Robust Design of Internal Models by Nonlinear Regression

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Abstract: This paper focuses on the design of robust internal model-based regulators for a class of nonlinear systems and exosystems such that the desired steady state control law fulfills a nonlinear regression formula affine in possible uncertainties. The design methodology builds upon the ideas presented in [9] in the context of linear adaptive output regulation. The design methodology relies upon an assumption of left invertibility of a matrix. Through numerical analysis we show how the design methodology succeeds in relevant cases, such as the cases in which the desired steady state is generated by uncertain Van der Pol, Duffing, and Lorentz oscillators.

1. INTRODUCTION

The paper deals with the problem of output regulation for a class of nonlinear systems under the assumption that the desired steady state input needed to enforce a regulation error that is identically zero fulfills a regression formula parametrized by possible uncertainties in a linear way. The design procedure builds upon the general ideas proposed in [9] in which the theory of nonlinear high-gain observers, proved to be effective in [2] within the context of nonlinear output regulation, is used in order to design a nonlinear adaptive internal model which does not rely upon an “explicit” adaptation law. The goal of [9] was to enrich the design strategies for adaptive output regulation by presenting a tool alternative to the ones already available in literature (see [11], [3] and [4] among others) more inspired by standard adaptive design strategies and relying upon an “explicit” estimation of the uncertainties. The main assumption of [9] is that the desired steady state input able to enforce an identically zero regulation error fulfills a linear regression law. This assumption practically limited the applicability of the tool to the cases in which the desired steady state input is generated by an uncertain linear oscillator, as originally proposed in [12]. In this paper we make a step further by showing how nonlinear regression laws can be successfully dealt with, by thus enlarging the class of exosystem that can be handled. In order to prove the effectiveness of the design strategy, the particular cases in which the desired steady state input is generated by the Van der Pol, the Duffing, and the Lorentz uncertain nonlinear oscillators are specifically addressed. The presented methodology strongly relies on high-gain tools and on the design of high-gain observers [see [5]]. It thus expected that the method presents intrinsic limitations in applicative fields where the dimension of the exosystem, and thus of the internal model, is large and the measured variables are affected by high-frequency noise.

The paper frames in the already rich literature on nonlinear output regulation pioneered with the work [8] and still a field of active research nowadays (see [6] for general presentation of the problem and of the techniques, and [7] for applicative scenarios).

2. PROBLEM DESCRIPTION AND PRELIMINARIES

We consider the class of nonlinear systems described by the following normal form

\[
\begin{align*}
\dot{z} &= f(w,z,e) \\
\dot{e} &= q(w,z,e) + b(w,z,e)u
\end{align*}
\]

with state \((w,z,e) \in \mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}\), control input \(u \in \mathbb{R}\) and with \(f(w,z,e)\), \(q(w,z,e)\) and \(b(w,z,e)\) that are smooth functions of their arguments. The function \(b(w,z,e)\), referred to as high-frequency gain, is assumed to be bounded from below by a positive number \(\delta\), namely \(b(w,z,e) \geq \delta\) for all \((w,z,e) \in \mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}\). As usual in output regulation, the exogenous variable \(w\) is generated as solution of an exosystem having the form

\[
\dot{w} = s(w).
\]

The initial conditions \((w(0), z(0), e(0))\) of the system range in a known compact set \(W \times Z \times E \subset \mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}\), with \(W\) assumed to be invariant for \((2)\).

Note that (1) has relative degree 1 between the input \(u\) and the output \(e\). All the results presented in the paper can be extended to the case of higher relative degree by mean of standard arguments that are omitted.

The problem of nonlinear output regulation amounts to design a output feedback smooth controller of the form

\[
\begin{align*}
\dot{\xi} &= \varphi(\xi,e) \\
u &= \psi(\xi,e)
\end{align*}
\]

with state \(\xi \in \mathbb{R}^d\) and a compact set \(\Xi \subset \mathbb{R}^d\), such that the closed loop trajectories originating from \(W \times Z \times E \times \Xi\) are bounded and the error \(e(t)\) asymptotically decays to 0 uniformly in the initial conditions.
By following [1], we assume the existence of a function $\pi: W \to \mathbb{R}^n$ solution of the regulator equation
\[ L_s(w)\pi(w) = \ldots \] have omitted the argument $w$ of $u^\star$.

