On componentwise ultimate bound minimisation for switched linear systems via closed-loop Lie-algebraic solvability

Rahmat Heidari * Maria M. Seron * Julio H. Braslavsky ** ***
Herman Haimovich ***

* Priority Research Centre for Complex Dynamic Systems and Control, The University of Newcastle, Callaghan NSW 2308, Australia (heidari.rahmat@gmail.com, maria.seron@newcastle.edu.au).
** Australian Commonwealth Scientific and Industrial Research Organisation (CSIRO), Division of Energy Technology, Newcastle, NSW 2300, Australia (julia braslavsky@csiro.au).
*** CIFASIS-CONICET; Depto. de Control, FCEIA, Univ. Nacional de Rosario, Argentina. (haimo@fceia.unr.edu.ar).

Abstract: We present a novel state feedback design method for perturbed discrete-time switched linear systems. The method aims at achieving (a) closed-loop stability under arbitrary switching and (b) minimum of ultimate bounds for specific state components. Objective (a) is achieved by computing state feedback matrices so that the closed-loop $A$ matrices generate a solvable Lie algebra (i.e. admit simultaneous triangularisation). Previous results derived an iterative algorithm that computes the required feedback matrices, and established conditions under which this procedure is possible. Based on these conditions, objective (b) is achieved by exploiting available degrees of freedom in the iterative algorithm.

Keywords: Switched systems, Eigenstructure assignment, Ultimate bounds.

1. INTRODUCTION

In the last decade there has been increasing research activities in the areas of stability and stabilisability of switched systems; see, for example, Liberzon [2003], Shorten et al. [2007], Lin and Autsakis [2009]. A problem of interest is that of stability under arbitrary switching between subsystems, which consists in obtaining conditions that guarantee stability of the switched system for every switching signal. Finding these conditions in general is difficult except for special cases where the subsystems are pairwise commutative, symmetric or normal [Liberzon, 2003]. An equivalent condition to stability under arbitrary switching is the existence of a common Lyapunov function for all subsystems. As a special case, one can study the existence of a common quadratic Lyapunov function (CQLF).

While most efforts on stability and stabilisation of switched systems deal with asymptotic stability of the origin (as equilibrium point of the system), it might not be possible to achieve asymptotic stability in some situations, such as, for example, when the switched system is subject to non-vanishing disturbances. In these cases, the concern is practical stability of the system in the sense that its trajectories ultimately lie in a bounded region sufficiently close to the origin. In order to have an acceptable system performance, it is desirable for the system ultimate bounds, characterising these regions, to be sufficiently small. However, for a discrete-time system no feedback law can produce arbitrarily small closed-loop ultimate bounds since the latter are bounded from below by the effect of the disturbance on the state equations. For switched systems, this limitation could be more severe as each subsystem may have different disturbance characteristics.

In this paper, we address closed-loop stability under arbitrary switching and ultimate bound minimisation simultaneously, for discrete-time switched systems. To this purpose, our first contribution is to derive conditions in terms of eigenstructure of the perturbed switched system in order for the trajectories of one component of the state to lie within the minimum possible ultimate bound. Then, we exploit an algorithm from Haimovich and Braslavsky [2013], which iteratively seeks feedback matrices for each subsystem, and a common transformation matrix so that the closed-loop $A$ matrices are stable and can be simultaneously transformed into upper-triangular by means of the computed transformation. Simultaneous triangularisation of the closed-loop $A$ matrices (generation of a solvable Lie algebra) in addition to stability of each subsystem guarantees the existence of a CQLF [Liberzon, 2003]. Haimovich and Braslavsky [2013] give specific conditions on the number of states, inputs and subsystems under which the Lie-algebraic state feedback design is possible for almost every set of system parameters. The main contribution of the present paper is then to show how the degrees of freedom that remain available when the aforementioned conditions are satisfied can be exploited in order to minimise the ultimate bound for one arbitrary state component.

Notation. The index set $\{1,2,\ldots,N\}$ is denoted $N$. The nullspace of matrix $A$ is denoted $\ker A$ and its image,
ing $A$. For $x \in \mathbb{C}^{n \times m}$, its $j$-th row is denoted $x(j,:)$, its transpose $x'$, its conjugate transpose $x^*$ and its Moore-Penrose pseudoinverse $x^+$. $d(S)$ denotes the dimension of the vector space $S$. The $j$-th component of $x_k \in \mathbb{C}^n$ is denoted $x_{j,k}$. An eigenvalue $\lambda \in \mathbb{C}$ is stable if $|\lambda| < 1$.

