Null Space Method for Selecting Optimal Measurement Combinations as Controlled Variables

Vidar Alstad† and Sigurd Skogestad*

Department of Chemical Engineering, Norwegian University of Science and Technology, Trondheim, Norway

The issue in this paper is to select controlled variables \( c \) as combinations of the measurements \( y \). The objective is to obtain self-optimizing control, which is when we can achieve near-optimal steady-state operation with constant setpoints for the controlled variables, without the need to reoptimize when new disturbances perturb the plant. The null space method yields locally optimal controlled variables \( c = Hy \) that are linear combinations of measurements \( y \). The requirement is that we at least have as many measurements as there are unconstrained degrees of freedom, including disturbances, and that the implementation error is neglected. The method is surprisingly simple. From a steady-state model of the plant, the first step is to obtain the optimal sensitivity matrix \( F \), with respect to the disturbances. The optimal matrix \( H \) satisfies \( HF = 0 \); therefore, the next step is to obtain \( H \) in the left null space of \( F \). As an illustration, the method is used to obtain temperature combinations for control of a Petlyuk distillation column.

1. Introduction

For cases with unconstrained control degrees of freedom (inputs \( n \)), an important issue is to decide what to control. Although not widely acknowledged by control theorists, controlling the right variables is a key element in overcoming uncertainty in operation.\(^1\)\(^2\) This applies also when using advanced control (e.g., model predictive control (MPC)) or real-time optimization (RTO). This paper focuses on the interaction between the local optimization layer and the feedback control layer (see Figure 1), and, more specifically, on the selection of the controlled variables \( c \) that link these layers. Two subproblems are important here:

1. Selection of the controlled variables \( c \). This is a structural decision that is made before the control strategy is implemented.

2. Selection of setpoints \( c_\ast \). This is a parametric decision that can be done both online and offline.

Here, we focus on the first structural problem of finding the controlled variables and we will assume constant nominal optimal setpoints. As Figure 1 shows, there are two sources of uncertainty that will make a constant setpoint policy nonoptimal: (i) disturbances \( d \) (these are the external unmeasured disturbances, including parameter variations), and (ii) implementation error \( u \) (this includes the sum of the effect of the measurement error for \( y \) and the control error).

Single measurements or functions or combinations of the measurements may be used as controlled variables \( c \). The objective is to obtain self-optimizing control, which is when we can achieve near-optimal steady-state operation with constant setpoints for the controlled variables, without the need to reoptimize when new disturbances perturb the plant. The use of single measurements is simple and is the preferred choice if the loss is sufficiently small. However, for some applications, there may not exist any self-optimizing single measurements, and one may consider measurement combinations. In this paper, we consider linear combinations, that is, \( c = Hy \), where \( H \) is a constant matrix.

Ideas related to self-optimizing control have been presented repeatedly in the process control history, but the first quantitative treatment was that of Morari et al.\(^1\) Skogestad\(^2\) defined the problem more carefully, linked it to previous work, and was the first to include the implementation error as well. He mainly considered the case where single measurements are used as controlled variables; that is, \( H \) is a selection matrix where each row has a single digit (one, 1) and the rest are zeros (0). The loss with a constant setpoint policy for expected disturbances and implementation errors was evaluated using a “brute-force” approach. An important advantage of a brute-force evaluation is that one can also identify controlled variables that may yield infeasability for certain disturbances or implementation errors. This was also considered in more detail by Larsson et al.\(^3\) for the Tennessee—Eastman challenge problem and Govatsmark and Skogestad,\(^4\) who suggested adjustment of the setpoints to achieve feasibility. However, the computational load of the “brute-force” method can be very large, so local methods based on linearizing the behavior around the steady-state are attractive. Skogestad\(^5\) introduced the approximate maximum gain rule as a simple method for selecting controlled variables. In the

\[ \text{Figure 1. Block diagram of a feedback control structure including an optimizer layer.} \]
2. Problem Formulation

