Comparison of differentiation schemes
for the velocity and acceleration
estimations of a pneumatic system

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Abstract: In this paper, various type of differentiators are studied. In the context of velocity
and acceleration estimations of a pneumatic system, a comparison is made between numerical
methods, based on classical or algebraic approaches, and a high order sliding mode differentiator.

Keywords: Differentiation, high order sliding mode, pneumatic actuator.

1. INTRODUCTION

These last decades, numerous results have been proposed for the control of electropneumatic systems, the main
part of these results being based on state feedback approaches: input-output linearization based control (Brun et al. (1999)), sliding mode control (Girin et al. (2009); Laghrouche et al. (2006); Plestan et al. (2013); Shteessel et al. (2012); Smaoui et al. (2006 b); Taleb et al. (2013)), backstepping control (Smaoui et al. (2006 a,b), etc. However, these controllers require the measurement of state
variables. In practice, some state variables are not easy to be measured directly by the sensors, like the velocity and the acceleration, whereas they are required to compute the control law. In order to overcome this difficulty and also with the objective of minimizing the number of sensors, some states estimation schemes can be proposed (Yan et al. (2014)). One solution is the use of nonlinear state observer. In Bornard et al. (1991), a high gain observer is proposed for a class of nonlinear system. A sliding mode observer is used by Pandian et al. (2002), so as to estimate the chamber pressure for pneumatic actuators.

An alternative method to estimate the system state is the numerical differentiation. According to the studies in Gauthier and Kupka (1994), for an observable nonlinear system, any state variables is a function of finite number of time derivatives of the control and output variables. Furthermore, the use of numerical differentiation schemes enables a model-independent derivation. In recent years, numerous technics have been proposed for the problem of numerical differentiation. A robust exact differentiator based on 2-sliding algorithm (see Levant (1993)) is proposed by Levant (1998). It allows to estimate the first order derivative of a bounded noisy signal. Such a differentiator is used by Smaoui et al. (2005) for the acceleration estimation of a pneumatic system. The sampling feature of the differentiation computation is also taken into account by Plestan and Glumineau (2010). The sliding mode differentiator is generalized to the higher-order sliding mode differentiator (Levant (2003)), which allows to estimate the k-th order derivative of a bounded noisy signal. Moreover, the so called chattering phenomenon is reduced through the high order sliding mode theory. Another kind of differentiator based on algebraic parametric estimation technics is proposed by Mboup et al. (2007): a truncated Taylor expansion and calculations in operational domain are used to obtain the approximations of the finite order derivatives of a noisy signal. In Liu et al. (2009), the error analysis for such a differentiator is done.

The objective of the paper is to apply some differentiation approaches for the velocity and acceleration estimations of a pneumatic system. Four methods are considered here. Two of them are based on classical approaches: one is based on the backward-difference formula, the other is based on a three-point formula. The third differentiator is developed from results of Mboup et al. (2007) by using an algebraic approach. Finally, the higher-order sliding mode differentiator proposed by Levant (2003) is considered. The paper is organized as follows. In the second section, numerical differentiation methods are presented. The differentiator based on high order sliding mode is exposed in the third section. In the fourth one, the experimental set-up of the pneumatic system is described. Furthermore, experimental results of velocity and acceleration estimations are presented and a comparison between the four approaches is made with different working conditions.

2. NUMERICAL DIFFERENTIATION

2.1 Classical approach

The principle of the classical numerical differentiation method presented in this section is to estimate the time derivative of a function \( f(t) \) by calculating the derivative of an interpolating polynomial that fits \( f(t) \) over an interval \( I \).
The next theorem establishes a formula to approximate \( f'(t_0) \) from the sampling points at instants \( t_{-j}, \ldots, t_0, \ldots, t_k \) (denote \( I = [t_{-j}, t_k] \)).

