Solution of Static Reduced Decoupling Problem for Linear Systems

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Abstract: We propose a structural solution by non-regular static state feedback to the diagonal, or row by row decoupling problem for linear systems. Without being completely general, this solution concerns the Reduced Morgan’s Problem, that is we do not want it increases the essential orders of the original system. The solution that we propose relies on properties between, on one hand, some partial infinite structures extracted from well chosen interactors and on the other hand, the controllability indices of a specific shifted system. To my knowledge, there was at this date only very partial solutions to this problem deemed structurally hard.

Keywords: linear systems, non regular control, state feedback, diagonal decoupling, structural properties.

1. INTRODUCTION

The diagonal decoupling of linear systems by static state feedbacks \( u = Fx + Gv \), or Morgan’s Problem, is a very difficult control problem when \( G \) should be not invertible. Yet such non regular static solutions can exist even when there is none regular. Such non regular feedbacks can transform all the structures underlying the regular solutions to control problems. The challenge is to find new structures that will allow us to solve the decoupling Problem. The non-regular decoupling by dynamic state feedback was solved by Dion and Commault [1988] due to new invariants, the essential orders (E.O.). Now, non regular decoupling always comes down to a problem of increasing structures. The relative simplicity of the dynamic case comes from the facts that the E.O. are the minimal infinite structure to achieve for decoupling and that it is always possible to get it if there is a solution. To make this increase, dynamic feedbacks use integrators that are external to the system. This is no more the case for static feedbacks that must only use internal dynamics. We consider here the Static Reduced Morgan’s Problem (SRMP), say the decoupling without increasing the E.O.: it provides insight into the complex mechanisms of structural changes by non-regular controls and, from the practical point of view, this restricted problem is not without interest: indeed, if a static solution exists with an increasing of the E.O., there always exists a dynamic one without this increasing od E.O.. There are so far only very partial results for the SRMP: when it is sufficient to increase only one element of the infinite structure (Herrera H and Lafay [1993]), or for trivial internal structures, (Zagalak et al. [1998]),(Lafay [2013]). The specific difficulties of SRMP are of two kinds: firstly, the increases of infinite structure for solving SRMP depend on the order of the outputs of the system, while the sum of the sizes of these increases does not depend of this order.

This lock has been lift in Lafay [2013], where it is proved that there is a unique ”minimal list of decoupling indices” such that if SRMP has a static solution, one solution exists with this list. Secondly we must take into account internal couplings which are unobservable from the outputs to be decoupled. For that, we develop a non trivial general formulation of SRMP inspired by (Herrera H and Lafay [1993]). The general solution of SRMP relies on the controllability of a well defined shifted system. We propose here only the structural aspects of decoupling, without considerations on the internal stability.

2. NOTATIONS AND BACKGROUND

2.1 Notations

Let \( \Sigma(C,A,B) \), denoted shortly \( \Sigma \), a linear system whose state is supposed to be measured or reconstructible:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t).
\end{align*}
\]

\( x \in X \subset \mathbb{R}^n, u \in U \subset \mathbb{R}^m \) and \( y \in Y \subset \mathbb{R}^p. \) \( B \) is monic and \( C \) epic. \( \Sigma \) is controllable and right invertible. \( \mathcal{V}^* \) and \( \mathcal{R}^* \) are the supremal \((A,B)\)invariant and the supremal controllability subspaces in the kernel of \( C \). As the structure of \( \mathcal{V}^*/\mathcal{R}^* \) does not affect the decouplability by static state feedbacks (cf the Morse’s canonical form Morse [1973]) we can assume without any loss of generality that \( \mathcal{V}^* = \mathcal{R}^* \), say that \( \Sigma \) has no finite zeros (they concern only internal stability of the state, which is not addressed here). A set of \( p \) elements is noted \( \{\bullet\}_p \). Polynomial and rational functions in variable \( s \) are respectively noted \( \{\bullet\}_s \) and \( \{\bullet\}(s) \). We note \( \partial_p[s] \) the degree of \( p[s] \) and \( \partial_s M[s] \) the highest degree of the \( 1 \)th column of matrix \( M[s] \). \( f^{(n)}(t) \) is the derivative of order \( n \) of \( f(t) \). \( I_p \) is the identity matrix of order \( n \) and \( \text{diag}(a_i) \), the \( p \times p \) matrix which diagonal terms are \( a_i \), for \( i \leq p \).
Definition 1. The interactor of $\Sigma$ (Wolovich and Falb [1976]), is the unique $p \times p$ triangular and non-singular polynomial matrix $\Phi[s] = [\varphi_{i,j}[s]]$ such that there exists a biproper $m \times m$ (non unique) matrix $B(s)$ satisfying: $T(s) = C(slA - A)^{-1}B = [\Phi^{-1}[0]]B(s)$ where:

- $\varphi_{i,i} = s^{d_{i}}, i = 1, \ldots, p, d_{i}$ being positive integers
- $\varphi_{i,j}$ is zero or $\varphi_{i,j}/s^{d_{j}}$ is divisible by $s$, $\forall i > j$.