We assume that the operational goal is to use the degrees of freedoms \( u \) to minimize the cost \( J \), while satisfying equality and inequality constraints. The (original) constrained steady-state optimization problem can, for a given disturbance \( d \), be formulated as

\[
\min_{x,u} J(x, u, d) = \frac{1}{2}(c - c_{\text{opt}})^T J_{cc}(c - c_{\text{opt}})
\]

subject to

\[
\begin{align*}
    f(x, u, d) &= 0 \\
    g(x, u, d) &\leq 0 \\
    y &= f(x, u, d)
\end{align*}
\]

where \( x \in \mathbb{R}^n_u, u \in \mathbb{R}^n_u, \) and \( d \in \mathbb{R}^n_d \) are the states, inputs, and disturbances, respectively; \( f \) is the set of equality constraints corresponding to the model equation; \( g \) is the set of inequality constraints that limits the operation (e.g., physical limits on temperature measurements or flow constraints); and \( y \) represents the measurements.

When solving this optimization problem, we generally find that some constraints are active, corresponding to \( g_i(x, u, d) = 0 \). To achieve steady-state optimal operation, we assume here that we control all the active constraints and that the set of active constraints does not change (see assumption A3 presented later in this work). Controlling the active constraints consumes a corresponding number of the degrees of freedom (inputs), and we consider, in the following, the remaining uncontrolled reduced-space optimization problem where the scalar cost function \( J \) is to be minimized with respect to the \( n_u \) remaining degrees of freedom (inputs) \( u \):

\[
\min_u J(u, d)
\]
A3. **Active constraint control:** We assume that the same active constraints remain active for all values of the disturbances and that we control these constraints.

A4. **No implementation error:** The implementation error is the sum of the control error and the effect of the measurement error. The assumption of no steady-state control error is satisfied if we use a controller with integral action. It is a more serious assumption to neglect the measurement error, so the method implicitly assumes that the measurements have been carefully selected.

We then have the following result.

**Theorem 1: Null space method.** Assume that we have \( n_\upsilon \) independent unconstrained free variables \( \mathbf{u} \), \( n_d \) independent disturbances \( \mathbf{d} \), \( n_\gamma \) independent measurements \( \mathbf{y} \), and we want to obtain \( n_c = n_n \) independent controlled variables \( \mathbf{c} \) that are linear combinations of the measurements

\[
\mathbf{c} = \mathbf{H}\mathbf{y}
\]

Let

\[
\mathbf{F} = \frac{\partial \mathbf{y}^{\text{opt}}}{\partial \mathbf{d}^{\text{opt}}}
\]

be the optimal sensitivity matrix evaluated with constant active constraints. If \( n_n \geq n_\upsilon + n_\gamma \), it is possible to select the matrix \( \mathbf{H} \) in the left null space of \( \mathbf{F} \), \( \mathbf{H} \in \mathbb{R}^{n_\gamma \times (\mathbb{F}^T)} \), such that we get

\[
\mathbf{HF} = 0
\]

With this choice for \( \mathbf{H} \), fixing \( \mathbf{c} \) (at its nominal optimal value) is first-order optimal for disturbances \( \mathbf{d} \); that is, the loss is zero as long as the sensitivity matrix \( \mathbf{F} \) does not change.

Proof. We first prove that selecting \( \mathbf{H} \) such that \( \mathbf{HF} = 0 \) gives zero disturbance loss. For small disturbances, the optimal change in the measurements to a change in the disturbances can be written as

\[
\mathbf{y}^{\text{opt}}(\mathbf{d}) - \mathbf{y}^{\text{opt}}(\mathbf{d}^*) = \mathbf{F}(\mathbf{d} - \mathbf{d}^*)
\]

where

\[
\mathbf{F} = \begin{bmatrix}
\frac{\partial y_1^{\text{opt}}}{\partial d_1} & \ldots & \frac{\partial y_1^{\text{opt}}}{\partial d_{nd}} \\
\vdots & \ddots & \vdots \\
\frac{\partial y_{n_n}^{\text{opt}}}{\partial d_1} & \ldots & \frac{\partial y_{n_n}^{\text{opt}}}{\partial d_{nd}}
\end{bmatrix}
\]

