Characterization of power systems near their stability boundary by 
Lyapunov direct method

Igor B. Yadykin∗  Dmitry E. Kataev ∗  Alexey B. Iskakov ∗  
Vladislav K. Shipilov ∗∗

Abstract: We propose a new method for the small-signal stability analysis of power systems 
based on the spectral decomposition of a square $H_2$ norm of the transfer function. Compared 
with the dynamics of $H_2$ and $H_\infty$ norms of the transfer functions, the analysis of the behavior 
of individual eigen-components allows the earlier identification of the pre-fault condition 
ocurrence. Since each eigen-component is associated with a particular eigenvector, the potential 
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1. INTRODUCTION

The small-signal stability analysis continues to be among 
major problems of the control theory (Lyapunov (1947); Andreyev (1976); Kwakernaak, Sivan (1991); Polyak, 
Scherbakov (2002); Boyd et al. (1994)). It is of supreme theoretical and practical interest in electrical engineering, 
aerospace technology and power industry (Vostrikov et al. (2006)). For example, the simplified model of complex 
power grid can be composed of large number of the oscillatory 
systems, representing elastically connected generator 
groups (Kundur (1994)). The oscillatory systems have the 

meets with difficulties being applied to the large-scale power systems. The modified Arnoldi method (Arnoldi 
(1951); Kundur et al. (1990)) and the matrix sign function technique (Misrikhanov, Ryabchenko (2008)) are the ef-
cient algorithms for computing the ill-stable eigenvalues. Another approach employs the computation of the domi-
nant pole spectrum of a power system (Martins (1997)).

In this paper we propose a new method for the small-
signal stability analysis of the power systems based on the 
spectral decomposition of a square $H_2$ norm of the transfer 
function. We analyze the dynamic behavior of individual 
eigen-components. The proposed method can be consid-
ered as a special case of a more general approach of Grami-
ans and sub-Gramians (Yadykin (2010); Yadykin, Galaviev 
(2013)) for solving the matrix differential and algebraic 
Lyapunov equations, based on the spectral decomposition 
of the Lyapunov integral, the Laplace transform and the 

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sub-Gramian approach

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proposed method can be applied to the stability analysis of a real small power grid at Russky Island (Grobbovy et al. (2013)).

2. PROBLEM STATEMENT

Let us consider a mathematical model of the power system defined by a nonlinear algebro-differential system of equations

\[ \dot{x}(t) = f(x, u, t), \quad x(t_0) = 0, \]
\[ M(x, t) \dot{x}(t) = N(x, t) u(t). \]  

(1)

The linearized model of this system about the equilibrium point can be represented as a linear algebro-differential system of equations

\[ \dot{x}(t) = A_1 x(t) + B u(t), \quad x(t_0) = 0, \]
\[ M(x, t) \dot{x}(t) = N u(t). \]  

(2)

For a nonsingular matrix \( N \) this system can be written as

\[ \dot{x}(t) = A x(t), \quad x(t_0) = 0, \]
\[ A = A_1 + B N^{-1} M. \]  

(3)

Let us consider a fully controllable and observable continuous linear time-invariant system with one input and one output defined by real matrices \( A_{n \times n}, B_{n \times 1}, C_{1 \times n} \)

\[ \dot{x} = A x + B u, \quad x(0) = 0, \]
\[ y = C x, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^1, \quad y \in \mathbb{R}^1. \]  

(4)

The finite and infinite controllability Gramians of this system in the time domain are defined (Talbot (1959); Hanzon, Peeters (1996)) as

\[ P^C(t) = \int_0^t e^{A T} B B^T e^{A^T} d\tau, \]
\[ P^C(\infty) = \int_0^\infty e^{A T} B B^T e^{A^T} d\tau. \]  

(5)

Direct substitution reveals that these Gramians are solutions of the differential and algebraic Lyapunov equations (Antoulas (2005)) as

\[ \frac{dP(t)}{dt} = AP(t) + P(t) A^T + B B^T, \quad P(0) = 0_{n \times n}, \]
\[ AP(\infty) + P(\infty) A^T + B B^T = 0. \]  

(6)

A degree of system (4) stability can be defined as

\[ d = \max_i \{ \text{Re}(\lambda_i) \} \]  

(8)

where \( \lambda_i \) are the system eigenvalues.

