &= \begin{bmatrix} C & 0 & 0 \\ CA & 0 & CB_d \end{bmatrix} \begin{bmatrix} x \\ \sigma \\ d \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \bar{y} \end{bmatrix} + \begin{bmatrix} n^y_P \\ n^y_a \end{bmatrix}
\end{align*}
\]
(6)
This system can be discretized to
\[
\begin{align*}
\tilde{x}_{k+1} &= \Phi \tilde{x}_k + \Gamma u_k \\
\tilde{y}_k &= \tilde{C} \tilde{x}_k + \tilde{D} u_k + \tilde{n}^y
\end{align*}
\]
(7)
Here \(\tilde{x}_k = (x_k, \sigma_k, d_k)\), \(\tilde{y}_k = (y^P_k, y^D_k)\) and \(\tilde{n}^y = (n^y_P, n^y_a, n^y_b)\).

2. Define the LQ-objective for the control problem,
\[
\min_U J(U, x(0)) = \sum_{i=0}^{\infty} x_k^T Q x_k + \Delta u_k^T R \Delta u_k
\]
subject to \(x_0 = x(0)\) and equation (7) for \(k = 0, 1, 2, \ldots\),
(8)
where \(U \triangleq (u_0, u_1, u_2, \ldots)\).

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3. Convert (8) to a finite optimization problem by using for \( k \geq N \), \( u_k = -K_{\text{LQR}}x_k \). This gives an objective function on the form

\[
J(u_0, u_1, \ldots, u_{N-1}, x_0) = x_N^T P x_N + \sum_{i=0}^{N-1} x_i^T Q x_i + \Delta u_k^T R_{\Delta} \Delta u_k,
\]

(9)

where \( P \) is a solution of a Lyapunov equation, see Scokaert and Rawlings [1998].

4. Substitute the model equations into the objective function, to get an objective on the form (2), with

\[
\frac{J_{uv}}{2} = \begin{bmatrix} G^T \Phi, \ldots, G^T \Phi_{N-1} \end{bmatrix} \begin{bmatrix} P \Phi, \ldots, P \Phi_{N-1} \end{bmatrix} \Phi
\]

(10)

where

\[
M = \begin{bmatrix} -1 & \cdots & -1 \\ \vdots & \ddots & \vdots \\ -1 & \cdots & -1 \end{bmatrix} \in \mathbb{R}^{n_u(N-1) \times n_u N},
\]

and

\[
\frac{J_{ud}}{2} = \begin{bmatrix} \Gamma^T \\ \vdots \\ \Gamma^T \end{bmatrix} \begin{bmatrix} P \Phi, \ldots, P \Phi_{N-1} \end{bmatrix} \Phi
\]

(12)

Here \( u = (u_0, u_1, \ldots, u_{N-1}) \) and \( d = x(0) \).

5. We now let the “measurements” in Theorem 1 include the process outputs and the inputs, \( y = (y^P, y^D, u_k, \ldots, u_{k+N-1}) \). These variables can be written as

\[
y = G^y u + G^y_d d
\]

(13)

with

\[
G^y = \begin{bmatrix} \tilde{D} \& 0 \\ I \& 0 \\ 0 \& I \end{bmatrix}; \quad G^y_d = \begin{bmatrix} \tilde{C} \& 0 \end{bmatrix},
\]

(14)

where 0 is a matrix of zeros of appropriate dimensions and \( I \) is an identity matrix of appropriate dimensions.

6. We can now compute the sensitivity matrix \( F = -(G^y J_{ud}^{-1} J_{ud} - G^y_d) \) and use (5) in Theorem 1 to find the optimal \( H \). This convex optimization problem can be solved for example with cvx, a package for specifying and solving convex programs [Grant and Boyd, 2008], with the following Matlab\textsuperscript{TM} code:

```matlab
cvx_begin
variable H(N*nu,ny+nu*N);
minimize norm(H*Ftilde,'fro')
subject to
    H*Gy == sqrtm(Juu);

end
```

The optimal \( H \) combines \( H y \) such that when controlled to the constant setpoint of 0 gives minimum operational loss from the optimal solution, which is defined by the solution of (8) when the full state vector \( x(0) \) is available for measurement.

7. From Alstad et al. [2008] we have that for an optimal \( \hat{H} \), \( H = \hat{H} \hat{H}^T \) will still be optimal with respect to the optimization problem in (5) provided that the \( n_e \times n_e \) \( D \)-matrix is non-singular. Let \( \hat{H} = [H^y \ H^d] \). For linearly independent inputs we have that \( H^u \) is non-singular, hence another optimal \( H = (H^u)^{-1} \hat{H} = (H^u)^{-1} H^y I \).

The \( H \) matrix is a \( N_{nu} \times (3n_y + N_{nu}) \) matrix. The first \( n_u \) rows of \( H y = 0 \) has this information:

\[
K_P y^P + K_1 y^1 + K_D y^D + I u_k + 0 u_{k-1} + \cdots + 0 u_{k+N-1} = 0
\]

(15)

We solve for \( u_k \) and finally get the LQ-optimal multi-variable PID controller:

\[
u_k = -(K_P y^P + K_1 y^1 + K_D y^D)\]

(16)

This is the MIMO PID approximation of the original LQ problem. To guarantee closed loop stability a separate analysis is required.

Conclusions

In this extended abstract we outlined how to find a multi-variable LQ-optimal PID controller based on convex optimization. This is a significant contribution because previous work indicates that this problem is non-convex. Examples will be given in the presentation.

References