**Theorem 1.** (Burden and Faires (2011)) Let \( n = j + k \) and \( f(t) \in C^{n+1}(I) \). Then, the first order derivative at \( t = t_0 \), \( f'(t_0) \) is given by the following \((n+1)\)-point formula

\[
f'(t_0) = \frac{k}{h} \sum_{i=-j}^{k} f(t_i) \mathcal{L}_{n,i}(t_0) + \frac{h}{2} f''(\xi),
\]

where \( \xi \in I \) is an unknown instant and the Lagrange polynomial \( \mathcal{L}_{n,i} \) associated with \( t_{-j}, \ldots, t_k \), is defined, for \( i = -j, \ldots, k \) by

\[
\mathcal{L}_{n,i}(t) = \left( \prod_{l=-j, l \neq i}^{k} \frac{t - t_l}{t_i - t_l} \right). \tag{2}
\]

**Proof.** By the Lagrange interpolation theorem, for all \( t \in I \), there exists \( \xi \in I \) such that the following equation holds

\[
f(t) = \sum_{i=-j}^{k} f(t_i) \mathcal{L}_{n,i}(t) + \frac{(t - t_{-j}) \ldots (t - t_k)}{(n+1)!} f^{(n+1)}(\xi). \tag{3}
\]

Then, the \((n+1)\)-point differentiation formula can be obtained by differentiating both sides of (3) at \( t = t_0 \).

**Remark 1.**
- With two sampling points, \( i.e. \, j = -1 \) and \( k = 0 \), one obtains the classical backward-difference formula

\[
f'(t_0) = \frac{f(t_0) - f(t_0 - h)}{h} + \frac{h}{2} f''(\xi), \tag{4}
\]

where \( h = t_0 - t_{-1} \) represents the sampling period. The term \( \frac{h}{2} f''(\xi) \) which is proportional to the sampling period \( h \), gives the approximation accuracy.
- In order to improve the accuracy, more sampling points are considered. Assume now that \( j = -2, k = 0 \) and the sampling period is uniform (i.e. \( t_0 - t_{-1} = t_{-1} - t_{-2} = h \)). Then, one gets the three-point formula

\[
f'(t_0) = \frac{3f(t_0) - 4f(t_0 - h) + f(t_0 - 2h)}{2h} + \frac{h^2}{3} f^{(3)}(\xi). \tag{5}
\]

In this expression, the accuracy is proportional to \( h^2 \). Since the sampling period \( h \) is generally smaller than 1, one gets a better accuracy.

Note that the smaller \( h \) is, the better accuracy one gets. However, in the case of a noisy signal \( f(t) \), the differentiation becomes more sensitive to this noise. From a practical point of view, in controlled systems, the sampling period should be tuned to satisfy the control law. So as to improve these differentiators, one adds a parameter \( H = nh \) where \( n \) is a positive integer. From expressions (4)-(5), one thus deduces two formulations for the estimate of the first derivative \( f'(t) \) of a signal \( f \) at current time \( t \).

\[
\hat{f}'(t) = \frac{f(t) - f(t - H)}{H}, \tag{6}
\]

and

\[
\hat{f}'(t) = \frac{3f(t) - 4f(t - H) + f(t - 2H)}{2H}. \tag{7}
\]

Expressions (6)-(7) will be experimentally tested on the pneumatic system (Section 4), in order to estimate the velocity and the acceleration.

### 2.2 Algebraic approach

An alternative approach to estimate the derivatives of a (possibly noisy) signal is proposed by Mboup et al. (2007). It is based on Taylor expansion and Laplace transform and provides the advantage of simultaneously getting estimates of the higher order derivatives of the signal. Next theorem presents this differentiator.

**Theorem 2.** (Mboup et al. (2007)) Let \( N \) be a positive integer. Assume that \( y(t) = f(t) + n(t) \) is a noisy signal defined on \([0, +\infty]\), which consists of a basic signal \( f(t) \) and a noise \( n(t) \).

Then, the estimates of the \( i \)-th order time derivatives \( \hat{f}(i)(0), \, i = 0, \ldots, N \) of \( f(t) \) at \( t = 0 \) are given by the following general expression

\[
P(T) \begin{bmatrix} \hat{f}(0) \\ \vdots \\ \hat{f}^{(N)}(0) \end{bmatrix} = \int_0^T Q(\tau)y(\tau)d\tau \tag{8}
\]

where \( T \) is the size of the estimation window. The nonzero elements of the triangular matrix \( P(T) \) are given, for \( i = 0, \ldots, N, \, j = 0, \ldots, N - i \), by

\[
P(T)_{i,j} = \frac{(N - j)!}{(N - i - j)!} \frac{T^{i+j+1}}{(i + j + 1)!}, \tag{9}
\]

and the elements of the integral term are

\[
Q(\tau)_i = \sum_{l=0}^{i} \binom{i}{l} \left( N^2 + 1 \right) (T - \tau)^l (-\tau)^{i-l} \tag{10}
\]

with \( i = 0, \ldots, N \).