is the optimal sensitivity matrix evaluated at the nominal point \( \mathbf{d}^* \). In eq 7, we have only included the first-order term in the Taylor expansion, so eq 7 is valid for small disturbances such that second- and higher-order terms in \( (\mathbf{d} - \mathbf{d}^*) \) can be neglected, or, equivalently, as long as \( \mathbf{F} \) does not change. From eq 6, the corresponding optimal change in the controlled variables is \( \mathbf{c}^{\text{opt}}(\mathbf{d}) - \mathbf{c}^{\text{opt}}(\mathbf{d}^*) = \mathbf{H}(\mathbf{y}^{\text{opt}}(\mathbf{d}) - \mathbf{y}^{\text{opt}}(\mathbf{d}^*)) \) and by inserting eq 7 we get

\[
\mathbf{c}^{\text{opt}}(\mathbf{d}) - \mathbf{c}^{\text{opt}}(\mathbf{d}^*) = \mathbf{HF}(\mathbf{d} - \mathbf{d}^*)
\]

which gives the requirement

\[
\mathbf{HF}(\mathbf{d} - \mathbf{d}^*) = 0
\]

This must be satisfied for any \((\mathbf{d} - \mathbf{d}^*)\), so we must require that

\[
\mathbf{HF} = 0
\]

Next, we must prove the conditions under which this is possible. To satisfy \( \mathbf{HF}(\mathbf{d} - \mathbf{d}^*) = 0 \), we must select \( \mathbf{H} \) such that \( \mathbf{H} \in \mathbb{R}^{n_\gamma \times (\mathbb{F}^T)} \). The rank of the \( n_\gamma \times n_y \) matrix \( \mathbf{H} \) is \( n_\gamma \) (because \( n_\gamma \geq n_\gamma \), and the controlled variables are independent). The rank of the \( n_\gamma \times n_y \) matrix \( \mathbf{F} \) is \( n_y \) (because \( n_y \geq n_y \)) and the disturbances are assumed to be independent. The fundamental theorem of linear algebra says that the left null space of \( \mathbf{F} \) \((\mathbb{F}^\top) \) has a rank of \( n_y \) (where \( r = n_d \) is the rank of \( \mathbf{F} \). To be able to find an \( \mathbf{H} \) of rank \( n_d \) in the null space of \( \mathbf{F} \), we must then require \( n_y \leq n_d \) or, equivalently, \( n_y \geq n_n + n_d \).

The proposed null space method is optimal only locally. It is globally optimal in cases where the sensitivity matrix \( \mathbf{F} \) is not dependent on the operating point (disturbances), for example, for a system with a quadratic cost objective and linear model equations (see eq 13, presented later in this work). Nevertheless, based on several case studies, this does not seem to be an important limitation in most practical cases.

**Obtaining \( \mathbf{F} \).** The optimal sensitivity matrix \( \mathbf{F} \) may be computed from the Hessian matrices \( \mathbf{J}_{uu} \) and \( \mathbf{J}_{ud} \) and steady-state gain matrices \( \mathbf{G}^2 \) and \( \mathbf{G}^3 \), using

\[
\mathbf{F} = -(\mathbf{G}^2 \mathbf{J}_{uu}^{-1} \mathbf{J}_{ud} - \mathbf{G}^3)
\]

However, in practice, it is usually easier to obtain \( \mathbf{F} \) directly, by optimizing for the selected disturbances using a nonlinear steady-state model of the plant. Note that we do not necessarily need an explicit representation of the model equations, because we can find \( \mathbf{F} \) numerically. For example, we may use one of the commercial steady-state process simulators, such as Aspen Plus or Hysys. In theory, one may even obtain \( \mathbf{F} \) from experiments on a real operating plant; however, it seems unlikely that this methodology will be sufficiently accurate.