2. TIGHTEST ULTIMATE BOUNDS

Consider a perturbed discrete-time switched linear system

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u_{\sigma(k)}(k) + H_{\sigma(k)}d(k)$$

(1)

where the switching function $\sigma(\cdot)$ takes values in $\mathcal{N}, x \in \mathbb{R}^n, u_k \in \mathbb{R}^{m_i}, A_i \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times m_i}$ and have full column rank, and $H_i \in \mathbb{R}^{n \times \varepsilon},$ for every $i \in \mathcal{N}$. The disturbance variable $d \in \mathbb{R}^\varepsilon$ is componentwise bounded by $|d(\cdot)| \leq d$, where $d \in \mathbb{R}^\varepsilon$ is a nonnegative vector. We are interested in state-feedback control design of the form

$$u_{\sigma(k)}(k) = K_{\sigma(k)}x(k)$$

(2)

so that the resulting perturbed closed-loop system

$$x(k+1) = A_{cl}(\sigma(k))x(k) + H_{\sigma(k)}d(k),$$

(3)

with $A_{cl} = A_i + B_iK_i$, simultaneously admits a CQLF and exhibits the minimum possible ultimate bound for one state component.

Componentwise ultimate bound minimisation for non-switched discrete-time systems has recently been studied in Heidari et al. [2013], where conditions were derived so that the ultimate bound on one (or more) state components is minimised to its least possible value via eigenvalue-eigenvector assignment. For discrete-time switched systems, there are also limitations on the lowest achievable ultimate bound for any state component. Indeed, the ultimate bound associated to a specific state component of (3) can never be smaller than the effect of the perturbation on that component. This is formalised in the following result.

**Lemma 1.** An ultimate bound on the $j$-th state component of the switched system (3) can never be smaller than

$$b_j^\text{min} = \max_{i \in \mathcal{N}} \max_{|d| \leq d} [d(\cdot)] [H_{i}(J_{\sigma,j})d(\cdot)].$$

(4)

**Proof.** An ultimate bound on the $j$-th state component can never be smaller than that corresponding to the case when the $j$-th row of $A_{cl}^i$ is zero for every $i \in \mathcal{N}$. Then, the result follows from direct analysis of (3).

We remark that the expression (4) is independent of the feedback matrices $K_i$, for all $i \in \mathcal{N}$, because it corresponds to the case when the $j$-th row of every $A_{cl}^i = A_i + B_iK_i$ is zero. Lemma 2 below gives conditions on a common transformation $V$ and resulting transformed matrices $M_i$, such that

$$A_{cl} = A_i + B_iK_i = VM_iV^{-1}$$

(5)

has its $j$-th row equal to zero $\forall i \in \mathcal{N}$ and $M_i$ is upper triangular.

**Lemma 2.** The $j$-th ultimate bound of the discrete-time switched system (1) can be minimised to its minimum possible value (4) if there exist feedback matrices $K_i$ for all $i \in \mathcal{N}$ and an invertible $V$ such that $M_i = V^{-1}(A_i + B_iK_i)V$ are stable, upper triangular, and have the form

$$M_i = \begin{bmatrix} \Delta_i & \delta_i \\ 0 & 0 \end{bmatrix},$$

(6)

where $\Delta_i$ is the $(n-1)$th leading principal of the upper-triangular matrix $M_i$, $\delta_i$ is an arbitrary vector and the transformation matrix $V$ is such that its $j$-th row has a nonzero element at the last column and is zero everywhere else, that is,

$$V_{(j,:)} = [0_{1 \times n-1} \ldots V_{j,n}]^T, \quad V_{j,n} \neq 0.$$

(7)

**Proof.** Using (6) and (7), the $j$-th row of the closed-loop matrix of each subsystems is

$$[A_{cl}^i]_{(j,:)} = [A_i + B_iK_i]_{(j,:)} = [VM_iV^{-1}]_{(j,:)} = V_{(j,:)}M_iV^{-1} = [0_{1 \times n-1} \ldots V_{j,n}]^T \Delta_i \delta_i V^{-1} = 0_{1 \times n}$$

(8)

and hence, the ultimate bound on the $j$-th state component is minimum as in (4).

□

Haimovich and Braslavsky [2013] developed an algorithm that iteratively seeks feedback matrices $K_i$ and the transformation $V$ so that $M_i = V^{-1}(A_i + B_iK_i)V$ are stable and upper triangular. In the next section, we modify this algorithm in order to achieve the additional conditions of Lemma 2 and hence yield closed-loop matrices $A_{cl}^i = A_i + B_iK_i$ with zero $j$-th row.

3. STABILISATION AND ULTIMATE BOUND MINIMISATION BY FEEDBACK

Haimovich and Braslavsky [2013] give conditions on the number of states $n$, the number of subsystems $N$, and the number of inputs of each subsystem $m_i$, $i \in \mathcal{N}$, so that the stabilising feedback matrices $K_i$ and the simultaneous triangularisation transformation $V$ will exist for almost every set of system parameters, i.e. for almost all possible entries of the matrices $A_i$ and $B_i$, for all $i \in \mathcal{N}$. When these conditions are satisfied, Haimovich and Braslavsky [2013] also show that, in addition, the closed-loop eigenvalues for every subsystem can be arbitrarily selected. In this section, we modify the feedback design algorithm of Haimovich and Braslavsky [2013] so that all available degrees of freedom are exploited to achieve minimum ultimate bounds. These degrees of freedom consist in the selection of some closed-loop eigenvalues and the construction of a unitary matrix with specific properties.