The proposed methodology relies upon the following crucial assumption.

Assumption 1 (minimum-phase): The set $\text{graph}(\pi) = \{(w, z) \in W \times \mathbb{R}^n : z = \pi(w), e = 0\}$ (6) 

invariant for (1)-(2).

The design methodology presented in paper relies upon high-gain arguments to make the set $\mathcal{M}$ semiglobally attractive. For this reason the following minimum-phase assumption will be used.

\[ \tau(w) = \begin{bmatrix} \tau_0(w) \\ \vdots \\ \tau_{d-1}(w) \end{bmatrix}, \quad \begin{bmatrix} u^\star(w) \\ L_{s(w)}^{-1}u^\star(w) \end{bmatrix} \] (14)

and let $\phi_s: \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz and bounded function that agrees with $\phi(t)$ on $\tau(W)$. Then, it is possible to show that if the triplet $(F(\cdot), \gamma(\cdot), G)$ is chosen so that

\[ F(\xi) = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_{d-1} \end{bmatrix} - G\xi_0 \] (15)

$\gamma(\xi) = \xi_0$, and

\[ G = [g_0\lambda_0, g_2^2\lambda_1, \ldots, g_2^d\lambda_{d-1}]^T \] (16)

where $(\lambda_0, \lambda_1, \ldots, \lambda_{d-1})$ are coefficients of an Hurwitz polynomial and $g$ is a design parameter, then there exists a $g^* > 0$ such that for all $g \geq g^*$ the triplet in question fulfills the properties of Proposition 1.

In the next section we present a simple condition under which a function $\phi$ fulfilling (13) can be found, and thus a regulator of the form (9) can be designed.

3. DESIGN IN CASE OF NONLINEAR REGRESSION LINEARLY PARAMETRIZED IN THE UNCERTAINTIES

The starting point in the design methodology is the existence of a regression formula that governs the $k$-th time derivative of the desired steady state input $u^\star$. The formula is specified in the next Assumption. For ease of notation, here and in the following we let $u^{*(a,b)} := (u^{*(a)}, \ldots, u^{*(b)})^T$, with $0 \leq a < b$, the vector of time derivatives of $u^\star$.

Assumption 2. There exist $k > 0$, $p > 0$, locally Lipschitz functions $h: \mathbb{R}^k \to \mathbb{R}$ and $L: \mathbb{R}^p \to \mathbb{R}^p$ such that

\[ u^{*(k)}(w) = h(u^{*[0,k-1]}(w)) + L(u^{*[0,k-1]}(w)) \mu \quad \forall w \in W, \] (17)

where $\mu \in \mathbb{R}^p$ is a vector of uncertainties.

In the second part of this section we show how the previous assumption is fulfilled in a number of relevant cases.

By differentiating $i \geq 0$ times relation (17) and collecting the resulting equations, we obtain the following set of equations

\[ u^{*[k,k+i]}(w) = H_i(u^{*[0,k+i-1]}(w)) + A_i(u^{*[0,k+i-1]}(w)) \mu \] (18)

where

\[ H_i(u^{*[0,k+i-1]}(w)) = \text{col}\left[ L_0(u_{[0,k-1]}^*) \ldots L_i(u_{[0,k+i-1]}^*) \right] \]

\[ A_i(u^{*[0,k+i-1]}(w)) = \text{col}\left[ L_0(u_{[0,k-1]}^*) \ldots h_i(u_{[0,k+i-1]}^*) \right] \] (19)

where $L_0(\cdot) = L(\cdot), h_0(\cdot) = h(\cdot), L_j(\cdot), h_j(\cdot), j = 0, \ldots, i-1$, and where for compactness we have omitted the argument $w$ of $u^\star$.