$\Phi[s]$ is invariant under the action of the group $(T, F, G)$, $T$ and $G$ being changing of bases on $X$ and $U$, and $F$ is a state feedback. Without any loss of generality, we can suppose that $\Phi$ is row reduced (r.r.), (Lafay et al. [1992]). Four lists of integers characterize partially the structure of $\Sigma$, (see Morse [1973], Commault et al. [1986]):

- $\{e_{i}\}_{i \in p}$: controllability indices of $(A, B)$ ,
- $\{n_{i}\}_{i \in p}$: Morse’s list of the orders of the zeroes at infinity. If the interactor of $\Sigma$ is r.r., $d_{i} = n_{i}'$ $i \leq p$.
- $\{e_{i}\}_{i \in p}$: Morse’s list providing the structure of $R^{*}$, $\{n_{i}'\}_{i \in p}$ the p E.O. of the outputs $y_{i}(t)$: $n_{i} = \partial_{e_{i}}\Phi[s]$, $i \leq p$.

These lists are invariant under the action of the group $(T, F, G, \Pi)$, where $\Pi$ is a permutation of the outputs of $\Sigma$ and $G$ is invertible (Herrera H et al. [1997]).

An extended system $\Sigma_{e}(C_{e}, A, B)$ is defined in Herrera H et al. [1997] by adding $m-p$ fictitious outputs, to reflect the structure of $R^{*}$.

Proposition 1. The $m \times m$ interactor of $\Sigma_{e}$, called “extended interactor of $\Sigma_{e}$”, has the structure:

$$\Phi_{e}[s] = \begin{bmatrix} \Phi_{e,1}[s] & \cdots & \Phi_{e,m}[s] \\ \Phi_{e,2}[s] & \cdots & \Phi_{e,m}[s] \\ \cdots & \cdots & \cdots \\ \Phi_{e,m}[s] & \cdots & \Phi_{e,m}[s] \end{bmatrix} = \begin{bmatrix} \varphi_{e,1,1}[s] & \cdots & \varphi_{e,1,m}[s] \\ \varphi_{e,2,1}[s] & \cdots & \varphi_{e,2,m}[s] \\ \cdots & \cdots & \cdots \\ \varphi_{e,m,1}[s] & \cdots & \varphi_{e,m,m}[s] \end{bmatrix} ,$$

where

$$\Phi_{e,1}[s] = \Phi_{e}[s], \text{then if } \Phi \text{ is r.r.}, \varphi_{e,1,i}[s] = s^{n_{i}}, i \leq p,$$

$$\Phi_{e,2}[s] = \text{diag}(s^{n_{i}})_{i \leq p},$$

polynomials $\varphi_{e,2,j}[s] = \sum_{i=1}^{\ell_{j}} a_{i,j} s^{n_{i}}$ of $\Phi_{e,2}[s]$ are zero or verify: $\partial_{\varphi_{e,2,j}} = 1$ and $h \leq \sigma - 1$ .

Note $\Phi_{e}[s] = W(s)\text{diag}(s^{n_{i}})_{i \leq p}$. $W(s)$ is a proper matrix, of rank at infinity $k$, called “proper part of $\Phi_{e}[s]$”. and:

Proposition 2. (Dion and Commault [1988]) There exists output’s permutation($\Pi$) such that the $p \times p$ interactor $\Phi_{n}$ of $\Sigma_{e}(\Pi, A, B)$ has the structure:

$$\Phi_{n} = \begin{bmatrix} \Phi_{n,1}[s] & \cdots & \Phi_{n,m}[s] \\ \Phi_{n,2}[s] & \cdots & \Phi_{n,m}[s] \\ \cdots & \cdots & \cdots \\ \Phi_{n,m}[s] & \cdots & \Phi_{n,m}[s] \end{bmatrix} = \begin{bmatrix} \varphi_{n,1,1}[s] & \cdots & \varphi_{n,1,m}[s] \\ \varphi_{n,2,1}[s] & \cdots & \varphi_{n,2,m}[s] \\ \cdots & \cdots & \cdots \\ \varphi_{n,m,1}[s] & \cdots & \varphi_{n,m,m}[s] \end{bmatrix} ,$$

where

$$\Phi_{n,1}[s] = \Phi_{e}[s], \text{then if } \Phi_{n,1}[s] \text{ is r.r.}, \varphi_{n,1,i}[s] = s^{n_{i}}, i \leq p,$$

$$\Phi_{n,2}[s] = \text{diag}(s^{n_{i}})_{i \leq p},$$

polynomials $\varphi_{n,2,j}[s] = \sum_{i=1}^{\ell_{j}} a_{i,j} s^{n_{i}}$ of $\Phi_{n,2}[s]$ are zero or verify: $\partial_{\varphi_{n,2,j}} = 1$ and $h \leq \sigma - 1$ .