Consider the discrete-time switched linear system (1) with state-feedback law (2), yielding the closed-loop system (3). The proposed modified algorithm is shown below as Algorithm ITBF in Figure 1. This algorithm is an extension of the algorithm in Haimovich and Braslavsky [2013], where the main modifications are: (a) the state component to be minimised, namely $j$, has to be supplied as input data, (b) the common eigenvector assignment (CEA) procedure of Haimovich and Braslavsky [2013] is replaced by the common shifted eigenvector assignment (CSEA) procedure in (10), and (c) the unitary matrix construction (14) has to satisfy the additional constraints (15). As in Haimovich and Braslavsky [2013], the proposed algorithm seeks feedback matrices $K_i$ so that the closed-loop matrices $A_{cl}^i$ in (3) are stable and simultaneously triangularisable, but with the additional property that the condition in Lemma 2 is fulfilled.

A brief description of the algorithm is as follows. After initialisation, the algorithm iterates the following steps: common eigenvector computation for the internal subsystems identified by $A_i^f, B_i^f$ [performed by procedure CSEA...
Algorithm ITBF: Iterative triangularisation and ultimate bound minimisation by feedback

Data: $A_i \in \mathbb{R}^{n_i \times n_i}, B_i \in \mathbb{R}^{n_i \times m_i}$ for $i \in \mathcal{N}$, and $j$
Output: $K_i$ for $i \in \mathcal{N}$
Initialisation: $A_i^1 \equiv A_i, B_i^1 \equiv B_i, K_i^0 \equiv 0, U_1 \equiv I, \ell \leftarrow 0, k^1 \leftarrow j$
repeat
\[ \ell \leftarrow \ell + 1, n_\ell \leftarrow n - \ell + 1, \]
\[ [v_\ell, \{F_i^{(\ell \mid \ell + 1)}\}] \leftarrow \text{CSEA}(\{A_i^{(\ell \mid \ell + 1)}\}_{i = 1}^{n}, \{B_i^{(\ell \mid \ell + 1)}\}_{i = 1}^{n}, k^\ell), \]
\[ A_i^{\ell + 1} \leftarrow A_i^\ell + B_i^1 F_i^\ell, \]
\[ K_i^\ell \leftarrow K_i^\ell - 1 + F_i^\ell \left( \sum_{\gamma = 1}^{\ell} U_{i,\gamma}^* \right), \]
\[ V_{(i,:)} \leftarrow \left( \sum_{\gamma = 1}^{\ell} U_{i,\gamma} \right) v_i^\ell. \]
if $\ell < n$ then
Construct a unitary matrix (14) satisfying (15), with arbitrary $k^\ell \neq 1$, and assign (16)–(19):
\[ [v_1^\ell, \ldots, v_{n_\ell}^\ell] \in \mathbb{C}^{n_\ell \times n_\ell}, \]
where $v_{k^\ell, i}^\ell = 0$ for $i = 1, \ldots, n_\ell, i \neq k^\ell, v_{k^\ell, k^\ell}^\ell \neq 0$ (14)
\[ k^{\ell + 1} \leftarrow k^\ell - 1 \]
\[ U_{\ell + 1} \leftarrow [v_1^\ell, \ldots, v_{n_\ell}^\ell], \]
\[ A_i^{\ell + 1} \leftarrow U_{\ell + 1}^* A_i^{\ell + 1} U_{\ell + 1}, \]
\[ B_i^{\ell + 1} \leftarrow U_{\ell + 1} B_i^{\ell + 1}. \]
end if
until $\ell = n$
\[ K_i \leftarrow K_i^n \]

Fig. 1. ITBF algorithm.

in (10)], state feedback and transformation update [performed at (12) and (13)], and internal matrices’ update for the next iteration [at (14) to (19)]. The internal subsystem matrices change dimensions during the execution of the algorithm because one state dimension is eliminated at each iteration. During initialisation, the internal matrices for iteration $\ell = 1$ are set to coincide with the subsystem matrices: $A_i^1 = A_i, B_i^1 = B_i$. The CSEA procedure in (10) seeks a vector $v_i^1$ having specific structure (which will be explained later), and corresponding (internal) feedback matrices $F_i^1$, so that $v_i^1$ is a feedback-assignable eigenvector common to all internal subsystems, with corresponding stable eigenvalues. That is, if Procedure CSEA is successful, then $v_i^1$ satisfies $\|v_i^1\| = 1$ and $(A_i^1 + B_i^1 F_i^1) v_i^1 = \lambda_i^1 v_i^1$ for some scalars $\lambda_i^1$ satisfying $|\lambda_i^1| < 1$ for all $i \in \mathcal{N}$.