In this paper we consider two problems. The first problem is to find the degree of stability (8) of the system (4) with a Hurwitz matrix \( A \). A straightforward calculation of eigenvalues for the matrix \( A \) could be a challenging problem for high-order systems. We examine another approach, that of solving algebraic matrix Lyapunov equations

\[ (A + dI)^T V + V (A + dI) = -B B^T \]  

(9)

with a positive real parameter \( d \) (Ahmetzyanov et al. (2012)). It is well known that a system (4) has a degree of stability exceeding \( d \) if and only if a positive definite solution \( V \) of the system (9) exists for any positive definite matrix \( BB^T \) (Andreyev (1970)). Therefore the degree of stability can be found by solving the system of equations (9) at increasing \( d \) until its positive definite solution \( V \) ceases to exist.

The second problem is to obtain a stability index suitable for the power systems, which estimates the contribution of the individual ill-stable eigenmodes to the risk of stability loss. This contribution can be determined as a corresponding term in the spectral expansion of the Frobenius norm of the system transfer function. This representation allows the identification of the most dangerous modes in terms of the greatest contributions to the asymptotic variation of the system energy over an infinite time interval. Such modes will constitute the major part of the transfer function Frobenius norm. We are looking for the eigenmodes decomposition based on a solution of the corresponding differential or algebraic Lyapunov equation. Note that finding of the degree of stability \( d \) can be considered as a special case of this problem.

3. SPECTRAL DECOMPOSITION OF \( H_2 \) NORMS OF TRANSFER FUNCTIONS

In order to analyze the behavior of the ill-stable dynamical system we obtain a spectral decomposition of the \( H_2 \) norm of its transfer function by matrix \( A \) eigenmodes and analyze the properties of this decomposition.

It is well known, that the matrix resolvent can be expanded (Faddeev, Faddeeva (1963); Hanzon, Peeters (1996)) as

\[ (I s - A)^{-1} = \sum_{j=0}^{n-1} s^j A_j + N^{-1}(s), \]  

(10)

where \( N(s) = a_n s^n + \cdots + a_1 s + a_0 \) is the characteristic polynomial of the matrix \( A \) and the matrices \( A_{j \times n} \) are referred to as Faddeev matrices and can be found (Kwakernaak, Sivan (1991)) in the following form:

\[ A_j = \sum_{i=j+1}^{n} a_i A^{i-j}. \]  

(11)

Then the transfer function of the system can be written as

\[ W(s) = \frac{y(s)}{u(s)} = C(I s - A)^{-1} B = \frac{C A_{n-1} B s^{n-1} + \cdots + C A_1 B s + C A_0 B}{N(s)}, \]  

(12)

\[ = \frac{b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}{N(s)}. \]  

(13)

For convenience, we introduce the following notations:

\[ W(s) \equiv \frac{b_n}{N(s)} M(s), \quad B^T \equiv [b_{n-1} \ b_{n-2} \ldots \ b_1 \ b_0]. \]  

(13)

The following theorem characterizes an eigenmode decomposition of the square \( H_2 \) norm of the system transfer function in the frequency domain.

**Theorem 1.** Let us consider a fully controllable and observable [stable] continuous linear dynamical system (4). Let \( A \) be a real square matrix with multiple eigenvalues \( s_\delta \) with multiplicities \( m_\delta, m_1 + m_2 + \ldots + m_q = n \). Then,
the square $H_2$ norm of the system transfer function (12) is given by
\[
\|W(s)\|^2_2 = \sum_{i=1}^{q} G_i \sum_{j=1}^{m_i} \frac{(-1)^m_j}{(m_j-k)!(k-1)!} \int_{0}^{\infty} \left( \frac{s^k}{N(s)} \right)_{s=s_j} ds^k \times \sum_{j=0}^{n-1} \sum_{j=0}^{n-1} \frac{m_j^{-k}}{ds^m_j-k} \left( \frac{s^l}{N(s)} \right)_{s=s_j} \times b_j b_j^* .
\]

Proof. Follows from Theorem 1 in (Yadykin (2010)).

Corollary. For a complex square matrix $A$ the square $H_2$ norm of the system transfer function is given by a formula similar to (14) with the replacement of $b_j$ by $b_j^*$. If the matrix $A$ spectrum contains only simple eigenvalues $s_k$, then an expression (14) for the square $H_2$ norm of transfer function takes the following form:
\[
\|W\|^2_2 = \sum_{k=1}^{n} G_k,
\]
\[
G_k = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} s_k^i (-s_k)^j \prod_{j \neq k} N(s_i) N(-s_k) b_j b_j^* ,
\]
where $N'(s)$ is the derivative of $N(s)$ and $G_k$ represents an eigen-part of the square $H_2$ norm of the transfer function corresponding to the particular eigenvalue $s_k$. Theorem 1 allows the square $H_2$ norm of the transfer function to be represented as a sum of the quadratic forms
\[
\|W\|^2_2 = \sum_{k=1}^{n} G_k = \sum_{k=1}^{n} b(s_k) b(-s_k) N(s_k) N(-s_k),
\]
where $b(s) \equiv b_n s^n + \cdots + b_1 s + b_0$.