There are always permutations $\Pi_{m}$ such that $\Phi_{n,1}[s]$ is the unique minimal list $\{(\delta_{i})_{p \times k}\}$ of decoupling indices of $\Sigma_{e}$. $\delta_{i}$ is defined in (Lafay [2013]) using the notion of “minor” list:

Definition 2. Let two lists of integers $\{(\delta_{i})_{k \times 1}\}$ and $\{(\gamma_{i})_{k \times 1}\}$ such that $\sum_{1}^{k} \delta_{i} = \sum_{1}^{k} \gamma_{i}$. Note $\{(\delta_{i})_{k \times 1}\}$ upper, $\{(\gamma_{i})_{k \times 1}\}$ lower, their dual lists, $\{(\delta_{i})_{k \times 1}\}$ "minor" the list $\{(\gamma_{i})_{k \times 1}\}$ if, for $i = 1, \ldots, \text{sup}($upper$) \text{sup}($lower$)$, we have:

$$\sum_{j=1}^{i} \delta_{j} \leq \sum_{j=1}^{i} \gamma_{j}.$$
3. MORGAN’S PROBLEM

3.1 Formulation of the problem and consequences

The static diagonal decoupling without stability of $\Sigma$, or Static Morgan’s Problem, can be stated as follows: Given $\Sigma$ described by (1), does it exists a static state feedback $u = Fx + Gu = Fx + G_1v_1 + ... + G_pv_p, v_p \in U$ and $G_i \in \mathbb{R}^{m \times 1}$ such that, for any $i \in 1, 2, ..., p, v_i$ controls the scalar output $y_i$ without affecting the $p - 1$ other outputs $y_j$. If such a feedback exists, the transfer matrix $TF_G(s)$ of $\Sigma(C, A + BF, BG)$ is $TF_G(s) = diag(s^{-r_1}, ..., s^{-r_r}), r_i \in \mathbb{R}$. If $G$ is regular, this problem has a solution if and only if the ordered lists $\{n'_i\}_p$ and $\{n_{ie}\}_p$ are the same. So, the interactor of $\Sigma$ is diagonal (Commault et al. [1986]). If $\{n'_i\}_p \neq \{n_{ie}\}_p$, $\Sigma$ will be decouplable if and only if it is possible to increase the structure at infinity so that it matches to (new) E.O. This can only be achieved by a non regular feedback, say with a loss of inputs. With such controls, the list of E.O. of $\Sigma$ is not always the minimal infinite structure to reach for the static decoupling.

Static Restricted Morgan’s Problem, (SRMP), is the particular case of Static Morgan’s Problem where the E.O. of $\Sigma$ should be the infinite structure of the decoupled system. Finally, as any non regular decoupling reduces to increase infinite structure, the solution will be based on a Theorem of Loiseau [1988] that we remind now:

**Theorem 1.** Let a linear system, $\{n'_i\}$ it’s infinite structure and $\{\sigma_i\}$ it’s I2 Morse’s list. Let $\{p_i\}$ a list of integers. Note $\{v_i\}, \{o_i\}$ and $\{\pi_i\}$ the dual lists of, respectively, $\{n'_i\}, \{\sigma_i\}$ and $\{p_i\}$. Let $\{\Gamma_i\}$ obtained by arranging the differences $(\pi_i - v_i)$ in a non increasing order. Then, there exists a static state feedback such that the structure at infinity of the closed loop system is the list $\{p_i\}$ if and only if:

$$v_i - v_j \geq \pi_i - \pi_j, \forall i, j \geq 1,$$

$$\sum_{i=1}^{j} a_i \geq \sum_{i=1}^{j} \Gamma_i, \forall j \geq 1.$$  

Let us consider now the dynamic solution of decoupling:

3.2 The Dynamic Morgan’s Problem (DMP)

**Proposition 3.** (Dion and Commault [1988]). DMP is solvable if and only if $\Sigma$ is right invertible and $m - p \geq p - k$. The final interactor is $diag\{s^{n_{ie}}\}_p$ and the system with entries $\{v_1, v_2, ..., v_{p-k}, u_{p-k+1}, ..., u_p\}$ is regularly decouplable. So we need $p - k$ independent chains of integrators coming from a dynamic extension of dimension $\sum_{i=1}^{p} n_{ie} - \sum_{i=1}^{p} n'_i$.

Note that the right invertibility of $\Sigma$ expresses the independence of it’s outputs. So this property will be also necessary for the SRPM.