Existence of such $v_i^1$ is ensured by the structural condition of Haimovich and Braslavsky [2013], as we next explain. Define $m_i^\ell \equiv \text{rank}(B_i^\ell)$, and factor $B_i^\ell = B_i^\ell r_i^\ell$, where $r_i^\ell : \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{m_i^\ell}$ has full row rank and $b_i^\ell : \mathbb{R}^{m_i^\ell} \rightarrow \mathbb{R}^{m_i}$ has full column rank. Note that $\text{im}(B_i^\ell) = \text{im}(b_i^\ell)$. Let $\Lambda^\ell$ be a vector with components $\lambda_i^\ell, i \in \mathcal{N}$, i.e.
\[ \Lambda^\ell \equiv [\lambda_1^\ell, \lambda_2^\ell, \ldots, \lambda_N^\ell]^T, \]
and build the matrices
\[ R_i(\Lambda^\ell) \equiv \begin{bmatrix} \lambda_1^1 A_i^1 - A_i^1 \\ \vdots \\ \lambda_N^1 A_i^1 - A_i^1 \end{bmatrix}, B_i \equiv \text{blkdag}[b_1^\ell, \ldots, b_N^\ell], \]
where blkdiag denotes block diagonal concatenation.

Lemma 3. (Structural condition) (Haimovich and Braslavsky [2013]) Let
\[ p_\ell \equiv n_\ell + \sum_{i=1}^{N} m_i^\ell - N n_\ell. \]
Then,
\[(a) d(\ker Q_i(\Lambda^\ell)) \geq p_\ell \text{ for every choice of } \Lambda^\ell \text{ as in (20).} \]
\[(b) \text{A vector that can be assigned by feedback as a common eigenvector with corresponding eigenvalues } \lambda_i^\ell \text{ for } i \in \mathcal{N} \text{ exists if and only if } d(\ker Q_i(\Lambda^\ell)) > 0. \]
Consequently, if $p_\ell > 0$,
then a feedback-assignable common eigenvector exists for every choice of corresponding eigenvalues.
\[(c) \text{If } Q_i(\Lambda^\ell) w_\ell = 0 \text{ with } w_\ell \neq 0 \text{ partitioned as } w_\ell = [v', u_1', \ldots, u_N'], \text{ then } v' \neq 0, (24)\]
\[(A_i^1 + B_i^1 F_i^1) v = \lambda_i^1 v, \text{ for } i \in \mathcal{N}. \]
for every $F_i^1$ satisfying $F_i^1 v = u_i$. For each $i \in \mathcal{N}$ one such $F_i^1$ is $F_i^1 = (r_i^1)^T u_i^T v_i^T$.

If the structural condition (23) holds, the nullspace of $Q_i(\Lambda^\ell)$ is not empty and, thus, we can find $w_\ell \in \ker Q_i(\Lambda^\ell)$. Let $d(\ker Q_i(\Lambda^\ell)) = \xi_\ell \geq p_\ell$, define $d_\ell \equiv n_\ell + \sum_{i=1}^{N} m_i^\ell$ and let $W_\ell \in \mathbb{C}^{d_\ell \times d_\ell}$ be a basis for $\ker Q_i(\Lambda^\ell)$. Then, from Lemma 3(c), the vector $w_\ell \neq 0$ has the form
\[ w_\ell = W_\ell \alpha_\ell \]
where $\alpha_\ell \in \mathbb{C}^{d_\ell}$ is an arbitrary vector. Once (26) is obtained, a feedback-assignable common eigenvector $v_i^1$ provided by procedure CSEA at iteration $\ell$ of algorithm ITBF can be computed by selecting the first $n_\ell$ components of $w_\ell$ to construct $v$ [cf. (24)] and then letting
\[ v_i^\ell = v / \|v\|. \]

In summary, the Algorithm ITBF (shown in Figure 1) seeks for feedback matrices $K_i$ so that
\[(1) \text{the closed-loop matrices } A_i^{\ell + 1} = A_i + B_i K_i \text{ are stable and simultaneously triangularisable, and} \]
\[(2) \text{the } j\text{-th ultimate bound, with } j \text{ an arbitrary state index, is minimised to its smallest value.} \]

Algorithm ITBF runs Procedure CSEA on its internal system matrices $A_i^{\ell + 1}$ and $B_i^{\ell + 1}$. This procedure selects the common eigenvector such that the resulting triangularising transformation matrix $V$ fulfills the condition of Lemma 2.

3.1 Solvability of Algorithm ITBF

In this section we first revisit some results from Haimovich and Braslavsky [2013] that analyse the structural condition (23). Then we present a new result on how to exploit the available degrees of freedom in the ITBF algorithm to guarantee satisfaction of (23) at each iteration, which ensures the successful generation of the desired common triangularising transformation.

The structural condition (23) depends on $m_i^\ell$, the rank of $B_i^{\ell + 1}$. At the first iteration $\ell = 1$, $n_\ell = n$, $m_i^\ell = m_i$ and thus,
Procedure CSEA (Common Shifted Eigenvector Assignment)

Input: $A_i^f, B_i^f \in \mathbb{R}^{n_i \times n_r}, \{\alpha_i^f : i \in \mathbb{N}\}$, for $i \in \mathbb{N}$ and $k$
Output: $v_i^f$ with $v_i^f, v_i^{f+1} \neq 0$, $F_i^f$ for $i \in \mathbb{N}$

