A Gradient Method for the Static Output Feedback Mixed $H_2/H_\infty$ Control

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Abstract: This paper is concerned with the mixed $H_2/H_\infty$ control problem via static output feedback control. The main purpose of this paper is to give an iterative method for finding a sub-optimal static output-feedback controller for the mixed $H_2/H_\infty$ control problem. The contribution of this paper is to derive a gradient of the $H_2$ cost function. Using this gradient, we propose a gradient method for $H_2$ and mixed $H_2/H_\infty$ control problems. Numerical examples show the efficiency of our methods.

Keywords: Optimization based controller synthesis, Robust controller synthesis

1. INTRODUCTION

One of major requirements for designing control systems is to achieve optimal performance and robust stabilization against uncertainty simultaneously. Since $H_2$ and $H_\infty$ norms are measures for these requirements such control systems can be designed through the so-called mixed $H_2/ H_\infty$ control problem which is an important example of multi-objective control problems. On the other hand some practical limitations, e.g., we can only measure part of state variables, make us use an output-feedback controller. Hence, the output-feedback mixed $H_2/H_\infty$ control problem is very important control problem from a point of view of practical applications. However, it is difficult to obtain the globally optimal solution, because this control problem is described as bilinear matrix inequality (BMI) problem.

Recently, many sub-optimization methods for multi-objective control problems have been proposed Chilali et al. [1996]–Shimomura [2005]. One well-known technique is to fix some variables so as to reduce BMI problems to LMI problems. Another well-known technique is to use common Lyapunov variables at the expense of conservatism Chilali et al. [1996]–Scherer et al. [1997]. Moreover, some techniques using non-common Lyapunov variables are proposed Ebihara and Hagiwara [2004]–Shimomura [2005]. However there is no efficient method for obtaining the globally optimal solution of multi-objective control problems and there are few methods which guarantee the properties of obtained controller.

In this paper, we tackle the mixed $H_2/H_\infty$ controller design with static output feedback. The purpose of this paper is to give a sub-optimization method for this control problem. The main contribution of this paper is to derive a gradient of the $H_2$ cost function. Using the gradient, we propose an iterative method for the mixed $H_2/H_\infty$ control problem, which guarantees that the obtained controller is a locally optimal controller or on the boundary of the $H_\infty$ norm constraint. Numerical examples show the efficiency of our methods.

The following notations are used in this paper: $A(p,q)$-th element of a matrix $M$ is shown as $M_{pq}$, $\text{He}\{M\}$ and $\{A \ B \ C \ \ B^T \ C \}$ denote $M + M^T$ and the symmetric matrix $\{A \ B \}$, respectively.

2. PROBLEM FORMULATION

In this paper, consider the following LTI system:

$$\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + B_1 w_1(t) + B_2 w_2(t), \\
z_1(t) &= C_1 x(t) + D_1 u(t), \\
z_2(t) &= C_2 x(t) + D_2 u(t), \\
y(t) &= C x(t),
\end{align*}$$

where $x$ is the plant state, $w_i(i=1,2)$ are any exogenous inputs, $u$ is the control input, $z_i(i=1,2)$ are the performance outputs, and $y$ is the measured output. Throughout this paper, the following assumptions are made:

1. $(A,B,C)$ is stabilizable and detectable.
2. $D_1^T D_2 = I$.

Let us consider the static output-feedback controller:

$$u(t) = Ky(t).$$

Via the output feedback control low the closed-loop system is described as

$$\begin{align*}
\dot{x}_c(t) &= A_c x_c(t) + B_1 w_1(t) + B_2 w_2(t), \\
z_1(t) &= C_{cl1} x_c(t), \\
z_2(t) &= C_{cl2} x_c(t),
\end{align*}$$

where

$$A_c := A + BKC, \ C_{cl_i} := C_i + D_i KC, \ (i = 1, 2).$$

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For this system we define the mixed $H_2/H_\infty$ control problem as follows.

The mixed $H_2/H_\infty$ control problem 1: Given an achievable $H_\infty$ norm bound $\gamma$, find a controller that satisfies

$$\min_K \|T_{z_2w_2}(K)\|_2 \text{ s.t. } \|T_{z_1w_1}(K)\|_\infty < \gamma, \tag{5}$$

where $\| \cdot \|_2$ and $\| \cdot \|_\infty$ denote the $H_2$ and $H_\infty$ norms, respectively, and $T_{z_iw_i}$ ($i = 1, 2$) denote the closed-loop transfer functions from $w_i$ to $z_i$.

3. PRELIMINARIES

For $\|T_{z_2w_2}(K)\|_2$ and $\|T_{z_1w_1}(K)\|_\infty$ the following lemmas hold Boyd et al. [1994].