The proposed methodology relies upon the following crucial assumption.
Assumption 3: There exists a $m \geq p$ and $\epsilon > 0$ such that

$$\det(A_m^T(u^*_{[0,k+m-1]}(w)) A_m(u^*_{[0,k+m-1]}(w))) \geq \epsilon$$

for all $w \in W$. \( \triangleright \)

The previous assumption implies that

$$\text{rank}(A_m(u^*_{[0,k+m-1]}(w))) = p \quad \forall w \in W$$

and, in turn, that the uncertain vector $\mu$ can be obtained from (18) as a function of $u^*$ and its first $(k + m)$-th time derivatives. In particular, taking the $(m+1)$-th time derivative of (17) and replacing $\mu$ with the estimation obtained by left-inverting (18) for $i = m$, one obtains

$$u^*(m+1) = h_m(u^*_{[0,k+m-1]}) + L_{m+1}(u^*_{[0,k+m-1]}).$$

where $A_m^T$ represents a pseudoinverse of $A_m$ given by

$$A_m^T(\cdot) = [A_m^T(\cdot) A_m(\cdot)]^{-1} A_m(\cdot).$$

This relation, in turn, is equivalent to (13) for an appropriately defined $\phi(\cdot)$ with $d = m + k + 1$.

In the remaining part of the section we show how the previous assumptions are fulfilled in a number of relevant cases in which $u^*$ is generated by nonlinear oscillators. The three cases of Van der Pol, Duffing, and Lorentz uncertain oscillators are considered and are dealt with in the following subsections.

3.1 Van der Pol Oscillator

As exosystem (2) consider the Van der Pol oscillator described by

$$\begin{align*}
\dot{w}_1 &= w_2 \\
\dot{w}_2 &= -\omega^2 w_1 + \epsilon (1 - w_1^2) w_2
\end{align*}$$

in which $\omega$ and $\epsilon$ are uncertain parameters, and consider the case in which the desired steady state input $u^*(w) = w_1$, the set $W$ is the omega limit set where the steady state trajectories of the Van der Pol take place. It turns out that

$$\ddot{u}^*(w) = -u^*(w) \omega^2 + (1 - u^2(w)) \dot{u}^2(w) \epsilon$$

and thus Assumption 2 is fulfilled with $\kappa = 2$, $h(\cdot) = 0$, $L(\cdot) = (-u^*(w), (1 - u^2(w)) \dot{u}^2(w))$ and $\mu = (\omega^2, \epsilon)^T$. We start now to take time derivatives of (21) to identify an $m \geq 2$ for which Assumption 3 is fulfilled. By differentiating once, we obtain

$$u^*_{[0,2]}(w) = A_1(u^*_{[0,2]}(w)) \mu$$

where

$$A_1(u^*_{[0,2]}) = \begin{bmatrix}
-u^* & (1 - u^2) u^* \\
-u^* & -2 u^* u^2 - u^2 \dot{u}^* \\
\end{bmatrix}.\quad (23)$$

It turns out that there are points of $W$ where $A_1$ is singular (see Fig. 1). By thus taking a further derivative we obtain

$$u^*_{[0,3]}(w) = A_2(u^*_{[0,3]}(w)) \mu$$

with

$$A_2(u^*_{[0,3]}) = \begin{bmatrix}
-u^* & (1 - u^2) u^* \\
-u^* & -2 u^* u^2 - u^2 \dot{u}^* \\
-u^* u^3 - 2 \ddot{u}^3 - 6 u^* \dot{u}^2 \ddot{u}^* - u^2 u^{(3)}
\end{bmatrix}.\quad (25)$$

A numerical analysis of the minors of $A_2$ (see Fig. 2) reveals that the matrix has rank 2 for all $w \in W$ and thus assumption 3 is fulfilled.

Fig. 1. Determinant of $A_1(u^*_{[0,2]})$ on the limit cycle ($\omega^2 = 1$ and $\epsilon = 1$).