Factor $B_i^f = b_i^f \alpha_i^f$ with $b_i^f \in \mathbb{R}^{n_i \times m_i^f}$ and $m_i^f = \text{rank}(B_i^f)$; 

if $p_i = n_i + \sum_{i=1}^{N} m_i^f - N n_r > 1$ then 

if $\ell < n$ then 

- Select $\lambda_i^f \in \mathbb{R}$ stable and construct $\Lambda^\ell$ as in (20); 
- Construct $Q_i(\Lambda^\ell)$ as in (21) and compute $W^\ell$ as in (26); 
- Select $\alpha^f$ as in (40), Lemma 8(b) satisfying Theorem 6(b); 
- Compute $v^\ell$ as in (26) and partition it as in (24); 
- $v_i^f = v/\|v\|$; 

else if $\ell = n$ then 

- Select $\lambda_i^f = 0$ and construct $\Lambda^\ell$ as in (20); 
- Select $v_i^n = 1$; 

end if 

end if 

else 

end if 

end if 

Fig. 2. CSEA when the structural condition is satisfied. 

$p_i = (1 - N)n + \sum_{i=1}^{N} m_i^f$. At subsequent iterations, $m_i^{f+1}$ depends on the vector $v_i^f$ given by Procedure CSEA as 

$m_i^{f+1} = \begin{cases} 
    m_i^f & \text{if } v_i^f \notin \text{img} \ B_i^f, \\
    m_i^f - 1 & \text{if } v_i^f \in \text{img} \ B_i^f. 
\end{cases}$ (28) 

From (28), $m_i^{f+1} = m_i^f - 1$ when $m_i^f = n_i$, because $v_i^f \in \mathbb{R}^{n_i} = \text{img} \ B_i^f$. The next lemma follows from (22) and (28).

**Lemma 4.** (Haimovich and Braslavsky [2013]). Consider Algorithm ITBF at iteration $\ell$ and $p_i$ as in (22), with $m_i^f = \text{rank}(B_i^f)$. Then, $p_i^{\ell+1} \geq p_i - 1$, with equality if and only if 

$v_i^f \in B_i^f$, with $B_i^f = \bigcap_{i \in \mathbb{N}} B_i^f$ and $B_i^f \subseteq \text{img} \ B_i^f$. (29) 

From Lemma 4, if at iteration $\ell$, $p_i = 1$ and $v_i^f \in \bigcap_{i \in \mathbb{N}} B_i^f$, then $p_i^{\ell+1} = p_i - 1 = 0$ and no common eigenvector can be found; hence, the ITBF algorithm terminates unsuccessfully. We thus want to avoid this situation.

Let $S_i^f$ denote the set of vectors $v \in B_i^f$ for which there exist a matrix $F_i^f$ and a stable scalar $\lambda$ so that 

$(A_i^f + B_i^f F_i^f)v = \lambda v.$ (30) 

That is, $S_i^f$ is the set of feedback-assignable eigenvectors for $(A_i^f, B_i^f)$, with associated stable eigenvalue, which are contained in $B_i^f$.

Suppose $\{S_i^f : i \in \mathbb{N}\}$ are transverse subspaces (a generic property of randomly selected subspaces, see, Haimovich and Braslavsky [2013, Lemma 3]). Define the quantities 

$q_i^f \doteq \text{d}(S_i^f), \quad q_i \doteq n_i + \sum_{i \in \mathbb{N}} q_i^f - N n_r, \quad S_i^f \doteq \bigcap_{i \in \mathbb{N}} S_i^f, \quad \rho_i^f \doteq \text{d}(S_i^f).$ (31)–(32) 

The next lemma relates the quantities $p_i, q_i$ and $\rho_i^f$, defined in (22) and (31)–(32), and is central for the solvability of the proposed algorithm at all iterations. The proof follows from arguments in Haimovich and Braslavsky [2013].

**Lemma 5.** Let $p_i > 0, \{S_i^f : i \in \mathbb{N}\}$ be transverse and $(A_i^f + B_i^f F_i^f)$ be controllable. Then, $p_i \geq q_i^f = \max\{0, q_i\}$, with $p_i = q_i^f$ if and only if $m_i^f = n_i$ for all $i \in \mathbb{N}$.

If the common eigenvector $v_i^f$ lies within $B_i^f$, then $v_i^f \in B_i^f$ as in (29), and the structural condition $p_i^{\ell+1} = p_i - 1 > 0$ may not be satisfied. To avoid this situation and guarantee that (23) continues to be satisfied, we provide in part (b) of Theorem 6 below a new result on exploiting the degrees of freedom to choose the common eigenvector such that $v_i^f \notin B_i^f$ and hence, $p_i^{\ell+1} \geq p_i$.

**Theorem 6.** Let $\{S_i^f : i \in \mathbb{N}\}$ be transverse, $q_i \geq 0$ and $(A_i, B_i)$ be controllable for all $i \in \mathbb{N}$. Then, 

(a) $p_i > 0$ for $\ell = 1, \ldots, n$.

(b) It is always possible to select $\alpha^f$ in (26) such that $p_i$ is non-decreasing for $\ell = 1, \ldots, n$ or, if for some $k < n$ we have $p_k < p_{k-1}$ then $p_k = n_k$ for $\ell = k, \ldots, n$.