**Remark 2.**
- A more general result is proved in Mboup et al. (2007) with an additional parameter \( \nu \). For simplicity, we assume \( \nu = N + 2 \) and we retain this choice throughout the remainder of the paper.
- The expansion (8) is not a causal differentiator: it requires the signal values \( y(t) \) for \( t > 0 \) in order to reconstruct the derivatives at \( t = 0 \). It means that the future signal values should be known to estimate the derivative in the current time.

So as to estimate the derivatives at the current time \( t \) from \( y(\tau), \, \tau < t \), one adapts the differentiator (8) by the following way.

**Corollary 1.** Consider the same assumptions as in Theorem 2. The estimates \( \hat{f}(i)(t) \) for \( i = 0, \ldots, N \), are given by the expression

\[
\tilde{P}(T) \begin{bmatrix} \hat{f}(0) \\ \vdots \\ \hat{f}^{(N)}(t) \end{bmatrix} = \int_0^T Q(\tau)y(t - \tau)d\tau \tag{11}
\]

with

\[
\tilde{P}(T)_{i,j} = (-1)^i P(T)_{i,j} \tag{12}
\]

and \( P(T)_{i,j}, \, Q(\tau)_i \) defined by (9) and (10).

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holds

Theorem 1.

however, in the case of a noisy signal \( f(t) \), the differentiation becomes more sensitive to this noise. From a practical point of view, in controlled systems, the sampling period should be tuned to satisfy the control law. So as to improve these differentiators, one adds a parameter \( H = nh \) where \( n \) is a positive integer. From expressions (4)-(5), one thus deduces two formulations for the estimate of the first derivative \( f'(t) \) of a signal \( f \) at current time \( t \).

\[
\hat{f}'(t) = \frac{f(t) - f(t - H)}{H}, \tag{6}
\]

and

\[
\hat{f}'(t) = \frac{3f(t) - 4f(t - H) + f(t - 2H)}{2H}. \tag{7}
\]

Expressions (6)-(7) will be experimentally tested on the pneumatic system (Section 4), in order to estimate the velocity and the acceleration.
Proof. Consider the expression (8), and replace \( y(\tau) \) by the signal \( g(\tau) \) defined as
\[
g(\tau) = y(t - \tau), \quad \tau \in [0, T].
\]
Then, one formally has
\[
g^{(i)}(0) = (-1)^i y^{(i)}(t), \quad i = 0, \ldots, N.
\]
It follows that the estimate of the \( i \)-th order time derivative from the observed signal \( g \) at the origin equals
\[
(-1)^i \hat{f}^{(i)}(t).
\]
The corollary is proved.

In the sequel, the objective is to estimate the velocity and the acceleration, then only the case \( N = 2 \) is considered. One gets
\[
\tilde{P}(T) \begin{pmatrix} \hat{f}(t) \\ \hat{f}'(t) \end{pmatrix} = \int_0^T Q(\tau)y(t - \tau)d\tau
\]
where
\[
\tilde{P}(T) = \begin{pmatrix} T - \frac{T^2}{2} & \frac{T^3}{6} \\ \frac{T^2}{2} & \frac{T^3}{6} \\ \frac{T^3}{6} & 0 \end{pmatrix},
\]
\[
Q(\tau) = \begin{pmatrix} 3(1 - \tau) & -\tau & \tau^2 \\ -\tau & 3(1 - \tau) & -\tau \\ \tau^2 & -\tau & 3(1 - \tau) \end{pmatrix}.
\]

Remark 3. For the numerical implementation, the integral \( \int_0^T Q(\tau)y(t - \tau)d\tau \) will be performed using the midpoint method.

The expression (13) will be implemented and tested on the experimental set-up of the pneumatic system (see Section 4).

3. DIFFERENTIATION BASED ON HIGH ORDER SLIDING MODE

The derivative estimation problem can also be solved by the arbitrary-order exact robust differentiator proposed by Levant (2003).