Numerically, the \( n_\gamma \times n_d \) matrix \( \mathbf{F} \) may be obtained by perturbing the disturbances \( \mathbf{d} \) and re-solving the optimization problem in eq 3 with the active constraints being constant:

1. Under nominal conditions \((\mathbf{d} = \mathbf{d}^*)\), use the steady-state model to obtain the nominal optimum \( \mathbf{y}^{\text{opt}}(\mathbf{d}^*) \) and identify the active constraints (finding the nominal optimum may be difficult, because the optimization problem is generally non-convex).

2. For each of the \( n_d \) disturbances, make a small perturbation \((d_i = d^*_i + \Delta d_i)\) and resolve the optimization with the constant active constraints to obtain \( \mathbf{y}^{\text{opt}}(d_i) \) (this is generally simple, because it is only a small perturbation to the existing nominal solution).

3. Compute \( \Delta y^{\text{opt}} = y^{\text{opt}}(d_i) - y^{\text{opt}}(d^*) \) and obtain \( \mathbf{F} \) numerically using eq 8. A minimum of \( n_d + 1 \) optimization runs are required; however, more runs may be needed to obtain an accurate estimate for \( \mathbf{F} \).

Ganesh and Biegler\(^{17} \) have provided an efficient and rigorous strategy for finding \( \mathbf{F} \), based on a reduced Hessian method. In addition, some process simulators have built-in optimizers from which the optimal sensitivity \( \mathbf{F} \) may be available.

The next step is to obtain \( \mathbf{H} \). Numerically, \( \mathbf{H} \) may be obtained from a singular value decomposition of \( \mathbf{F} \). We have \( \mathbf{HF} = 0 \) or, equivalently, \( \mathbf{F}^\top \mathbf{H}^\top = 0 \). Thus, selecting \( \mathbf{H}^\top \) as the input
singular vectors of $F^T$, corresponding to zero singular values in $F^T$, gives an orthogonal basis.

Example 1. Consider a simple example with one unconstrained degree of freedom $n_u$, $n_u = 1$, and one disturbance $d$, $n_d = 1$. The cost function to be minimized during operation (for varying $d$) is

$$J(u, d) = (u - d)^2$$

Nominally, $d^* = 0$. We have available two measurements:

$$y_1 = 0.9u + 0.1d$$

and

$$y_2 = 0.5u - d$$

Because $n_y = 2 = n_u + n_d$ and the two measurements are independent, it is possible to find a linear measurement combination

$$c = Hy = [h_1, h_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = h_1y_1 + h_2y_2$$

for which a constant setpoint gives zero disturbance loss, at least locally. We first must obtain the optimal sensitivity matrix $F$. Optimality is ensured when $(\delta J/\delta u) = 2(u - d) = 0$, which gives $u^{\text{opt}} = d$ and $F^{\text{opt}} = 0 \forall d$. The corresponding optimal outputs are

$$y_1^{\text{opt}} = d$$

and

$$y_2^{\text{opt}} = -0.5d$$

and (8) gives that $F^T = [1 \ 0.5]$. Alternatively, use (13) with $G^{\text{opt}} = [0.9 \ 0.5]$, $G_d^{\text{opt}} = [0.1 \ -1]$, $J_{uu} = 2$, and $J_{ud} = -2$. From the null space method, the optimal matrix $H$ must satisfy $H^T F = 0$, or

$$h_1f_1 + h_2f_2 = 0$$

$$h_1 + h_2(-0.5) = 0 \Rightarrow h_1 = 0.5h_2$$

The solution is non-unique. For example, selecting $h_2 = 1$ gives

$$c = 0.5y_1 + y_2$$

Keeping the controlled variable $c$ at its nominally optimal setpoint, $c_n = c^{\text{opt}}(d^*) = 0$, gives zero disturbance loss, as is easily verified. Generally, the loss will be zero only locally, i.e., for small changes in $d$, but, for this example, the cost function is quadratic with linear model equations, and the loss will be zero for any magnitude of the disturbance $d$.