(c) There exist feedback gains $K_i$ such that the set $\mathcal{Z} = \{A_i + B_i K_i : i \in \mathbb{N}\}$ consists of stable matrices and generates a solvable Lie-algebra. Hence, such that the closed-loop system admits a CQLF.

**Proof.** For the proof of (a) and (c) see Haimovich and Braslavsky [2013, Theorem 2]. Here we show that by proper selection of $\alpha^f$ in (26), $p_i$ is non-decreasing for all iterations until $p_i = n_i$ and remains equal to $n_i$ afterwards.

From Lemma 3(a), we have $\text{d}(\ker Q_i(\Lambda^\ell)) = q_i^f \geq p_i$. Thus, a basis for the nullspace of $Q_i(\Lambda^\ell)$ has the form (see (26))

$$W^\ell = \begin{bmatrix} v_1 & \cdots & v_{\ell^f} \\ u_1 & \cdots & u_{\ell^f} \end{bmatrix} = \begin{bmatrix} \lambda_1 & \cdots & \lambda_{\ell^f} \\ u_1 & \cdots & u_{\ell^f} \end{bmatrix}, \quad \text{rank}(W^\ell) = \ell^f \geq p_i.$$

(33) where the partition of each vector follows from (24). From (24) and (26), the common eigenvector is determined as

$$v = [v_1 \ldots v_{\ell^f}] \alpha^f.$$

(34)

First, we show that the subspace generated by the $v_r$ vectors is also of dimension $\ell^f$, i.e.

$$\text{rank}\{v_1 \ldots v_k \ldots v_{\ell^f}\} = \ell^f.$$

(35) 

Suppose, for a proof by contradiction, that $v_k$, for some $k \in \{1, \ldots, \ell^f\}$, is a linear combination of $v_r$, $r \neq k$, i.e.

$$v_k = \sum_{r=1, r \neq k}^{\ell^f} v_r \gamma_r.$$

(36) 

where at least one coefficient $\gamma_r$ is nonzero. Then, since $W^\ell$ in (33) is in the nullspace of $Q_i(\Lambda^\ell)$ we have

$$Q_i(\Lambda^\ell)v_k = 0, \quad r = 1, \ldots, \ell^f$$

(37) 

and by replacing (21) and (33) in (37), for $i \in \mathbb{N}$ we obtain

$$(\lambda_i^f I - A_i^f - B_i^f u_{\ell^f})v_r = 0, \quad r = 1, \ldots, \ell^f.$$ (38)

For $r = k$, using (36) in (38) we obtain

$$\begin{align*}
(\lambda_i^f I - A_i^f - B_i^f u_{\ell^f})v_k &= \sum_{r=1, r \neq k}^{\ell^f} (\lambda_i^f I - A_i^f) v_r \gamma_r - B_i^f u_{\ell^f} \\
&= b_i^f \sum_{r=1, r \neq k}^{\ell^f} u_{ir} \gamma_r - b_i^f u_{ik} = b_i^f \left( \sum_{r=1, r \neq k}^{\ell^f} u_{ir} \gamma_r - u_{ik} \right) = 0.
\end{align*}$$
Since the \( b^\ell_i \) matrices have full column rank, the above implies
\[
\sum_{r=1, r \neq k}^{\xi_\ell} u_{ir} \gamma_r - u_{ik} = 0. \tag{39}
\]
This means that \( u_{ik} = \sum_{r=1, r \neq k}^{\xi_\ell} u_{ir} \gamma_r \), for \( i \in \mathcal{N} \), which together with (36) yields \( w^\ell_k = \sum_{r=1, r \neq k}^{\xi_\ell} w^\ell_{r \gamma_r} \), i.e.
\[
\text{rank}(W^\ell) < \xi_\ell, \tag{33}
\]
which contradicts our assumption in (33). Hence, (35) holds and the common eigenvector \( v \) in (24) can be chosen in a space of dimension \( \xi_\ell \).

In Haimovich and Braslavsky [2013, Theorem 2] it is proved by induction that when \( \{S^i : i \in \mathcal{N}\} \) is transverse, \( q_1 > 0 \) and \( (A_i, B_i) \) is controllable, then for \( \ell = 1, \ldots, n \), \( \{S^\ell : i \in \mathcal{N}\} \) is transverse and \( (A^\ell_i, B^\ell_i) \) is controllable. Hence, from part (a) in the current theorem and Lemma 5, we know that \( p_\ell \geq p^\ell \) with equality if and only if \( m^\ell_i = n^\ell \) for \( i \in \mathcal{N} \). We consider two cases, \( p_\ell = p^\ell \) and \( p_\ell > p^\ell \).

When \( p_\ell = p^\ell \), then \( m^\ell_i = n^\ell \) for \( i \in \mathcal{N} \) and all (internal) input matrices are invertible. Then we have \( p_\ell = p^\ell = n^\ell \ leveraging the above transverse condition) and Theorem 6(b), since \( p_\ell = n^\ell \) and \( m^\ell_i = n^\ell \) holds for all \( i \in \mathcal{N} \), we can select \( \alpha^\ell \) as in Lemma 8 and Theorem 6(b). In the remainder of the proof, the iterative ultimate bound minimisation is explained.