**Theorem 3.** (Levant (2003)) Let the input signal \( f(t) \) be a function defined on \( [0, +\infty) \) and consisting of a bounded Lebesgue-measurable noise with unknown features and an unknown basic signal \( f_0(t) \), whose \( k \)-th time derivative has a known Lipschitz constant \( L > 0 \). Its time derivatives \( f_0^{(i)}(t), i = 0, 1, \ldots, k \), can be estimated by the differentiator
\[
\begin{align*}
\hat{z}_0 &= v_0 \\
v_0 &= -\hat{\lambda}_k L^{1/k+1}|z_0 - f|^{1/k+1} \times \text{sign}(z_0 - f) + z_1 \\
v_i &= -\hat{\lambda}_{k-i} L^{1/k-i+1}|z_i - v_{i-1}|^{1/k-i+1} \\
&\quad \times \text{sign}(z_i - v_{i-1}) + z_{i+1} \\
\hat{z}_k &= -\hat{\lambda}_0 L \text{sign}(z_k - v_{k-1}) \\
\end{align*}
\]
\[
i = 0, \ldots, k - 1
\]
with \( z_i \) the estimation of \( f_0^{(i)} \) and \( \hat{\lambda}_0, \ldots, \hat{\lambda}_k \) the differentiator parameters.

**Remark 4.** According to Levant (2003), a possible choice for \( k \leq 5 \) is \( \{\hat{\lambda}_i\}_{i=0}^{k-1} = 1.1, 1.5, 2, 3, 5, 8, \ldots \).

By substituting expressions \( v_i \) in (14), one gets the non-recursive form
\[
\begin{align*}
\dot{z}_i &= -\hat{\lambda}_k \times f_{(i+1)/(k+1)}|z_0 - f|^{(i-1)/(k+1)} \\
&\quad \times \text{sign}(z_0 - f) + z_{i+1} \\
\dot{z}_k &= -\hat{\lambda}_0 L \text{sign}(z_0 - f)
\end{align*}
\]

(15)

with \( \lambda_0, \lambda_1, \ldots, \lambda_k > 0 \) the new coefficients calculated from (14).

For the same reason as in the previous section, one takes \( k = 2 \). Then, the second order time differentiator reads as
\[
\begin{align*}
\dot{f}_0 &= -\hat{\lambda}_2 L^{1/3}|\dot{f}_0 - f|^{1/3} \text{sign}(\dot{f}_0 - f) + \dot{f}_0 \\
\ddot{f}_0 &= -\hat{\lambda}_1 L^{2/3}|f_0 - f|^{2/3} \text{sign}(f_0 - f) + f_0 \\
\dddot{f}_0 &= -\hat{\lambda}_0 L \text{sign}(f_0 - f)
\end{align*}
\]

(16)

with \( \dot{f}_0 \) the estimation of \( f_0 \) and \( \dddot{f}_0 \) the estimation of \( f''_0 \). As previously, the differentiator (16) will be applied on the pneumatic system in the following section.

4. APPLICATION TO THE PNEUMATIC SYSTEM

4.1 Description of the experimental set-up

The scheme of the pneumatic system is displayed in Fig.1. The system is composed by two antagonists actuators. The one, on left hand side, named “main” actuator, is a double acting electropneumatic actuator controlled by two servodistributors and is composed by two chambers denoted \( P \) and \( N \). The pneumatic jack horizontally moves a load carriage of mass \( M \). This carriage is coupled to the second electropneumatic actuator, the so-called “perturbation” one. The goal of this latter is to produce a dynamical load force on the main actuator.

An adaptive twisting controller proposed by Taleb et al. (2013) is used for the “main” actuator position control; it forces the “main” actuator to track a position reference with high accuracy, in spite of perturbation force produced by the “perturbation” actuator. From the fundamental mechanical theorem, the dynamics of the “main” actuator position read as
\[
\begin{align*}
\frac{dy}{dt} &= v \\
\frac{dv}{dt} &= \frac{1}{M} (S(p_P - p_N) - b_v v - F_{ext})
\end{align*}
\]

(17)

with \( y \) the piston position, \( v \) the piston velocity, \( S \) the piston surface, \( p_P, p_N \) the pressures in chamber \( P \) and \( N \), \( b_v \) the viscous friction coefficient, and \( F_{ext} \) the unknown perturbation force. The complete model has been given in Girin et al. (2009).

The “main” actuator controller requires real-time derivatives of the position: in the sequel one focuses on the estimations of the velocity and the acceleration for the “main” actuator. The differentiators (6), (7), (13) and (16) have been experimentally evaluated.

4.2 Experimental results

In the experimental tests, the position reference trajectory used by the controller designed in Taleb et al. (2013) is a sinusoidal signal \( y_{ref}(t) = 0.04 \sin(2\pi f_{eq} t) \). The frequency \( f_{eq} \) takes values from 0.1Hz to 1Hz. A sinusoidal external
force with a magnitude equal to 500 N and a frequency of 0.17Hz is produced by the perturbation actuator.