4. Discussion

4.1. Measurement Selection. One weakness of the null space method is that it does not consider the measurement error, or, more generally, the implementation error. If we have extra measurements, that is, $n_y > n_u + n_d$, then we have extra degrees of freedom in selecting $H$ that should be used to reduce the sensitivity to measurement error. A simple approach is to select a subset of the “best” measurements such that we get $n_y = n_u + n_d$, but which should these measurements be? This is outside the scope of this paper, and it will be treated in more detail a forthcoming publication on the extended null space method (also see Alstad13), but let us provide some results. Let the linear model be

$$\Delta y = G^T \Delta u + G_d^T \Delta d = \tilde{G}^T \begin{bmatrix} \Delta u \\ \Delta d \end{bmatrix}$$

(14)

where $y$ has been scaled, with respect to the expected measurement error, and $u$ and $d$ have been scaled such that they have similar effects on the cost. It can then be shown that a reasonable approach is to maximize the minimum singular value of the matrix $\tilde{G} = [G F]$ from the combined inputs and disturbances to the selected measurements. To understand why this is reasonable, we may imagine using the measurements to backcalculate the inputs and disturbances. For the case with $n_y = n_u + n_d$, $\tilde{G}$ is invertible and we get

$$\begin{bmatrix} \Delta u \\ \Delta d \end{bmatrix} = (\tilde{G}^T)^{-1} \Delta y$$

(15)

To avoid sensitivity to measurement errors in $y$, we want the norm of $(\tilde{G}^T)^{-1}$ to be small, which is equivalent to wanting a large minimum singular value, $\sigma(\tilde{G})$. From eq 15, we also see why it is reasonable to require $n_y \geq n_u + n_d$ in the null space method, because this is the requirement for being able to uniquely determine all independent variables (inputs and disturbances) from the measurements.

4.2. Freedom in Selecting $H$. Even for the case $n_y = n_u + n_d$, there are an infinite number of matrices $H$ that satisfy $HF = 0$. This stems from the freedom of selecting basis vectors for the null space. Let $H_b$ be one such matrix, i.e., $H_b F = 0$. For example, $H_b$ may consist of the one set of basis vectors that span the null space of $F^T$. Then, $H = CH_b$ also satisfies $HF = 0$, provided the $n_y \times n_y$ matrix $C$ is nonsingular.

Actually, the degrees of freedom in selecting $C$ (and $H$) are the same as the degrees of freedom that are used in steady-state decoupling (or similar) in control. The linear model for the selected controlled variables can be written as

$$\Delta c = H \Delta y = H G^T \Delta u + H G_d^T \Delta d = G \Delta u + G_d \Delta d$$

(16)

and the degrees of freedom in the matrix $C$ may be used to affect $G = HG^T$ and $G_d = HG_d^T$. For example, it is possible to select $H$ such that $G = I$, and we have a decoupled steady-state response from $u$ to $c$.

4.3. Disturbances. The minimum number of measurements required in the null space method $(n_u + n_d)$ may be large if we have many disturbances (a large $n_d$ value). Therefore, in practical applications, it is desirable to reduce the number of disturbances. Unfortunately, there does not seem to be any simple rigorous procedure for eliminating unimportant disturbances, although some approaches are discussed in Chapter 5 of the work by Alstad.15 It is obvious that we may eliminate disturbances $d_i$ that satisfy both of the following conditions: (1) no steady-state effect on the measurements ($y$ is independent of $d_i$, i.e., $G_d d_i = 0$), and (2) no steady-state effect on the optimal operation ($u^{\text{opt}}$ is independent of $d_i$).

It could be argued that we may eliminate all “unobservable” disturbances that satisfy condition 1, because we have no way of detecting them and, thus, correcting for them. However, such disturbances may affect the optimal operation and result in large losses; therefore, an analysis based on neglecting them may be highly misleading. To achieve acceptable operation in such cases, we must obtain additional measurements, for example, of the disturbance itself. One example would be a price change, as is discussed in more detail below. Also, we cannot eliminate
all disturbances that have no effect on optimal operation and thus satisfy condition 2. This is because the disturbance may affect a measurement, and controlling this measurement will then result in a loss.