Lemma 1. ($H_2$ norm optimization) For $\|T_{z_2w_2}(K)\|_2$, the following statements hold:

1. $K$ stabilizes the closed-loop system (3) and minimizes $\|T_{z_2w_2}(K)\|_2$.
2. $K = K^*_2$, where $K^*_2$ is the solution of the problem

$$\inf \text{ trace } Q \text{ s.t. } \begin{bmatrix} -Q & * \\ V B_2 & -V \end{bmatrix} < 0 \tag{6}$$

$$\text{He}(VA_d) + C_{d2}^T C_{d2} < 0, \tag{7}$$

$$Q > 0, V > 0 \tag{8}$$

where

$$A_d := A + B K C, \ C_{d2} := C_2 + D_2 K C.$$ \tag{9}

(3) $K = K^*_2$, where $K^*_2$ is the solution of

$$\inf J(K) := \|T_{z_2w_2}(K)\|_2^2 = \text{trace}(B_2^T G B_2), \tag{10}$$

s.t. $\text{He}(GA_d) + C_{d2}^T C_{d2} = 0. \tag{11}$

Lemma 2. ($H_\infty$ norm constraint) For $\|T_{z_1w_1}(K)\|_\infty$ the following statements hold:

1. $K$ stabilizes the closed-loop system (3) and achieves $\|T_{z_1w_1}(K)\|_\infty < \gamma.$
2. There exists $X$ which satisfies

$$\begin{bmatrix} \text{He}(A_d X) + B_1 B_1^T & * \\ C_{d1} X & -\gamma^2 I \end{bmatrix} < 0, \ X > 0. \tag{12}$$

Using Lemmas 1-(2) and 2-(2), the mixed $H_2/H_\infty$ control problem 1 can be often described as follows:

Mixed $H_2/H_\infty$ control problem 2 : Given an achievable $H_\infty$ norm bound $\gamma$, find a controller that satisfies

$$\inf \text{ trace } Q \text{ s.t. } (6), (7), (8), \text{ and } (12). \tag{13}$$

Since there are bilinear terms in (7) and (12), the mixed $H_2/H_\infty$ control problem 2 is a bilinear matrix inequality (BMI) problem. In general, it is difficult to obtain the globally optimal solution of BMI problem. Hence, many researchers have proposed interesting sub-optimization methods for such BMI problems Chilali et al. [1996]– Shimomura [2005]. Classically, the next iterative method which uses the property that BMI’s become LMI’s with some variables fixed is used for obtaining a sub-optimal solution:

4. GRADIENT METHOD FOR THE $H_2$ CONTROL PROBLEM

In this section, we derive the gradient of the $H_2$ cost function $J(K)$ with respect to the controller variable $K$. Using the gradient, we propose a gradient method for the $H_2$ control problem.

4.1 Gradient of the $H_2$ cost function

The next theorem gives a gradient of $H_2$ cost function $J(K)$:

Theorem 3. Let $K$ be a stabilizing controller. Then the partial differentiation of $J(K)$ with respect to $K_{pq}$ is given as follows:

$$\frac{\partial J(K)}{\partial K_{pq}} = 2 \text{trace}(MY), \tag{14}$$

where

$$M := (GB + C_2^T D_2 + C_1^T K) E_{pq} C,$$ \tag{15}

and $E_{pq} := \frac{\partial K}{\partial K_{pq}}$, i.e., $E_{pq}$ is the matrix such that $(p, q)$-th element is equal to 1 and the others are equal to 0, and $G$ and $Y$ are solutions of the Lyapunov equations (11) and

$$\text{He}(A_d Y) + B_2 B_2^T = 0, \tag{16}$$

respectively.

Proof: From (10), $\frac{\partial J(K)}{\partial K_{pq}}$ is given as follows:

$$\frac{\partial J(K)}{\partial K_{pq}} = \text{trace} \left( B_2^T \frac{\partial G}{\partial K_{pq}} B_2 \right). \tag{17}$$

For obtaining $\frac{\partial G}{\partial K_{pq}}$ differentiating (11) with respect to $(p,q)$-th element of $K$ to get the following Lyapunov equation:
He \left\{ \frac{\partial G}{\partial K_{pq}} A_{cl} + M \right\} = 0. \tag{18}

Since \( K \) is the stabilizing controller, \( \frac{\partial G}{\partial K_{pq}} \) is given by

\[ \frac{\partial G}{\partial K_{pq}} = \int_{0}^{\infty} e^{A_{cl}t} \text{He}[M] e^{A_{cl}t} dt. \tag{19} \]