The aim is to iteratively construct the columns of the matrix \( V \) through (13)-(17) to achieve the final form (7). Since the columns of \( V \) are the result of a product of matrices (c.f. (13)), the idea is to propagate the location of zero and nonzero elements in relevant rows of these matrices so that the end result is the \( j \)-th row of \( V \) having all zero elements except at the last column. At the first iteration, since \( U_1 = I_n \), to have \( V_{j,1} \neq 0 \), Procedure CSEA needs to select the common eigenvector such that \( v^{j,1}_1 \neq 0 \). Then, to construct a unitary matrix with \( v^{j,1}_1 \) as its first column, the \( j \)-th row of the unitary matrix has \( n - 1 \) zero elements and one nonzero element at its \( k^1 \)-th place, \( k^1 \neq 1 \). Thus, the \( j \)-th row of \( U_2 \) in (17) has \( n_j - 1 = n - 2 \) zeros and a nonzero entry at \( k^2 = k^1 - 1 \) [cf. (16)]. Hence,
\[
U_1 U_2 = I_n \begin{bmatrix}
\ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots \\
0 & \ddots & \ddots \\
0 & \ddots & \ddots \\
\end{bmatrix}_{n \times n_2}
\]
where \( * \) is a non-specified entry. It can be seen that the matrix \( U_1 U_2 \) has its \( j \)-th row with one nonzero element at the \( k^2 \)-th place \( (k^2 = k^1 - 1) \) and otherwise zero. Hence, to achieve \( V_{j,2} = 0 \), from (13) we need to have \( v^{j,2}_{k^2} = 0 \) for the common eigenvector \( v^j_2 \). Accordingly, the unitary matrix (14) constructed using this \( v^j_2 \) will have its \( j \)-th row with \( n_j - 1 \) zeros and one nonzero element at its \( k^2 \)-th place, \( k^2 \neq 1 \). Continuing with the same procedure, at iteration \( \ell \) the matrix \( \prod_{r=1}^{\ell} U_r \) is of size \( n \times n_{\ell} \) and has its \( j \)-th row equal to zero except for its \( k^\ell \)-th component. Thus, to have \( V_{j,\ell} = 0 \), the common eigenvector assignment should satisfy \( v^{j,\ell}_{k^{\ell}} = 0 \). At the last iteration, the matrix
\[
\prod_{r=1}^{n} U_r \text{ is of size } n \times 1 \text{ with its } j \text{-th element being nonzero:}
\]
Thus, in order to have $V_{i,n} \neq 0$, the scalar $v_i^n$ needs to be nonzero. Since we have a scalar system at this iteration, the common eigenvector can be taken to be $v_i^n = 1$ for arbitrary eigenvalues. From (13), multiplying $v_i^n$ by the vector $U_{i-1}^{n-1}$ displayed above results in $V_{i,n} \neq 0$. By executing the above procedure, the $j$-th row of $V$ takes the form (7). Setting the last eigenvalue of all subsystems to 0, all $M_i$ matrices take the form (6), and the $j$-th ultimate bound will be minimised to its lowest value (4).

**Remark 10.** If at any iteration $\ell$, $B_i^\ell$ for $i \in N$ have rank $n_\ell$, then the control input matrices are invertible and we can assign arbitrary eigenvalues for all subsystems with common eigenvector matrix $I_{n\ell}$, that is, all remaining iterations from $\ell$ to $n$ can be subsumed in one step by replacing $v_i^n$ in (13) with the matrix $V_i^n \equiv I_{n\ell}$. Since the matrix $\prod_{\ell=1}^{\ell=n_\ell} U_{i-1}^{n_\ell}$ is of size $n \times n_\ell$ with its $j$-th row having a nonzero $k_\ell$-th component and otherwise zero: $$\prod_{\ell=1}^{\ell=n_\ell} U_{i-1}^{n_\ell} = \begin{bmatrix} v_{j,0} \ldots v_{j,k_\ell} \ldots 0 \\ \vdots \vdots \vdots \\ 0 \ldots v_{j,n_\ell - 1} \end{bmatrix}_{n \times n_\ell},$$

by multiplying this matrix with the common eigenvector matrix $I_{n\ell}$, the last $n-\ell-1$ columns of the matrix $V_i^n$ in (13) are $V_i^n = \left( \prod_{\ell=1}^{\ell=n_\ell} U_{i-1}^{n_\ell} \right)_j$, and thus, the $j$-th row of $V$ has the form $V_i^n = \left[ 0_{1 \times (\ell-k_\ell)} v_{j,k_\ell}^\ell 0_{1 \times (n_\ell-k_\ell)} \right]$. Then, a property similar to that of Lemma 2 holds by assigning to zero the eigenvalue associated with $v_{j,k_\ell}^\ell$:

$$M = \begin{bmatrix} \Delta_{(\ell-k_\ell+1)}^{(\ell-k_\ell)} & \delta_{(\ell-k_\ell)}^{(\ell-k_\ell+1)} & \Delta_{(n-k_\ell)}^{(\ell-k_\ell)} \\ 0_{1 \times (\ell-k_\ell)} & 0_{1 \times (n-k_\ell)} & 0_{1 \times (n-k_\ell)} \\ 0_{(n-k_\ell) \times (\ell-k_\ell)} & 0_{(n-k_\ell) \times (n-k_\ell)} & 0_{(n-k_\ell) \times (n-k_\ell)} \\ \end{bmatrix}.$$  