In practice, with too few measurements, one may eliminate some disturbances and obtain the controlled variables \( \mathbf{c} = \mathbf{Hy} \) using the null space method, but, afterward, one should analyze the loss with all disturbances included. Alternatively, one may be able to obtain the optimal combination numerically using the exact local method of Halvorsen et al. or the extended null space method that will be presented in a forthcoming publication.

### 4.4. Physical Interpretation.

The proposed null space method yields controlled variables that are linear combinations of the available measurements. A disadvantage is that the physical interpretation of what we control is usually lost. This is by no means a fundamental limitation, because, in principle, we can control any signal from the process, as long as it is independent. Thus, if all measurements are regarded as signals, the concept of controlling a combination of signals may be easier to grasp. If possible, one can choose to combine measurements of one type, for instance, only temperatures (e.g., in a distillation column) or only mass flows. In any case, we can scale variables such that the resulting measurements are dimensionless, which is common in practice.

#### 4.4.1. Change in Active Constraints.

It has been assumed that the set of active constraints does not change. If they may change, then one must, for each set of active constraints, obtain new controlled variables \( \mathbf{c} = \mathbf{Hy} \) satisfying \( \mathbf{HF} = 0 \), where \( \mathbf{F} \) is evaluated with the given set of active constraints. This involves offline calculations. In addition, one must have an online strategy for identifying a change in active constraints change and some logic for switching to a new set of controlled variables. Thus, for cases where the active constraints shift frequently with the disturbances, other online methods may be better suited, for example, real-time optimization (RTO) combined with model predictive control (MPC). Alternatively, we could use the ideas of Arkun and Stephanopoulos in regard to how to handle varying active constraints.

#### 4.5. Nonobservable Disturbances and Price Changes.

Self-optimizing control is based on using feedback to detect disturbances and optimally adjust the inputs to achieve near-optimal operation. Thus, one must require that the disturbances are observable (visible) in the measurements \( \mathbf{y} \). One example of a “disturbance” that is not visible in the measurements is prices. However, prices \( (p_i) \) do enter in the objective function, because typically \( J = \sum p_i x_i \), and price changes will change the optimal point of operation.

To handle price changes (or, more generally, disturbances that are not observable in the measurements \( \mathbf{y} \)), one must assume that the price (disturbances) is known (measured). Price changes can then be handled in two ways:

1. Adjust the setpoints in a feedforward manner. Then, for a price change \( \Delta p \), we have that

\[
\mathbf{c}_s = \mathbf{c}_s(p^*) + \mathbf{HF}_p \Delta p
\]

2. Include the prices as extra measurements in \( \mathbf{y} \) and use the regular procedure of selecting self-optimizing control variables, as previously described.

The first approach is probably the simplest and most transparent.

### 4.6. Limitations.

As already mentioned, the proposed null space method is optimal only locally. It is globally optimal in cases where the sensitivity matrix \( \mathbf{F} \) is not dependent on the operating point. Nevertheless, based on several case studies, this does not seem to be an important limitation in most practical cases. Other possibly more-serious limitations for practical use are that (1) implementation errors are not explicitly handled (except through the selection of which measurements to use), (2) the active constraints are assumed not to change (as discussed previously, in more detail), (3) the optimal setpoint for \( c \) must be obtained (not really a limitation, but rather a fact), and (4) the derivation of the optimal measurement combination \( \mathbf{c} = \mathbf{Hy} \) is based on steady-state models, and one must later check that the resulting structure has acceptable dynamic controllability (this may usually be achieved using a cascade control structure where local controllers handle the dynamic response and \( c \) is controlled at steady state using the setpoints for the local controllers as manipulated variables).

### 5. Petlyuk Distillation Case Study

#### 5.1. Introduction.

The Petlyuk distillation column is an appealing alternative for the separation of ternary mixtures. Compared with the traditional configuration of two columns in series, typical savings on the order of 30% are reported in both energy and capital costs. However, the savings in energy may be difficult to achieve in practice, and the goal here is to suggest simple control policies. We are looking for a “self-optimizing” control structure that, despite external disturbances and measurements errors, gives near-optimal operation with constant setpoints.