Each of this forms corresponds to a particular root of the system characteristic equation. The coefficients of the quadratic form corresponding to the eigenvalue $s_k$ are given by a matrix
\[
\hat{G}_k = \left[ s_k^i (-s_k)^j \right]_{i,j}, \quad G_k = b^T \hat{G}_k b .
\]
The basic properties of the eigen-parts $G_k$ can be characterized by the following theorem.

Theorem 2. Let us examine a fully controllable and observable stable system (4) with a diagonalizable matrix $A$. Then the eigen-parts (15) of the square $H_2$ norm of the transfer function have the following properties.

(i) If the matrix $A$ is real, then both the eigen-parts $G_k$ corresponding to the real eigenvalues and the sum of eigen-parts $G_{k1} + G_{k1}^*$ corresponding to pairs of complex conjugate eigenvalues are real. Therefore, the square $H_2$ norm of the transfer function can be represented as a sum of real numbers:
\[
\|W\|^2_2 = \sum_{k=1}^{l} G_k + \sum_{k=1}^{m} (G_{k1} + G_{k1}^*)
\]

(ii) If the matrix $A$ is Hurwitz, then the eigen-parts $G_k$ can be found as
\[
G_k = -CA(k)BB^T [s_k I + A^T]^{-1} C^T
\]
\[
A(k) \equiv \text{Res } ((Is - A)^{-1}, s_k) = \sum_{i=0}^{n-1} s_k^i A_i N'(s_k)
\]

Proof. Follows from Theorem 3 in (Yadykin, Galayev (2013)).

The square $H_2$ norm of the transfer function $\|W\|^2_2$ can be interpreted as a measure of the output energy produced by a unit energy input. Then the absolute values of eigen-parts $|G_k|$ characterize the contribution of the eigenmodes to the total variation of the system output energy. Therefore the eigenmode decomposition (15) allows an identification of the most dangerous modes in terms of the greatest contributions to the total output energy of the system. Such modes will constitute the major part of the sum. Conversely, those modes not making a significant contribution can be regarded as posing no threat to a system stability. Since each eigen-part is associated with a particular eigenvector, the potential sources of the instabilities can easily be localized and tracked over time. The representation (15) has a particular importance for analyzing the behavior of a power system operating near its stability boundary.

4. ASYMPTOTIC BEHAVIOR OF TRANSFER FUNCTIONS NEAR THE STABILITY BOUNDARY

Let us examine the asymptotic behavior of the eigen-parts (15) when one or more eigenvalues approach the imaginary axis from the left. We call these eigenvalues ill-stable. Formally, ill-stable eigenvalues can be defined as eigenvalues with small negative real parts. The denominator polynomial in (15) takes the following form:
\[
N'(s_k) = \prod_{i \neq k} (s_k - s_i),
\]
\[
N(-s_k) = -2s_k \prod_{i \neq k} (-s_k - s_i),
\]
\[
N'(s_k) N(-s_k) = -2s_k \prod_{i \neq k} (s_k^2 - s_i^2), k = 1, 2, \ldots, n.
\]

For the nominator polynomial in (16) we obtain:
\[
b(s_k) = \prod_{j=1}^{n-1} (s_k - \beta_j),
\]
\[
b(-s_k) = \prod_{j=1}^{n-1} (-s_k - \beta_j),
\]
\[
b(s_k) b(-s_k) = \prod_{j=1}^{n-1} (\beta_j^2 - s_k^2), k = 1, 2, \ldots, n.
\]

where $\beta_j$ is a j-th root of the nominator polynomial $b(s)$ of the transfer function (12).

Let us consider some specific cases when ill-stable eigenvalues approach the imaginary axis from the left.
Case A. There is one real ill-stable eigenvalue $s_k = -\alpha$ with $\alpha \to +0$. All the other eigenvalues are fixed. From (16, 20-21) we obtain:

$$||W||^2_2 \sim G_k \sim \frac{b_0^2}{2\alpha \prod_{i\neq k} s_i^2} \to \infty (\alpha \to +0).$$

(22)

The square $H_2$ norm of the transfer function is directly proportional to the constant term $b_0^2$ of the numerator polynomial and is inversely proportional to the absolute value $\alpha$ of the ill-stable real eigenvalue.