The eigenvalues of the upper triangular matrices $\Delta_{(\ell-k_\ell+1)}^{(\ell-k_\ell)}$ and $\Delta_{(n-k_\ell)}^{(\ell-k_\ell)}$ are stable and can be arbitrarily chosen. 

4. NUMERICAL EXAMPLE

Consider a switched system with $N = 2$ and matrices

$$A_1 = \begin{bmatrix} 1.0393 & -4.5216 & 2.4237 & -2.4003 \\ -1.2962 & 2.3926 & 4.3745 & -2.5097 \\ -4.2177 & -4.6200 & 0.1336 & 4.9314 \\ -0.4365 & 5.4324 & -2.5910 & -1.4329 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 3.2700 & 2.0058 & -2.2526 & -1.9570 \\ -1.9192 & -2.5813 & -4.9389 & -2.0914 \\ -0.9764 & 2.5983 & -1.2299 & -2.5748 \\ 3.8423 & -2.9907 & -0.6307 & 4.3668 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 2.5286 & 2.3072 & 1.4007 \\ -3.8995 & -2.3882 & -3.6796 \\ 0.9705 & -0.4519 & -0.4718 \\ -0.6940 & -0.4904 & 1.5220 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 3.6019 & -0.1038 & 3.1666 \\ -1.0277 & -2.3019 & -4.6737 \\ -0.2058 & 4.8974 & -1.6804 \\ 0.6500 & -3.1632 & 2.4875 \end{bmatrix},$$

$H_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \end{bmatrix},$ 

$d = 1.$

We aim to minimise the 4-th ultimate bound. At the first iteration, $q_1 = 0, p_1 = 2$ and, for arbitrary eigenvalues $\Lambda^1 = [\lambda_1^1 \lambda_2^1] = [0.0398 -0.1141]$, Procedure CSEA yields

$$V_{(-1)} = v_1^1 = \begin{bmatrix} 0.3553 \\ 0.7593 \\ 0.2840 \end{bmatrix},$$

with $\alpha_1^1 = [-0.8968 0.4424].$ (41)

The matrix (17) with $\hat{k}^1 = 3$ is $U_2 = \begin{bmatrix} -0.8445 & 0 & 0 \\ 0.5043 & 0 & -0.3363 \\ 0.8101 & 0.9418 \end{bmatrix}$. At the next iteration, $n_2 = 3$ and $m_2^2 = m_2^3 = 3$ and hence, the input matrices are invertible. Since $k_2^1 = k_2^1 - 1 = 2$, we need $\lambda_2^1 = 0$. Thus, assigning the remaining eigenvalues at $\Lambda^2 = [\lambda_1^2 \lambda_2^2] = [0.0969 0.9690]$,

$$\Lambda^3 = [\lambda_1^3 \lambda_2^3] = [0.7244 0.1337],$$

and computing the eigenvector matrix as in Remark 10, $V_2^3 = I_3$, yields

$$V_{(-2)} = \begin{bmatrix} 0.1337 \\ 0.5043 & -0.3363 \\ 0.8101 & 0.9418 \end{bmatrix}.$$ (42)

The resulting feedback gains and upper triangular closed-loop matrices are

$$K_1 = \begin{bmatrix} 0.1300 & 2.7027 & -0.3796 & -0.6779 \\ -1.0793 & -0.1629 & -0.6014 & 1.0049 \\ -0.0017 & -1.7852 & 1.3354 & 0.9558 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 2.4581 & -1.4435 & -0.9600 & 1.4290 \\ -0.0229 & -0.5822 & -0.0339 & 0.1581 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 0.0398 & -0.5862 & -0.8573 & 3.6896 \\ 0 & 0 & 0 & 0.7244 \\ -0.1141 & -0.0128 & 0.6379 & -6.2035 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0 & 0 & 0 & 0.1337 \end{bmatrix}.$$ (42)

The matrices $V$ (see (41) and (42)) and $M_i$ satisfy the conditions of Lemma 1 and Remark 10. For random disturbance and switching signals, the trajectories of the switched system depicted in Figure 3 show that the 4-th state is kept within the smallest possible bound.

![Fig. 3. Switched system state trajectories.](image)

5. CONCLUSION

This paper has derived conditions to achieve a minimum ultimate bound for one component of the state of a discrete-time switched linear system under arbitrary switching and non-vanishing perturbations. A procedure to satisfy the derived conditions has been presented via an iterative algorithm which simultaneously triangularises all subsystem matrices and gives the minimum achievable value for one of the ultimate bounds.

REFERENCES