4. THE SOLUTION OF SRMP

Let $\Sigma_m$ as in Subsection 2.3. We note $B_m = [B_B|B_r|B_r]$ and $u(t) = (u_k(t) u_s(t) u_i(t))^T$, where $B_b = [b_1, ..., b_{p-k}], B_s = [b_{p-k+1}, ..., b_p]$ and $B_r = B \cap \mathbb{R}_+^* = [b_{p+1}, ..., b_m]$. Define the $m \times m$ polynomial matrix $\Phi_{m,s}^S$ by:

$$\Phi_{m,s}^S[s] = \Phi_{m,s}^S[s] = \Phi_{m,s}^S[s] = \Phi_{m,s}^S[s] = \Phi_{m,s}^S[s],$$

$$\Phi_{m,s}^S[s] = \phi_{m,s}^S[s] \Phi_{m,s}^S[s] \Phi_{m,s}^S[s] \Phi_{m,s}^S[s] \Phi_{m,s}^S[s],$$

where

- $\Phi_{m,s}^S[s] = diag\{s^{n_{ie}}\}_p$
- $\Phi_{m,s}^S[s] = diag\{s^{n_{ie}}\}_p$
- $\Phi_{m,s}^S[s] = \phi_{m,s}^S[s] \Phi_{m,s}^S[s] \Phi_{m,s}^S[s] \Phi_{m,s}^S[s] \Phi_{m,s}^S[s],$
- $\Phi_{m,s}^S[s] = \phi_{m,s}^S[s] \Phi_{m,s}^S[s] \Phi_{m,s}^S[s] \Phi_{m,s}^S[s] \Phi_{m,s}^S[s],$
- $\Phi_{m,s}^S[s] = \phi_{m,s}^S[s] \Phi_{m,s}^S[s] \Phi_{m,s}^S[s] \Phi_{m,s}^S[s] \Phi_{m,s}^S[s],$

4.1 Shifted system

**Definition 3.** The shifted system $\Sigma_m^S(C_m^S, A_m^S, B_m^S)$ associated with $\Sigma_m$ is the invertible system of which $\Phi_m^S[s], \text{ equation (10)}$, is the extended interactor.

Extend $X$ by an dynamic extension $X_s$, of dimension $n_a = \sum_{i=p-k}^{p} \Delta_i$, composed of $p - k$ independent controllable and observable chains of integrators of lengths $\Delta_1, ..., \Delta_{p-k}$ with entries $w_i(t), ..., w_{p-k}(t)$ in $U_s$ and outputs $z_i(t), ..., w_{p-k}(t)$ in $Y_s$. The state, control and output spaces of $\Sigma_m$ complemented by this extension are $X_s = X \oplus X_s, U_s = U \oplus U_s$ and $Y_s = Y \oplus Y_s$. The shifted system is obtained by replacing the entries $u_k(t)$ of $B_b$ by $z_i(t)$ for $i \leq p - k$. Then:

$$A_m^S = \begin{bmatrix} A_m & A_m^1 & \cdots & A_m^k \\ \Delta_1 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ \end{bmatrix},$$

where

$$J_{\Delta_i} = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ \end{bmatrix},$$

$$B_m^S = B_m^S B_m^S B_m^S B_m^S,$$

$$B_a = diag\{b_{a,1}, b_{a,2}, ..., b_{a,n_a}\} \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix},$$

$$C_m^S = C_m^S \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}.$$
and the corresponding extended interactor of $\Sigma_m^{\Sigma^c}$ is:

$$\Phi_m^{S,\Sigma^c}[s] = \left[ diag\{s^{m_k}\}^{-1}_p \begin{pmatrix} 0 \end{pmatrix}_{p \times m - p} \right],$$ \hspace{1cm} (15)

$\Phi_m^{S,\Sigma^c}[s]$ comes from $\Phi_m^{S,\Sigma}[s]$ by eliminating, in each column, $j \leq p$, monomials of degree less than or equal to $n_{je}$.

4.2 A convenient formulation of SRMP

SRMP will be solved if it is possible to replace the $p - k$ independent external chains of DMP by $p - k$ independent chains extracted from $\Sigma$. A chain of length $L$ is activated only at its beginning by a function $f_a(t)$ and generate $f_c(t)$, i.e. $f_c(t) = f_a(t)$. Let a set of $q$ chains of integrators defined by $f_c(t) = f_q(t), i \leq q$. For SRMP, we require that each function $f_q(t)$ contains at least one input of $\Sigma$. These chains are independent if, $\forall j \neq i$, the term containing the inputs in $f_q(t)$ is not a linear combination (i.e.) of corresponding terms in $f_r(t)$, and if $f_q(t)$ is not a l.c. of $f_r(t)$. The following Lemma characterizes internal chains that will increase the infinite structure without changing the E.O.: it is a new formulation for SRMP, generalizing in a non trivial way Lemma 4.2 in (Herrera H and Lafay [1993]) valid when $k = p - 1$.