Case B. There are two real ill-stable eigenvalues $s_k = -\alpha_1$ and $s_l = -\alpha_2$ with $\alpha_1 \to +0$ and $\alpha_2 \to +0$. All the other eigenvalues are fixed. In this case we obtain:

$$||W||^2_2 \sim G_k + G_l \sim \frac{b_0^2}{2\alpha_1 \alpha_2 (\alpha_1 + \alpha_2) \prod_{i \neq k, l} s_i^2} \to \infty (\alpha_1, \alpha_2 \to +0).$$

(23)

The square $H_2$ norm of the transfer function is inversely proportional to the product of absolute values of ill-stable eigenvalues and their absolute values sum. In contrast to the previous case $||W||^2_2$ grows much faster due to the synergetic effect of two real ill-stable modes.

Case C. There is a pair of ill-stable complex conjugate eigenvalues $s_k = -\alpha + j\omega$ and $s_k^* = -\alpha - j\omega$ with $\alpha \to +0$. All the other eigenvalues are fixed. In a similar way we obtain:

$$||W||^2_2 \sim G_k + G_k^* \sim \frac{b(j\omega) b(-j\omega)}{4\alpha(\omega^2 + \alpha^2) \prod_{i \neq k} s_i^2} \to \infty (\alpha \to +0).$$

(24)

If the frequency $\omega$ also approaches zero value, then we obtain the following asymptotic expression:

$$||W||^2_2 \sim G_k + G_k^* \sim \frac{b_0^2}{4\alpha(\omega^2 + \alpha^2) \prod_{i \neq k} s_i^2} \to \infty (\alpha, \omega \to +0).$$

(25)

Therefore in the case of a pair of complex conjugate ill-stable low-frequency eigenvalues, $||W||^2_2$ is asymptotically inversely proportional to the product of these conjugate eigenvalues and the absolute value of their real part.

Case D. There is one real ill-stable eigenvalue $s_1 = -\alpha_0$ and one pair of ill-stable complex conjugate eigenvalues $s_2 = -\alpha + j\omega$ and $s_3 = -\alpha - j\omega$ with $\alpha_0, \alpha \to +0$. All the other eigenvalues are fixed. From (16) we obtain:

$$||W||^2_2 \sim G_1 + G_2 + G_3 \sim \frac{b_0^2}{2\alpha \prod_{i \neq 1,2,3} s_i^2} + \frac{b(j\omega) b(-j\omega)}{2\alpha \prod_{i \neq 1,2,3} (s_i^2 + \omega^2)} \to \infty (\alpha_0, \alpha \to +0).$$

(26)

If the frequency $\omega$ also approaches zero value, then we obtain the following asymptotic expression:

$$||W||^2_2 \sim G_1 + G_2 + G_3 \sim \frac{b_0^2}{2\alpha \prod_{i \neq 1,2,3} s_i^2} + \frac{b(j\omega) b(-j\omega)}{2\alpha \prod_{i \neq 1,2,3} (s_i^2 + \omega^2)} \to \infty (\alpha_0, \alpha \to +0).$$

(27)

Comparing this with the case A one can see that an additional pair of complex conjugate ill-stable eigenvalues significantly increases the contribution of a single ill-stable real eigenvalue into $||W||^2_2$. In particular, low-frequency ill-stable modes can heavily increase the $H_2$ norm of the transfer function.

The considered cases suggest several conclusions. The eigen-parts (15) of the square norm $||W||^2_2$ corresponding to the ill-stable modes have a similar asymptotic behavior. They infinitely grow near the stability boundary of the system. Our interpretation of this result is that the total energy of a dynamical system, operating under the pre-fault conditions, accumulates in the ill-stable modes. In this case the decomposition (15) of $||W||^2_2$ allows the identification of the most dangerous modes in terms of their greatest contributions to the total energy of the system. Such modes will constitute the major part of the sum. Conversely, those modes not making a significant contribution can be regarded as posing no threat to a system stability.

If there are several ill-stable modes they can increase the system energy up to a critical level much earlier due to their synergetic effect. In particular, according to (24, 26) a transfer function norm is inversely proportional to the frequency or even to the frequency squared. The lower the frequency of a given mode is, the stronger influence it has on the system transfer function norm. Therefore the ill-stable low-frequency modes can pose a special threat to the system stability because they can act as a catalyst increasing the energy in the system (Gaglioti et al. (2011)).

5. CASE STUDY

In order to illustrate how the proposed method can be applied to the stability analysis of power grids, we employ the Simulink model of a mini-grid at Russky Island (Grobovoy et al. (2013)). The one-line diagram of the power network model is presented in Fig.1.