3.2 Duffing Oscillator

We consider now the case in which $u^*(w)$ is generated by the Duffing oscillator modeled by

$$\begin{align*}
\dot{w}_1 &= w_2 \\
\dot{w}_2 &= -w_1^3 \alpha - w_1 \beta
\end{align*}$$

where $\alpha$ and $\beta$ are uncertain parameters and $u^*(w) = w_1$. the set $W$ is the limit cycle of the oscillator. It turns out that

$$\ddot{u}^*(w) = -u^3 \alpha - u^* \beta,$$
namely Assumption 2 is fulfilled with \( k = 2, h(\cdot) = 0, \ L(\cdot) = (\ -u^3(w), -u^*(w) \) \) and \( \mu = (\alpha, \beta)^T \). By differentiating once relation (27) we obtain \( u_{[2,3]}^* = A_1(u_{[0,1]}^*)\mu \) with

\[
A_1(u_{[0,1]}^*) = \begin{bmatrix}
-3u^3 w^2 & -u^* \\
-3u^* w^2 & -u^*
\end{bmatrix}
\]  

(28)

that is singular in some point of the limit cycle. Taking a further derivative we get \( u_{[2,4]}^* = A_2(u_{[0,2]}^*)\mu \) with

\[
A_2(u_{[0,2]}^*) = \begin{bmatrix}
-3u^3 w & -u^* \\
-3u^* w & -u^*
\end{bmatrix}
\]  

(29)

that is still rank-deficient. By thus taking a further derivative we get \( u_{[2,5]}^* = A_3(u_{[0,3]}^*)\mu \) with

\[
A_3(u_{[0,3]}^*) = \begin{bmatrix}
-3u^* w & -u^* \\
-3u^* w & -u^*
\end{bmatrix}
\]  

(30)

which, finally, has rank 2 (see Figures 3-4).

Fig. 3. Limit cycle for the Duffing oscillator with \( \alpha = 1, \ \beta = -2 \). In the red points at least one minor out of six of matrix \( A_3(u_{[0,3]}^*) \) is not singular.

3.3 Lorenz Oscillator

As a third example we consider the case in which \( u^* \) coincides with the \( w_1 \) component of the Lorenz oscillator described by

\[
\begin{aligned}
w_1 &= \sigma(w_2 - w_1) \\
w_2 &= w_1 \rho - w_2 - w_3 \\
w_3 &= w_1 w_2 - \beta w_3
\end{aligned}
\]  

(31)

where \((\sigma, \rho, \beta)\) are positive uncertain parameters. We let the set \( W \) coincide with the Lorenz attractor by assuming a persistence of excitation condition of the oscillator. Specifically we assume there exists a \( \epsilon > 0 \) such that

\[ w_1^2 + w_2^2 = \|u_{[0,1]}^*\|^2 \geq \epsilon \quad \forall w \in W. \]

Hence Assumption 3 is fulfilled with \( k = 2, \ h(\cdot) = 0, \ L(\cdot) = (\ -u^3(w), -u^*(w) \) \) and \( \mu = (\alpha, \beta)^T \). By differentiating once relation (27) we obtain

\[
A_1(u_{[0,1]}^*) = \begin{bmatrix}
-3u^* w & -u^* \\
-3u^* w & -u^*
\end{bmatrix}
\]  

(32)

with \( c_1 := \sigma(\rho - 1), \ c_2 := -\sigma \). Furthermore,

\[
w_3 = u^{*3}(w) + u^*(w)\dot{u}^*(w) - \beta w_3.
\]

By differentiating once more (32) and using the previous expression of \( w_3 \), we obtain

\[
u^{*3} = -u^{*2}u^* - c_1 u^* + c_2 u^* + c_3 u^* \]  

(33)

Relations (32) and (33) can be compactly rewritten as

\[
u_{[2,3]}^* = \rho(u_{[0,2]}^*) + C(\rho, \sigma) \varphi(u_{[0,2]}^*) + M(\sigma, \beta) u_{[0,1]}^* w_3
\]

where

\[
\varphi = \begin{bmatrix}
\rho(u_{[0,2]}^*) \\
u^* \end{bmatrix}, \rho = \begin{bmatrix}
\dot{u}^* \\
\dot{u}^* - u^{*2}\tilde{u}^*
\end{bmatrix}
\]  

(34)

and

\[
C := \begin{bmatrix}
c_1 & c_2 & 0 \\
0 & c_1 & c_2
\end{bmatrix}, \ M := \begin{bmatrix}
c_2 & 0 \\
0 & -c_2\beta
\end{bmatrix}.
\]  