Lemma 1. Let a right invertible system $\Sigma_m$ with $R^* = V^*, k$ the rank of $W(s)$ and $\{\Delta_l\}_{l=0}^{m}$ its decouplability indices (minimal decoupling indices of $\Sigma$). Then, SRMP has a solution if and only if it is possible to extract, from $R^* p - k$, independent chains of integrators of lengths $\Delta_1,...,\Delta_{p-k}$ described by $f_q(t)^{\Delta_1} = f_{\Delta_1}(t), i \leq p - k$ such that:

(a) The output $f_q(t)$ of each chain is only function of $x(t)$ and these $p - k$ functions are independent.

(b) For $\forall j \leq p - k$, the chain $f_q(t)$ of each chain does not contain derivatives of $u_j(t), j = p - k + 1,...,p$.

(c) For $\forall i \leq p - k$, the chain $f_q(t)$ of each chain does not contain derivatives of inputs $u_j(t)$ of order greater than or equal to $\Delta_j, j \leq p - k$.

Proof 1. 1F. Let a state feedback $(F, G)$ which decouples $\Sigma_m$. From Dion and Commault [1988] this feedback is equivalent to the precomparator $C(s) = (I_m - F(sI_n - A_m)^{-1}B)^{-1}G = \begin{bmatrix} W_1, m(s) \\ X(s) \end{bmatrix}$, where $X(s)$ is a proper $m - p \times p$ matrix and, noting $d = s^{-1}$, the $p \times p$ proper part $W_1, m(s)$ of the interactor of $\Sigma_m$ can be written as:

$$W_1, m(s) = \begin{bmatrix} d\Delta_1 & \vdots & 0 \\ (\hat{\varphi}_{2,1} & \vdots & \hat{\varphi}_{2,p-k} \\ \vdots & \vdots & \vdots \\ (\hat{\varphi}_{p-1} & 0 & 0 & \cdots & 1) \end{bmatrix},$$ \hspace{1cm} (16)

and $\partial \hat{\varphi}_{i,j}[d] < \Delta_j = n_{je} - f_i$ for $i \leq p - k$.

$$G = \lim_{s \to \infty} W_1, m(s) = \begin{bmatrix} 0 \end{bmatrix}_{p \times k}, \begin{bmatrix} 0 \end{bmatrix}_{k \times p}$$

where $g_{i,j} \in \mathbb{R}$, (Herrera H [1992]) .

So, the feedback $u(t) = Fx(t) + Gu(t)$ is given by:

$$u_i(t) = \begin{cases} F_i x(t), & i = 1, ..., p - k \\ F_i x(t) + \sum_{j=1}^{p-k} g_{i,j} v_j(t) + v_{i}(t), & i = p - k + 1, ..., p \end{cases}$$ \hspace{1cm} (17)

As $\Sigma_m(F, G) = (C_m, A_m + B_m F, B_m G)$ is assumed to be decoupled without changing the E.O., we have:

$$\begin{cases} y_i(\Delta_j)(t) = u_i(\Delta_j)(t) = v_i(t), & i = 1, ..., p - k \\ y_i(0)(t) = u_i(0)(t) = v_i(t), & i = p - k + 1, ..., p \end{cases}$$ \hspace{1cm} (18)

Therefore, equations (18) and (17(a)) describe $p - k$ chains of lengths $\Delta_1,...,\Delta_{p-k}$ generating independent functions $F_i x(t)$. Moreover, from (17b), these chains are not activated by derivatives of $u_{p-k+1}(t), ..., u_{p}(t)$ and for $i$ and $j \leq p - k$, the $i$th chain is not actived by derivatives of $v_j(t), i \neq j$. But, by (17b), if the chains should only make $\Sigma_m$ regularly decouplable, the $i$th chain can be actived by derivatives of $u_i(t)$ of order less than $\Delta_i$ for $i \leq p - k$, because each chain must generate a function of $x(t)$, say:

$$f_i(t) = f_i(\Delta_i^k)$$

From Loiseau [1988], since any increasing of infinite structure can only be done using entries of $R^*$, entries $v_1(t), ..., v_{p-k}(t)$ should contain independent l.c. of entries of $R^*$.

**Only if:** Assume that these $p - k$ independent chains can be constructed. For $i \leq p - k$, each chain is characterized by its length $\Delta_i$, its entry $f_i(t)$ and its output $f_i(t)$ linked by $f_i(\Delta_i^k) = f_i(t)$. As $f_i(t)$ is uniquely a function of $x(t)$, we can write $f_i(t) = F_i x(t)$, where $F_i \in \mathbb{R}^{1 \times n}$, and as the chains are independent, rank $F = [F_1 \cdots F_{p-k}] = p - k$.