First, the frequency of reference trajectory is fixed to be 0.5Hz. Fig. 2 shows the measured position signal, which overlaps with the reference trajectory. The velocity is not measured directly; therefore, the analytic time derivative of $y_{ref}$ is used as the standard to compare with the estimations. The objective is to get velocity and acceleration estimations with good filtering and minimal delay. In order to get the best performance, the differentiator parameters have been tuned after a series of experimental tests. The optimal configurations were obtained as

- Classical differentiators (6) and (7) $H = 100h$; 
- Algebraic differentiator (13) $N = 2, T = 200h$; 
- High order sliding mode differentiator (16) $L = 10$

with the sampling period $h = 1$ ms.

Fig. 3 shows the velocity estimations obtained by these four differentiators. In these working conditions, the estimation by algebraic differentiator is smoother. The effect of the measure noise is rejected due to the numerical integration in (13). However, its advantage is not obvious. In order to estimate the acceleration, the classical differentiators have been implemented in series as a second order differentiator. The acceleration estimations results are shown in Fig. 4. It appears that the three points formula makes wild oscillations, whereas the algebraic differentiator gives well filtered and more accurate results.

### Influence of the sampling period

The sampling period plays an important role in the numerical differentiation problem. In order to analyze the influence of the sampling period, some tests have been made with $h = 5$ ms (Fig. 5-6). With the increase of the sampling period, the derivative estimations are more filtered, but a delay is unavoidable. To evaluate the performances of the differentiators, the mean square error and the standard deviation of the estimation error are computed for the steady state (on the time interval $[5, 20]$ sec) as follows.

- Mean square error
\[ MSE = \frac{1}{N} \sum_{i=1}^{N} (ex_i)^2 \]  
- Standard deviation of estimation error
\[ STD = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (ex_i - E(ex))^2} \]

with $ex$ the estimation error of the velocity or acceleration and $N$ the hits during $[5, 20]$ sec.

In this paper, four differentiation approaches – classical differentiators, algebraic differentiator and high order sliding mode differentiator – are studied in the context of velocity and acceleration estimations for a pneumatic system. The classical differentiators are easy to implement. However, their estimation results are sensitive to the measurement noises and a delay is unavoidable for the high
Fig. 3. **Velocity estimation.** Time derivative of position reference ($\dot{y}_{ref}$-dotted line) (m/s) and velocity estimations ($\hat{v}$-solid line) (m/s) versus time (sec), with $h = 1$ms. Estimation by (a): backward differentiator; (b): three points differentiator; (c): algebraic differentiator; (d): sliding mode differentiator.

Fig. 4. **Acceleration estimation.** Second order time derivative of position reference ($\ddot{y}_{ref}$-dotted line)(m/s) and acceleration estimations ($\hat{a}$-solid line)(m/s) versus time (sec), with $h = 1$ms. Estimation by (a): double backward differentiator; (b): three points differentiator; (c): algebraic differentiator; (d): sliding mode differentiator.

Fig. 5. **Velocity estimation.** Time derivative of position reference ($\dot{y}_{ref}$-dotted line) (m/s) and velocity estimations ($\hat{v}$-solid line) (m/s) versus time (sec), with $h = 5$ms. Estimation by (a): backward differentiator; (b): three points differentiator; (c): algebraic differentiator; (d): sliding mode differentiator.

Fig. 6. **Acceleration estimation.** Second order time derivative of position reference ($\ddot{y}_{ref}$-dotted line)(m/s) and acceleration estimations ($\hat{a}$-solid line)(m/s) versus time (sec), with $h = 5$ms. Estimation by (a): double backward differentiator; (b): three points differentiator; (c): algebraic differentiator; (d): sliding mode differentiator.

Fig. 7. Evaluation of mean square error of velocity estimation $(m/s)^2$ versus frequency (Hz).

Fig. 8. Evaluation of Standard deviation of velocity estimation error (m/s) versus frequency (Hz).

frequency working condition. Algebraic and high order sliding mode differentiators appear to be the most efficient solutions. Indeed, the algebraic differentiator does not require any information about the input signal. For its part, the high order sliding mode differentiator is a closed-loop one which shows its advantage in the high frequency working condition. Furthermore it is less sensitive to the change of sampling period. However, an upper bound for Lipschitz's constant of higher order derivatives is required.
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