The Petlyuk column has six sections and may be implemented as a “divided wall” column, as illustrated in Figure 2. The boilup and reflux streams are split at the dividing wall with split fractions \( R_1 = V_2/V_1 \) and \( R_i = L_i/L_3 \), respectively. With a given feed and pressure, the Petlyuk column has five steady-state degrees of freedom. For example, these may be selected as

\[
\mathbf{u}^T_{all} = [L \ V \ S \ R_L \ R_V]
\]
The minimum boilup \( V_{\text{min}} \) with an infinite number of stages is \( V_{\text{min}} = 0.5438 \); therefore, the nominal optimal boilup of 0.5811 is \( \sim 6\% \) higher than the theoretical minimum.

Because the objective is to minimize the boilup, which also is an input, one may mistakenly believe that one can use an open-loop approach, where the optimal value for the boilup is calculated and implemented in the column, \( V = V^{\text{opt}} \). However, Halvorsen and Skogestad noted that such an approach is impossible (or at least very difficult):

(1) The operation is infeasible for \( V < V^{\text{opt}} \), so we must ensure that \( V \geq V^{\text{opt}} \).

(2) The optimal value of \( V \) varies, with respect to the disturbances, and may be difficult to find, requiring a detailed model and a direct measurement of the disturbances (or a very accurate estimate) to be viable. This is unrealistic in most cases.

(3) The measurement or estimation of \( V \) may be difficult to achieve (measuring vapor flow); thus, it may be sensitive to measurement error when trying to implement the optimal \( V \).

Thus, the approach here is to use self-optimizing control. As candidate measurements (\( y \)), we include all flows (ratioed to the feed), as well as the temperature at all stages. This gives \( \sim 60 \) measurements. The component compositions in each product stream are also measured; however, because of the fact that they are active constraints (and, thus, are constant), they are not useful for self-optimizing control and are not included in \( y \).

**Alternative 1: Two Degrees of Freedom.** We first consider using the two available unconstrained degrees of freedom \( (n_u = 2) \) to control (and fix) two measurement combinations. The two unconstrained degrees of freedom could, for example, be \( u = [R_L, R_V] \) (but the specific choice does not actually matter for the steady-state analysis). The assumed disturbance vector is (these are determined to be the most important disturbances, from those listed in Table 1)

\[
d_{\text{def}} = \begin{bmatrix} z_A \\ z_B \\ q \\ x_{B,S} \end{bmatrix}
\]

(19)

where the subscript “tdf” denotes that the are two degrees of freedom. The last entry represents the composition offset for the side-stream product. The feed rate \( F \) is not included, because we have chosen to use only intensive variables when forming...
the controlled variables (with a constant column efficiency, a feed rate change is automatically compensated for at steady state by fixing intensive variables).

To use the null space method, we need from Theorem 1 to combine the best subset of 6 out of the ~60 candidate measurements. We use the measurement selection approach mentioned in the discussion section. This results in the following six temperature measurements:

\[ y_{\text{df}} = \begin{bmatrix} T_{37} \\ T_{11} \\ T_{43} \\ T_{25} \\ T_4 \\ T_9 \end{bmatrix} \]  

The location of the selected measurements is shown in Figure 3. Note that the majority of measurements are located in the bottom portion of the column, whereas only two measurements are located above the feed point. The sensitivity matrix \( F \) was obtained numerically by perturbing each of the four disturbances and reoptimizing. The null space method in Theorem 1 gives the optimal measurement combinations (controlled variables):

\[ c_{\text{df},1} = -0.472T_{37} + 0.312T_{11} + 0.113T_{43} - 0.457T_{25} + 0.561T_4 - 0.378T_9 \]  

\[ c_{\text{df},2} = 0.185T_{37} - 0.376T_{11} - 0.667T_{43} - 0.524T_{25} + 0.154T_4 + 0.285T_9 \]