Now, as the chains are extracted from $R^*$, each function $f_i(t)$ contains at least one l.c. of the entries $u_{p+1}(t), ..., u_m(t)$ and these l.c. are independent: this implies that dim $B \cap R^* \geq p - k$ (a necessary and sufficient condition for DMP). For static decoupling, we must add:

$$\sum_{i=1}^{p-k} \Delta_i \leq \sum_{i=1}^{m-p} \sigma_i.$$ By assumption, no derivative of $u_{p-k+1}(t), ..., u_{p}(t)$ appears in functions $f_i(t).$ However there may be terms which depend on $x(t)$ and/or on $u_{p-k+1}(t), ..., u_{p}(t)$, and/or for $i \leq p - k$, on derivatives of entries $v_i(t)$ of order less than $\Delta_i$, (from the fact that $f_i = F_i x(t)$). So, for $i \leq p - k$, the general form of $f_i(t)$ is:

$$f_i^p(u_{p+1}(t), ..., u_m(t)) + \sum_{j=1}^{p-k} f_{i, p-j}(u_{p-j+1}(t), ..., u_p(t))$$

$$+ \sum_{j=1}^{p-k} \sum_{j=1}^{p-k} \Psi_{i, j}[s] y_j(s)$$ \hspace{1cm} (19)

where $\partial \Psi_{i, j}[s] < n_{je}$.

Define the non regular feedback $u(t) = F x(t) + G v(t)$ by:

$$F = \begin{bmatrix} F_{p-k \times n} \\ F_{0 \times m - p \times n} \end{bmatrix}$$

and $G = \begin{bmatrix} 0_{p-k \times k} \\ 0_{p-k \times p-k} \\ 0_{m-p \times k} \end{bmatrix}$.
Here \( \text{rank } F = p - k \) and \( \text{rank } G_0 = p - k \leq m - p \).

The possibly non regular feedback \( (F_0, G_0) \) acts only on \( \mathcal{R}^* \) to create the \( p - k \) independent chains of integrators. Noting \( B_p = \text{Im} B \cap \mathcal{R}^* \) and \( x_{R_p}(t) \) trajectories in \( \mathcal{R}^* \), this part of feedback is such that: \( F_0 x_{R_p}(t) + B_p G_0 u_s(t) = \begin{bmatrix} f_{a_1}(\cdot) \\ \vdots \\ f_{a_{m-k}}(\cdot) \end{bmatrix} \), where \( u_s(t) = \begin{bmatrix} v_{p-k+1}(t) \\ \vdots \\ v_p(t) \end{bmatrix} \).

Then, as \( y_{f_i}^j(t) = u(t) = f_{a_i}(t) \) for \( i \leq p - k \), we obtain:

\[
L(y_{f_i}^j(t)) = f_i^p x(s) \Delta_i \overset{\Delta_i}{=} f_i^p \Delta_i = f_i^p (\bullet) + f_{a_2}(\bullet) + \sum_{j=1}^{p-k} \psi_{i,j}(s) y_j(s),
\]

Then, outputs \( y(s) \) and new entries \( e(s) \) are linked by:

\[
\begin{bmatrix} H_{11} & (0)_{p-k \times k} \\ H_{21} & H_{22} \end{bmatrix} y(s) = V(s) = \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix},
\]

where (21)

1. The \( p - k \times p - k \) polynomial matrix \( H_{11} \) is given by:

\[
H_{11} = \begin{bmatrix} s^{n_{i,1}} - \psi_{1,1} \\ \vdots \\ s^{n_{i,j}} - \psi_{i,j} \\ \vdots \\ s^{n_{p-k,i}} - \psi_{p-k,p-k} \end{bmatrix},
\]

2. \( H_{21} = \begin{bmatrix} \psi_{m,1}(s) \\ \vdots \\ \psi_{m,j}(s) \end{bmatrix} \) is a \( k \times p - k \) polynomial matrix,

3. \( H_{22}[s] = \text{diag} \{ s^{p_{1}}(s), \ldots, s^{p_{m}}(s) \} \).

4. \( V_1(s) = \begin{bmatrix} f_{a_1}(\bullet) + f_{a_2}(\bullet) \\ \vdots \\ f_{a_{p-k}}(\bullet) + f_{a_{p-k}}(\bullet) \end{bmatrix} \) and

\[
V_2(s) = \begin{bmatrix} v_{p-k+1}(s) = u_{p-k+1}(s) \\ \vdots \\ v_p(s) = u_p(s) \end{bmatrix}.
\]

As the polynomial of highest degree of each column of \( H[s] \) is the diagonal polynomial, the infinite structure coincides with E.O.: this system is regularly decouplable without increasing E.O.. This ends the proof of Lemma 1.