Substituting (19) in (17) to get

\[ \frac{\partial J(K)}{\partial K_{pq}} = \text{trace} \left( B_{1}^{T} \int_{0}^{\infty} e^{A_{cl}t} \text{He}[M] e^{A_{cl}t} dt B_{2} \right) \]

\[ = \text{trace} \left( \text{He}[M] \int_{0}^{\infty} e^{A_{cl}t} B_{2}B_{2}^{T} e^{A_{cl}t} dt \right) \]

\[ = 2 \text{trace}(MY), \tag{20} \]

where

\[ Y := \int_{0}^{\infty} e^{A_{cl}t} B_{2}B_{2}^{T} e^{A_{cl}t} dt, \tag{21} \]

and since \( A_{cl} \) is stable \( Y \) is the unique solution of (16).

4.2 A Gradient method for the \( H_2 \) control problem

Using (14), the \((p,q)\)-th element of a descent direction \( \Delta K \) is defined as

\[ \Delta K_{pq} := -2 \text{trace}(MY). \tag{22} \]

Then, a gradient method for the \( H_2 \) control problem is proposed as follows:

Algorithm 1: Gradient Method for the \( H_2 \) control problem

Step 1 Find \( K_0 \) which stabilizes the closed-loop system (3) and let \( i := 0 \). For example, an exterior-point approach Kami and Nobuyama [2004] can be used for finding \( K_0 \).

Step 2 Get \( G_i \) and \( Y_i \) which are the solutions of

\[ \text{He}[G_i A_{cl}] + C_{cl}^{T} C_{cl} = 0, \tag{23} \]

and

\[ \text{He}[A_{cl} Y_i] + B_{2}B_{2}^{T} = 0, \tag{24} \]

respectively, where

\[ A_{cl} = A + BK_{i}C, C_{cl} = C_{2} + D_{2}K_{i}C. \tag{25} \]

Step 3 Calculate the partial derivative of \( J(K) \) with respect to \( K_{pq} \) via (14) and define the descent direction \( \Delta K \via (22). If \( \Delta K \) is a zero matrix then let \( K^* := K_i \) and exit. Otherwise go to the next step.

Step 4 Let \( K_{i+1} := K_i + d\Delta K \), where \( d > 0 \) is a step size which is the solution of

\[ \min_{\delta} \| T_{z_{2}w_{2}}(K_i + d\Delta K) \|_2 \text{ s.t.} \]

\[ \| T_{z_{1}w_{1}}(K_i + d\Delta K) \|_{\infty} < \gamma. \tag{28} \]

Step 5 For sufficiently small \( \varepsilon_1 \) and \( \varepsilon_2 \), if \( |d_i| < \varepsilon_1 \) and \( \| T_{z_{1}w_{1}}(K_{i+1}) \|_{\infty} > \gamma - \varepsilon_2 \), then let \( K^* := K_{i+1} \) and exit. Otherwise let \( i := i + 1 \) and go to Step 2.

Lemma 4. Algorithm 1 has the next property:

(1) \( J(K_i) \geq J(K_{i+1}) \) holds i.e., \( \| T_{z_{2}w_{2}}(K_i) \|_2 \) is monotonically decreasing.

(2) \( K^* \) is a locally optimal solution of the \( H_2 \) control problem.

Proof: Obvious from the construction of Algorithm 1.

Remark 1 It is difficult to get the globally optimal solution of the problem (26), because the search area for \( d_i \) is not bounded. Therefore, when Algorithm 1 is implemented we limit the search area of \( d_i \) to \( 0 \leq d_i \leq d \), where \( d > 0 \) is a prescribed upper bound of \( d_i \), i.e., the next problem is solved by grid search instead of (26):

\[ \min_{0 \leq d_i \leq d} \| T_{z_{2}w_{2}}(K_i + d\Delta K) \|_2. \tag{27} \]

Remark 2 In the case that \( C = I \), i.e., \( K \) is static state feedback controller, \( K^* \) is the globally optimal solution of the \( H_2 \) control problem, because a stationary point of \( J(K) \) is unique.

5. A GRADIENT METHOD FOR THE MIXED \( H_2/H_\infty \) CONTROL PROBLEM

In this section, we extend Algorithm 1 to an iterative method for the mixed \( H_2/H_\infty \) control problem. The key idea of the extension is to choose \( K_{i+1} \) on the descent direction so as to achieve the \( H_\infty \) norm constraint.

An iterative method for the mixed \( H_2/H_\infty \) control problem is proposed as follows:

Algorithm 2: Gradient Method for the mixed \( H_2/H_\infty \) control problem

Step 1 Find \( K_0 \) which achieves \( \| T_{z_{1}w_{1}}(K) \|_{\infty} < \gamma \) and let \( i := 0 \). For example, an exterior-point approach Kami and Nobuyama [2004] can be used for finding \( K_0 \).