In the existing power network at Russky Island, 35 kV rated voltage is used. The total length of the transmission lines is 19.65 km. In this investigation, the distributing network with the rated voltage of 10 kV and less are represented by the loads at the level of 35 kV, but some are combined with the nodes of the equivalent generators on the rated voltage of 10 kV or 6.3 kV. The power network model contains three two-winding power transformers, one three-winding transformer, and one autotransformer. Eight consumption loads represent the island electricity demands. Total electricity consumption in the model under investigation amounts to 45.65 MW. An electric power is generated in the system by four combined heat and power plants (CHPP). The CHPP-1 consists of five gas turbine with rated power 7.33 MWA, CHPP-2 is comprised
of two 2 MWA gas turbines, CHPP-3 consists of two gas turbines with rated power 7.33 MWA, and CHPP-4 has in its structure one 1.8 MWA gas turbine. For the purposes of the present examination, four equivalent generators represent these power generation units (G1, G2, G3 and G4 in Fig.1). Each of these is equipped by the excitation system and speed control which are available in the MATLAB software in the Simulink/SimPowerSystems package. The well-known Rowen’s model (Yee et al. (2008)) for gas turbine has been used. In our simulation, however, we used only one control channel and neglected both channels of the temperature control of the exhaust gases and acceleration loop.

This grid is obviously too small and underpowered to have a lack of static stability in a real life, but it is possible to overload a computer model for the testing purposes. We studied the limit of the system stability by simultaneously increasing all of the loads and the active power of each generator while keeping the ratios between these fixed. We define the power increase coefficient as $\gamma = \frac{P}{P_0}$, where $P$ and $P_0$ are the total active power and the initial total active power of the generators respectively. For each $\gamma$ we linearize the system at $t = 100s$ and obtain a 64th order linear model. For the transfer function calculations we assume all elements of the matrices $B$ and $C$ equal ones.

The behavior of the $H_2$ and $H_\infty$ norms of the transfer function as well as the behavior of the two largest eigen-parts (15) is shown in Fig.2 as functions of the power increase coefficient $\gamma$. The thick solid line represents the eigen-part corresponding to the inter-machine rotor angle oscillation between the generators G1 and G4 (or G1-G4 mode). The dotted line represents the eigen-part corresponding to the 4-inter-machine oscillatory mode (or 4-IM mode). For the most of loads the 4-IM eigenvalue is the closest to Im axis, but the G1-G4 mode becomes unstable for $\gamma \approx 7.25$ and the norm of the corresponding eigen-part approaches infinity. The $H_2$ and $H_\infty$ norms of the transfer function (shown in Fig.2 by the solid and dashed lines respectively) approach infinity at $\gamma \approx 7.25$ as well (Ahmetzyanov et al. (2012)).

One can see that the analysis of the behavior of G1-G4 eigen-part allows the identification of the pre-fault conditions much earlier than the analysis of the $H_2$ and $H_\infty$ norms of the transfer function. The proposed eigenmode decomposition also allows us to localize accurately the source of the potential instability (i.e., the rotor angle oscillation between G1 and G4 generators). Another interesting observation is that the eigen-part of the most dangerous mode seems to be a good approximation of the $H_\infty$ norm of the transfer function. This observation (purely experimental by the moment) seems to be reasonable since both of them represent the worst-case gain of the system.

\section{Conclusion}

In this paper we propose a new method for the small-signal stability analysis of power systems based on the spectral decomposition of a square $H_2$ norm of the transfer function. We analyze the dynamic behavior of individual eigen-components. Compared with the dynamics of the full expression for the $H_2$ and $H_\infty$ norms of the transfer functions, the analysis of the behavior of the individual eigen-parts allows the earlier identification of the pre-fault condition occurrence. In addition, since each eigen-part is associated with a certain characteristic state vector, the potential sources of instability can easily be located and tracked over time. The proposed method can be considered as a special case of a more general approach of Gramians and sub-Gramians proposed in (Yadykin (2010); Yadykin et al. (2013)). However, it is more convenient for the practical calculations and can be used to study the stability of large-scale dynamical systems.

The proposed decomposition has a particular importance for analyzing behavior of a power system operating near its stability boundary. We explore the asymptotic behavior of the ill-stable systems, when one or more eigenvalues approach the imaginary axis from the left. In this case the total energy of a dynamical system accumulates in the ill-stable modes, and it is sufficient to know only the corresponding eigenvalues, as they provide the principal contributions to the total energy of the system. Analyzing the asymptotic expressions, we observed that several ill-stable modes can increase the system energy up to a critical level much earlier due to their synergetic effect. In particular, the ill-stable low-frequency modes can pose a special threat to the system stability because they can act
as a catalyst increasing the energy in the system. Finally in our paper we illustrate how the proposed method can be applied to the stability analysis of a real small power grid at Russky Island.

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