(35)

By taking advantage from the persistence of excitation condition, the previous relation can be used to express \( u_3 \) as a function of \( u_{[0,3]}^* \), namely

\[
w_3 = \frac{1}{\|u_{[0,1]}^*\|^2} u_{[0,1]}^* T M^{-1} \left( \rho(u_{[0,2]}^*) + C \varphi(u_{[0,2]}^*) \right)
\]

or, equivalently,

\[
w_3 = \frac{u_{[0,1]}^* T \otimes \rho(u_{[0,2]}^*)^T}{\|u_{[0,1]}^*\|^2} \vect(M(\sigma, \beta)^{-1}) + \frac{u_{[0,1]}^* T \otimes \varphi(u_{[0,2]}^*)^T}{\|u_{[0,1]}^*\|^2} \vect(M(\sigma, \beta)^{-1} C(\rho, \sigma))
\]
where $\otimes$ denotes the Kronecker product and $\text{vect}(T)$ is the column vector obtained by taking row-wise the elements of the matrix $T$.

Furthermore, by taking another derivative of (33) we get

$$u^{(*)} = -3u^* u^2 - u^* u^* - u^{*(3)} +
\begin{align*}
c_1 u^* + c_2(u^{*3} + 4u^2 u^*) - c_2 \beta u^3 + \beta u^2 u^* \\
c_2(u^* - 2\beta u^* + \beta^2 u^*)w_3
\end{align*}$$

by which, using the expression of $w_3$ above and compacting the terms, we obtain

$$u^{(*)} = h(u_{[0,3]}^*) + L(u_{[0,3]}^*)\mu$$

with $\mu \in \mathbb{R}^{10}$ defined as

$$\mu := (\sigma, \beta \sigma \rho, \beta^2 \sigma \rho, \beta^3 \sigma \rho, \beta \sigma, \beta^2 \sigma, \beta^3 \sigma, \beta, \beta^2, \beta^3)^T$$

and where $h(\cdot)$ and $L(\cdot)$ are appropriately defined functions. This proves that Assumption 2 is fulfilled. To check if there exists a value of $\mu$ such that Assumption 3 is fulfilled, we go further by simplifying a bit the analysis by assuming that the parameter $\beta$ is known. This implies, by rearranging a bit the terms in (37), that the following relation

$$u^{(*)} = \tilde{h}(u_{[0,3]}^*) + \tilde{L}(u_{[0,3]}^*)\tilde{\mu}$$

holds, where $\tilde{h}$ and $\tilde{L}$ are known functions (dependent on $\beta$) and $\tilde{\mu}$ is defined as $\tilde{\mu} = (\sigma, \rho \sigma)^T$.

By differentiating once the equation (38) we get the following compact form

$$u_{[4,5]}^* = \tilde{H}_1(u_{[0,4]}^*) + \hat{A}_1(u_{[0,4]}^*)\tilde{\mu}$$

with

$$\tilde{H}_1(u_{[0,4]}^*) := \begin{bmatrix} \tilde{h}(u_{[0,3]}^*) \\ \tilde{h}_1(u_{[0,4]}^*) \end{bmatrix}$$

and

$$\hat{A}_1(u_{[0,4]}^*) := \begin{bmatrix} \tilde{L}(u_{[0,3]}^*) \\ \tilde{L}_1(u_{[0,4]}^*) \end{bmatrix}$$

To check whether the $2 \times 2$ matrix $\hat{A}_1(u_{[0,4]}^*)$ fulfills Assumption 3, we ran simulations with different values of the parameters and of initial conditions and we found that the matrix is singular in certain points of the Lorenz attractor. A further time derivative is thus taken by obtaining

$$u_{[4,5]}^* = \tilde{H}_2(u_{[0,5]}^*) + \hat{A}_2(u_{[0,5]}^*)\tilde{\mu}$$

in which

$$\tilde{H}_2(u_{[0,5]}^*) := \begin{bmatrix} \tilde{h}(u_{[0,3]}^*) \\ \tilde{h}_1(u_{[0,4]}^*) \end{bmatrix}$$