Step 2 Get \( G_i \) and \( Y_i \) which are the solutions of (23) and (24), respectively.

Step 3 Calculate the partial derivative of \( J(K) \) with respect to \( K_{pq} \) by (14) and define the descent direction \( \Delta K \via (22). If \( \Delta K \) is a zero matrix then let \( K^* := K_i \) and exit. Otherwise go to the next step.

Step 4 Let \( K_{i+1} := K_i + d\Delta K \), where \( d > 0 \) is a step size which is the solution of

\[ \min_{\delta} \| T_{z_{2}w_{2}}(K_i + d\Delta K) \|_2 \text{ s.t.} \]

\[ \| T_{z_{1}w_{1}}(K_i + d\Delta K) \|_{\infty} < \gamma. \tag{28} \]

Step 5 For sufficiently small \( \varepsilon_1 \) and \( \varepsilon_2 \), if \( |d_i| < \varepsilon_1 \) and \( \| T_{z_{1}w_{1}}(K_{i+1}) \|_{\infty} > \gamma - \varepsilon_2 \), then let \( K^* := K_{i+1} \) and exit. Otherwise let \( i := i + 1 \) and go to Step 2.

Lemma 5. Algorithm 2 has the next property:

(1) \( J(K_i) \geq J(K_{i+1}) \) holds i.e., \( \| T_{z_{2}w_{2}}(K_i) \|_2 \) is monotonically decreasing.

(2) \( K_i (i = 0, 1, 2, \cdots) \) achieve the \( H_\infty \) norm constraint, i.e., \( \| T_{z_{1}w_{1}}(K_i) \|_{\infty} < \gamma \).

(3) If Algorithm 2 stops at Step 2, then \( K^* \) is a locally optimal solution of the mixed \( H_2/H_\infty \) control problem.

(4) If Algorithm 2 stops at Step 5, then \( K^* \) is on the boundary of the \( H_\infty \) norm constraint of the mixed \( H_2/H_\infty \) control problem.
Proof: Obvious from the construction of Algorithm 2.

**Remark 3** From the same reason as described in Remark 1, when Algorithm 2 is implemented we limit the search area of \( d_i \) to \( 0 \leq d_i \leq \bar{d} \), i.e., the next problem is solved by grid search instead of (28):

\[
\begin{align*}
\min_{0 \leq d \leq \bar{d}} & \quad \|T_{z_2w_2}(K_i + d\Delta K)\|_2 \\
\text{s.t.} & \quad \|T_{z_1w_1}(K_i + d\Delta K)\|_\infty < \gamma.
\end{align*}
\]

(29)

**Remark 4** In the case that \( C = I \), i.e., \( K \) is static state-feedback controller, \( K^* \) satisfies a necessary condition for \( K \) to be the globally optimal solution of the mixed \( H_2/H_\infty \) control problem Kami and Nobuyama [2003].

6. NUMERICAL EXAMPLES

To demonstrate the efficacy of Algorithms 1 and 2, we consider two examples: one is output feedback case and the other is state feedback case. For both examples we consider the unconstrained \( H_2 \) control problem and the mixed \( H_2/H_\infty \) control problem.

6.1 Example 1: output feedback case

Let’s consider the following state space matrices:

\[
A = \begin{bmatrix} 0 & 1 \\ -0.5 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

\[
B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C = [1 \ 0], \quad \gamma = 1.
\]

For this example, the globally optimal \( H_2 \) value of the unconstrained \( H_2 \) control problem is 1.7579, and we set \( \bar{d} = 0.5 \).

Figures 1 shows behaviours of \( \|T_{z_2w_2}(K_i)\|_2 \) as a function of iteration number \( i \) on Classical iterative method and Algorithm 1. This figure shows that \( \|T_{z_2w_2}(K_i)\|_2 \) is monotonically decreasing as \( i \) increases and converges to the globally optimal \( H_2 \) value while Classical iterative method cannot improve \( \|T_{z_2w_2}(K_i)\|_2 \).

Figures 2 and Figure 3 show behaviours of \( \|T_{z_2w_2}(K_i)\|_2 \) and \( \|T_{z_1w_1}(K_i)\|_\infty \) as a function of iteration number \( i \) on Algorithm 2, respectively. Figure 2 also show a behaviour of \( \|T_{z_2w_2}(K_i)\|_2 \). Figures 2 shows that \( \|T_{z_2w_2}(K_i)\|_2 \) is monotonically decreasing as \( i \) increases while Classical iterative method cannot improve \( \|T_{z_2w_2}(K_i)\|_2 \). Figure 3 shows that \( \|T_{z_1w_1}(K_i)\|_\infty \) reaches the \( H_\infty \) norm bound as \( i \) increases, which implies that the obtained controller is on the boundary of the \( H_\infty \) norm constraint.