**Alternative 2: One Degree of Freedom.** Until now, we have assumed that the vapor split \( R_V \) is a degree of freedom during operation (available for manipulation); however, most likely, this is not possible in practice. Therefore, it is interesting to consider the case where \( R_V \) is fixed. In fixing \( R_V \), we add the implementation error of controlling \( R_V \) to the disturbance vector and get

\[ d_{\text{df}} = \begin{bmatrix} z_A \\ z_B \\ q \\ x_{A,S} \\ x_{B,S} \\ R_V \end{bmatrix} \]

The corresponding minimum number of measurements needed for the null space method is \( n_d + n_d = 1 + 5 = 6 \). The following subset of measurements was obtained:

\[ y_{\text{odf}} = \begin{bmatrix} T_{37} \\ T_{10} \\ T_{43} \\ T_{27} \\ T_5 \\ T_{12} \end{bmatrix} \]  

which is very similar to the two-degrees-of-freedom case. The optimal measurement combination from the null space method is

\[ c_{\text{odf}} = -0.388T_{37} - 0.658T_{10} + 0.192T_{43} - 0.047T_{27} + 0.448T_5 + 0.421T_{12} \]

**Loss Evaluation Using a Nonlinear Model.** The two previously mentioned control structures are compared with two alternative control structures (see Table 2). Alternative 3 is a control structure proposed by Halvorsen and Skogestad,\(^\text{20}\) where \( c_{\text{odf}} = (T_4 - T_{28}) + (T_{12} - T_{29}) \) is a symmetry measure of the temperature difference over the dividing wall, whereas alternative 4 is the “open loop” approach.

The nonlinear losses for the alternative control structures for different realistic magnitude of the disturbances and measurement errors are given in Table 3. In the table, we have also included losses for changes (‘‘disturbances’’) in the distillate and bottom product compositions (active constraints), which were not included in the original disturbances used to derive \( c_{\text{odf}} \). The conclusion is that the self-optimizing properties are excellent for both alternatives 1 and 2. When fixing two measurement combinations in alternative 1, the loss in energy usage (\( V \)) is \(<0.02\%\) for the disturbances considered and \( \approx 0.2\% \) for a disturbance in the bottom composition (which was not considered when deriving \( c_{\text{odf}} \)). The losses, with respect to implementation errors, are also very small. When fixing \( R_V \) and \( c_{\text{odf}} \) (alternative 2), the loss is \( \sim 10 \) times higher for the previously considered disturbances; however, it is still only \( \sim 0.2\% \) and, thus, is insignificant, from a practical point of view. Because the loss is so small, the strategy of fixing \( R_V \) and \( c_{\text{odf}} \) is clearly preferred for practical implementations. The losses for the other two control structures are higher, with a maximum disturbance loss of 11% for alternative 3 and 95% for alternative 4. Alstad\(^\text{13}\) also considered alternatives where \( R_V \) and a single
temperature was fixed. The best methodology was to fix a temperature just below the side stream with a maximum disturbance loss of ~1.4%.

In the work of Alstad, nonlinear closed-loop dynamic simulations are shown, which confirm the practical implementation of alternative 2.

6. Conclusion

This paper has introduced the null space method for selecting controlled variables \( c \). We consider a constant setpoint policy, where the controlled variables are kept at constant setpoints \( c \). We propose to select self-optimizing controlled variables as linear combinations \( c = H y \) of a subset of the available measurements \( y \). With no implementation error, it is locally optimal to select \( H \) such that \( HF = 0 \), where \( F = (\partial y / \partial d) \) is the optimal sensitivity, with respect to disturbance \( d \). However, ignoring the implementation error is a serious shortcoming for some applications. To compensate for this partially, it is important to use measurements \( y \) that are independent and not sensitive to measurement error. Another shortcoming is that a new set of controlled variables (for the unconstrained degrees of freedom) must be found for each possible set of active constraints. The global properties of the proposed variable combination \( c = H y \) must be evaluated by computing the loss for expected disturbances and implementation errors using the nonlinear model, and a controllability analysis should also be performed before implementation. The method has been illustrated on a Petyuk distillation example, where we find that the null space method yields controlled variables with very small losses.

Literature Cited


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