and

$$\hat{A}_2(u_{[0,5]}^*) := \begin{bmatrix} \tilde{L}(u_{[0,3]}^*) \\ \tilde{L}_1(u_{[0,4]}^*) \end{bmatrix}$$

with $\hat{A}_2(u_{[0,5]}^*)$ that is a $3 \times 2$ matrix. Numerical tests obtained with different values of the parameters and of the initial conditions showed that the three determinants of each minor of the matrix are never simultaneously zero, namely that the matrix has rank 2 on the Lorenz attractor for the numerical values used in the simulation. Assumption 3 is thus numerically verified and we obtain that relation (13) is fulfilled with a $\phi(\cdot)$ of the form

$$u^{(*)} = \tilde{h}_3(u_{[0,6]}^*) + \tilde{L}_3(u_{[0,6]}^*)\hat{A}_2(u_{[0,5]}^*)u_{[4,6]} - \tilde{H}_2(u_{[0,5]}^*)$$

where $\hat{A}_2$ is the left inverse of $\hat{A}_2$.

4. EXAMPLE

In this section we propose an example in order to validate numerically the proposed method. We consider as controlled plant, a linear oscillator

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u - w_1$$

forced by the control variable $u$ and by a matched exogenous disturbance $w_1$ generated by a Van der Pol exosystem modeled as in (20) with the two uncertain parameters $\epsilon$ and $\sqrt{\cdot}$ both equal to 1. The control goal is to regulate $x_1$ to zero by means of a state feedback control. Output feedback solutions can be easily obtained by the state feedback solution derived below by means of standard arguments that are here omitted. In this case we can define the error $e = x_1 + x_2$ whose dynamics are governed by $\dot{e} = e + u - w_1 - 2x_1$. The system is in the form (1) with $z = x_1$ governed by $\dot{z} = -z + e$ that trivially fulfills Assumption 1. The desired state state input is clearly $u^* = w_1$. The goal can be achieved using the regulator (9) with $d = 5, g = 10$ and $\lambda_0, \ldots, \lambda_3$ such that the polynomial $s^5 + \lambda_3 s^4 + \cdots + \lambda_2 s + \lambda_0 = 0$ is Hurwitz with two roots in $-1$ and three roots in $-2$. By following the theory in Section 3.1 the function $\phi(\cdot)$ in (13) is of the form

$$\phi(\xi) = \begin{bmatrix} \xi_3, \xi_4 - \xi_0 \xi_3 - 12 \xi_2^2 \xi_3 - 6 \xi_0 \xi_2^2 - 8 \xi_0 \xi_3 \xi_3 \end{bmatrix}$$

with the vector of estimated parameters given by

$$\hat{\mu} = A_2^T(u_{[0,3]}^*)(\xi_3, \xi_3, \xi_4)^T$$

where $A_2^T$ is the left inverse of $A_2$. This function, properly saturated over $\tau(W)$, has been used as $\phi_4(\cdot)$ in the expression (15) for $F$. As far as the stabilizer is concerned, the function $\kappa(\cdot)$ has been chosen as linear function $\kappa = 40$. As shown in Figure 5, the harmonic oscillator starts from an initial condition with $x_1 = 1$ and asymptotically converges to zero.

![Fig. 5. Regulation error $e(t)$](image)

In Figure 6 is shown the control input $u(t)$ asymptotically converging to the exogenous signal $w_1$.

5. CONCLUSIONS

The problem of designing internal model-based regulators has been considered for the class of systems (1) in the
special case in which the desired steady state input defined in (5) fulfills a regression law specified in (17). The crucial assumption behind the proposed methodology is Assumption 3. In the relevant cases in which (5) is generated as output of the Van der Pol, Duffing and Lorentz oscillators affected by uncertainties, we have shown the method applies and robust internal model-based regulator can be designed. Numerical analysis has been used to practically support the analysis. The presented methodology strongly relies on high-gain tools and on the design of high-gain observers (see [5]). It thus expected that the method presents intrinsic limitations in applicative fields where the dimension of the exosystem, and thus of the internal model, is large and the measured variables are affected by high-frequency noise.

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