Remark 2. Functions \( f_{a_i}(t) \) cannot contain derivatives or entries of \( \mathcal{R}^* \). If that were the case, for example, for the first chain, the effective length of this chain would be less than \( \Delta_1 \). Indeed, suppose that this chain of length \( \Delta_1 \) is activated by \( u_{p+j}(t) \) and by \( u_{p+j}(t) \), for \( j \leq m - p \). Then \( f_{x}^{(\Delta_1)}(t) = f_{a_i}(t) = u_{p+j}(t) + u_{p+j}(t) \) and \( f_{x}^{(\Delta_1)}(t) = \int f_{a_i}(t) dt = \int u_{p+j}(t) dt + \int u_{p+j}(t) dt = f(x(t)) + u_{p+j}(t) \). The effective length of the chain is \( \Delta_1 - 1 \).

Remark 3. Inputs \( u_{i}(t), \ldots, u_{p-k}(t) \) are suppressed, while inputs \( u_{p-k+1}(t), \ldots, u_{p}(t) \) are preserved.

Remark 4. If functions \( f_{a_i}(t) \) do not include derivatives of inputs \( u_i(t), i = 1, \ldots, p - k \), the system is decoupled.

Lemma 1 will help us to derive conditions on \( \Sigma_m \) for ensure that such \( p - k \) independent decoupling chains exist.

To characterize the maximal lengths of the chains of \( \mathcal{R}^* \) which are not activated by derivatives of entries \( u_i(t) \) of order higher or equal to \( \Delta_i, i \leq p - k \), it is possible to apply Theorem 4.1 in Herrera H and Lafay [1993] taking into account the following remark:

Remark 5. The chains of \( \mathcal{R}^* \) satisfying items (b) and (c) of Lemma1 for \( \Sigma_m \) are the same as the chains of \( \mathcal{R}^* \) which are not activated by derivatives of entries \( w_1, \ldots, w_{p-k}, u_{p-k+1}, \ldots, u_p \) for \( \Sigma_m \).

Theorem 4.1 in Herrera H and Lafay [1993] becomes:

Proposition 4. Let \( \Sigma_m^S \) be derived from \( \Sigma_m \) by reordering \( B_m^S \) as \( B_m^S = [B_{m_1}^S | B_{m_2}^S | B_{m_3}^S] \). Let \( \{c_i\}_m \) the controllability indices of \( (A_{m_1}^S, B_m^S) \). Then the set \( \{\alpha_i\}_m \) of maximal lengths of subchains of \( \mathcal{R}^* \) not activated by derivatives of inputs \( w_1, \ldots, w_{p-k}, u_{p-k+1}, \ldots, u_p \) is given by the controllability indices of the entries of \( \mathcal{R}^* \):

\[
\alpha_i = c_i, \quad \text{for } i \leq m - p. \tag{22}
\]

4.3 Main result: the solution of SRMP

Theorem 2. Let the right invertible system \( \Sigma \) be given with \( \mathcal{R}^* = V^* \) and \( k \) the rank at infinity of \( W(s) \). Let \( \Sigma_m \) deduced by regular state feedback from \( \Sigma \) such that the infinite structure \( \{\Delta_i\}(p-k) \) of \( W_m(s) \) of \( \Sigma_m \) is the minimal list of decoupling indices of \( \Sigma \). Let \( \{\alpha_i\}_m \) the controllability indices of the entries of \( \mathcal{R}^* \) for the shifted system \( \Sigma_m \) associated with \( \Sigma_m \). Then SRMP has a solution if and only if, for all \( i \geq 1 \),

\[
\sum_{j=1}^{\alpha_i} \gamma_j \geq \sum_{j=1}^{\alpha_{i-1}} \gamma_j + \sum_{j=1}^{\gamma_{i-1}} \gamma_j \tag{23}
\]

where \( \{\alpha_i\}_m \) the dual list of \( \{\alpha_i\}_m \) and \( \{\gamma_i\}_m \) the dual list of \( \{\Delta_i\}_m \).

Proof 2. Sufficiency:

Remark 6. As mentioned in Lafay [2013], Theorem 1 cannot be applied directly, but it suffices to apply it choosing for list \( \{n_i\}_p \) the list \( \{1\}_p \) and try to obtain \( \{1 + \Delta_i\}_p \). This amounts to build, from the \( m - p \) chains of integrators of lengths \( \sigma_i \), \( \mathcal{R}^* \) and \( p - k \) independent chains of lengths \( \{\Delta_i\} \). Note that condition (8) is still always true. So there remains only conditions (9).

Let us now return to SRMP. It is sufficient to replace the dynamic extension of Subsection 4.1 by \( p - k \) independent chains of lengths \( \{\Delta_i\}_m \) extracted from \( \mathcal{R}^* \) and satisfying Lemma 1. This is equivalent to create, from \( \Sigma_m^S \), \( p - k \) independent chains extracted from \( \mathcal{R}^* \) not activated by derivatives of entries \( w_1, \ldots, w_{p-k}, u_{p-k+1}, \ldots, u_p \) (cf Remark 5). By Proposition 4, such chains should be built from the maximal sub-chains of \( \mathcal{R}^* \) which are not individually activated by derivatives of entries \( w_1, \ldots, w_{p-k}, u_{p-k+1}, \ldots, u_p \).