6.2 Example 2: state feedback case

Let’s consider the system shown by figure 4, where \( m_1 = m_2 = 1 \) and the spring constant \( k \) is an uncertain parameter which satisfies \( 1 \leq k \leq 1.5 \). Moreover, a coefficient matrix of a control input includes 10% uncertainty. Then the state-space matrices of this system and the uncertainty structure are given as follows:

\[
A = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

\[
B_1 = \begin{bmatrix} 0 & 0 \\ -1.25 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

\[
C_1 = \begin{bmatrix} 1 & 0 \\ -0.25 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 \end{bmatrix},
\]

\[
D_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad C = \text{diag}(1,1,1,1).
\]

Then the condition for robust stability against \( \Delta(t) \) is given as \( \|T_{z_1w_1}(K_i)\|_\infty < 1 \). For this example, the globally optimal \( H_2 \) value of the unconstrained \( H_2 \) control problem is 1.4036, and we set \( \bar{d} = 0.5 \).

Figures 5 shows behaviours of \( \|T_{z_2w_2}(K_i)\|_2 \) as a function of iteration number \( i \) on Classical iterative method and Algorithm 1. Figures 2 and Figure 3 show behaviours of \( \|T_{z_2w_2}(K_i)\|_2 \) and \( \|T_{z_1w_1}(K_i)\|_\infty \) as a function of iteration.
In this paper, we consider the $H_2$ and mixed $H_2/H_\infty$ control problems with static output feedback. Firstly, we derived a partial differentiation of the $H_2$ cost function. Secondly, we proposed a gradient method for the $H_2$ control problem. This method guarantees that the obtained controller is a locally optimal solution of the $H_2$ control problem. Next, we modified the method to the mixed $H_2/H_\infty$ control. This method guarantees that the obtained controller is a locally optimal solution or on the boundary of the $H_\infty$ norm constraint of the mixed $H_2/H_\infty$ control problem. Finally, we gave numerical examples which showed the efficiency of the proposed methods.

Table 1. Comparison with the obtained $H_2$ norms

<table>
<thead>
<tr>
<th>Method</th>
<th>$|T_{z,w}(\Sigma)|_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Common Lyapunov Variables (Initial controller)</td>
<td>2.4153</td>
</tr>
<tr>
<td>Classical method</td>
<td>2.4153</td>
</tr>
<tr>
<td>Shimomura [2005]</td>
<td>2.0703</td>
</tr>
<tr>
<td>Kami and Nobuyama [2003]</td>
<td>1.4498</td>
</tr>
<tr>
<td>Proposed method</td>
<td>1.4939</td>
</tr>
</tbody>
</table>

number $i$ on Algorithm 2, respectively. From these figures the same result as described in Example 1 is obtained, i.e.,

- $\|T_{z,w_2}(K_i)\|_2$ is monotonically decreasing as $i$ increases in Algorithms 1 and 2 while Classical iterative method cannot improve $\|T_{z,w_2}(K_i)\|_2$.
- Algorithm 2 gives a controller on the boundary of the $H_\infty$ norm constraint.

Table 1 shows $H_2$ norms of the closed-loop system via the controllers obtained by Common Lyapunov Variables, Classical iterative method, Shimomura [2005], Shimomura and Fujii [2005], Kami and Nobuyama [2003], and Algorithm 2. Table 1 shows that the controller obtained by Algorithm 2 achieves lower $H_2$ norm than Common Lyapunov Variables, Classical iterative method, and Shimomura [2005] and achieves the almost same $H_2$ norm as those obtained by Shimomura and Fujii [2005] and Kami and Nobuyama [2003].

7. CONCLUSION

In this paper, we consider the $H_2$ and mixed $H_2/H_\infty$ control problems with static output feedback. Firstly, we derived a partial differentiation of the $H_2$ cost function. Secondly, we proposed a gradient method for the $H_2$ control problem. This method guarantees that the obtained controller is a locally optimal solution of the $H_2$ control problem. Next, we modified the method to the mixed $H_2/H_\infty$ control. This method guarantees that the obtained controller is a locally optimal solution or on the boundary of the $H_\infty$ norm constraint of the mixed $H_2/H_\infty$ control problem. Finally, we gave numerical examples which showed the efficiency of the proposed methods.

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