According to the shifted system, and Remark 5, integers \( \{\alpha_i\}_m \) represent equivalently the maximal lengths of sub-chains of \( \mathcal{R}^* \) not activated by derivatives of inputs \( w_{p-k+1}, \ldots, u_p \) and for \( i \leq p - k \) not activated by derivatives of inputs \( u_i \) of order greater than or equal to \( \Delta_i \). From Remark 6 we obtain (23) which is a sufficient condition for SRMP.

Necessity: The necessity comes from two facts. First: By Proposition 2, if there is a solution to SRMP, some of them require only (p-k) increases of infinite structure. Conditions (9) therefore requires that the minimal list of decoupling indices contains only (p-k)terms. Secondly: If there is no solution with these indices, it does not exist...\]
solution for any other list of decoupling indices coming from other permutations of outputs of $\Sigma$.

Corollary 1. The right invertibility of $\Sigma$ and condition (23) are necessary and sufficient conditions for the general Static Morgan’s Problem when $m - p = k$.

5. COMPARISONS

At our knowledge, the more advanced structural results on static Morgan’s problem are the following.

5.1 Herrera H and Lafay [1993]

SRMP is solved when $k = p - 1$.

Theorem 3. Let the right invertible system $\Sigma_m$ be given with $R^* = R^+$. Suppose $k = p - 1$. Let $\delta_1$ be the (nonzero)infinite structure of the interactor $\Pi_{m_k} [8]$, and $\{\alpha_i\}_{i=1}^{m-p}$ the controllability indices of the pair $(A_m, B_1[B_1])$ related with the columns of $B_f = B \cap R^*$. SRMP has a solution if and only if:

$$\sum_{j=1}^{m-p} \alpha_r, j > \delta_1.$$  \hspace{1cm} (24)

Let $\Sigma_m(C_m, A_m, B_m)$. Note $B_m = [b_1 | B_1 | B_k]$ as in Section 4. As $k = p - 1$, the list of decoupling indices contains only one term $\delta_1 = \sum_{i=1}^{m-p} n_i - \sum_{i=1}^{p} n'_i < n_{1e}$, and is minimal. A static solution will exist if and only if the sum of the lengths of the sub-chains of $R^+$ not activated by derivatives of entries $(u_2, ..., u_p)$ is greater than or equal to $\delta_1$. Consider now the permuted shifted system $\Sigma_m^S$ associated with $\Sigma_m$. By (11) and (4.1), we have:

$$A_m^S = \begin{bmatrix} A_m & A_m^S \end{bmatrix}$$

where $J_\delta$, is the upper $\delta_1$, Jordan bloc, and matrix $A_m^S = \begin{bmatrix} b_{r1(n+1)} \end{bmatrix}$ for $n \times \delta_1 - 1$.

$$B_m^S = \begin{bmatrix} B_{r2(n+1)} \end{bmatrix}$$

with $b_0 = [0 \ldots 0]$.

Let us compare the controllability indices $\{\alpha_i\}_{i=1}^{m-1}$ of $(A_m, [B_1[B_1]])$ and these $\{\alpha_i^S\}_{i=1}^{m-1}$ of $\Sigma_m^S$. By the Brunovsky’s procedure, Brunovski [1970], we have $c_i^S > n_{1e}$.

- if $\alpha_i \leq n_{1e}$, then $c_i^S = c_i^S + 1$;
- if $\alpha_i > n_{1e}$, then $c_i^S + 1 < \alpha_i$ but $c_i^S + 1 > n_{1e}$.

Then as $\delta_1 \leq n_{1e}$, the two Theorems are equivalent for the existence of a solution for SRMP.

5.2 Zagalak et al. [1998]

The authors consider systems $\Sigma$ with special conditions on dimensions as $m = 2p$ and $\sum_{i=1}^{p} \delta_i = \sum_{i=1}^{p} \sigma_i$, and mainly without couplings between $R^+$ and the blocks of infinite structure, ie $\Phi_{m2}^{(2)} = \Phi_{m2}^{(1)} = \Phi_{m2}^{(2)} = 0$ in the extended interactor of $\Sigma_m$. The problem was however difficult although these assumptions seem very simplistic. Certainly, the authors propose factorizations which must have a link with the minimal list of decoupling indices but, unless I have not properly understood their approach, this structural information does never appear explicitly.

6. CONCLUSION

In this paper, we propose a necessary and sufficient conditions for the row by row decoupling problem without modifying the essential orders. Even in this special form, the problem was recognized as structurally difficult. It remains to solve the general problem which is much more difficult because we do not know yet how to define minimal structures for the decoupled system.

REFERENCES


