Robust Control of Resistive Wall Modes in Tokamak Plasmas using \( \mu \)-synthesis

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Abstract:
In this work, \( \mu \)-synthesis is employed to stabilize a model of the resistive wall mode (RWM) instability in the DIII-D tokamak. The GA/Far-Tech DIII-D RWM model is used to derive a linear state space representation of the mode dynamics. The key term in the model characterizing the magnitude of the instability is the time-varying uncertain parameter \( c_{pp} \), which is related to the RWM growth rate \( \gamma \). Taking advantage of the structure of the state matrices, the model is reformulated into a robust control framework, with the growth rate of the RWM modeled as an uncertain parameter. A robust controller that stabilizes the system for a range of practical growth rates is proposed and tested through simulations.

1. INTRODUCTION

Nuclear fusion produces energy through fusing together the nuclei of two light hydrogen atom isotopes (e.g., deuterium and tritium). Such a process requires extreme temperatures to occur, since the nuclei need to overcome the Coulomb barrier (both nuclei carry positive charges) in order to fuse. The confinement of this high-temperature, ionized, hydrogen gas called plasma can be provided by a magnetic confinement device (e.g., a tokamak, which is in the shape of a torus).

One of the major non-axisymmetric instabilities in tokamaks is the resistive wall mode (RWM), a form of plasma kink instability whose growth rate is moderated by the influence of a resistive wall (Walker [2006]). This instability is present in sufficiently high pressure plasmas which causes the plasma to kink similar to that of a garden hose. In a kink mode, the entire plasma configuration deforms in a helically symmetric manner with an extremely fast growth time (a few microseconds). The mode spatial invariance, the mode spatial invariance, the state space model is parameterized with a scalar coupling coefficient \( \gamma \), which is directly related to the growth rate \( \gamma \) of the mode.

Although the plasma surface deformation cannot be directly measured in real time, the magnitude and phase of the deformation can be diagnosed from measurements by a set of 22 magnetic field sensors composed of poloidal magnetic field probes and saddle loops, which measure radial flux. A set of 12 internal feedback control coils (I-coils) can then be used to return the plasma to its original axisymmetric shape. Fig. 1 shows the arrangement of coils and sensors. Using an estimator, the 22 outputs are reduced to 2 outputs that represent the RWM.

The GA/Far-Tech DIII-D RWM model replaces the spatial perturbation of the plasma with an equivalent perturbation of surface current on a spatially fixed plasma boundary and represents the resistive wall using an eigenmode approach (Fransson [2003], In [2006]). The spatial and current perturbations are equivalent in the sense that they both produce the same magnetic field perturbation. Observations from experiments show that the mode spatial structure remains unchanged. Based on the surface current representation of the mode, a state-space model of the plant can be derived from Faraday’s Law, with states consisting only of the surrounding wall current and the external control coil currents. Since the plasma is represented as a single mode, and due to the mode spatial invariance, the state space model is parameterized with a scalar coupling coefficient \( c_{pp} \), which is directly related to the growth rate of the mode.

In this work, \( \mu \)-synthesis is employed to stabilize a model of the resistive wall mode (RWM) instability in the DIII-D tokamak. The GA/Far-Tech DIII-D RWM model is used to derive a linear state space representation of the mode dynamics. The key term in the model characterizing the magnitude of the instability is the time-varying uncertain parameter \( c_{pp} \), which is related to the RWM growth rate \( \gamma \). Taking advantage of the structure of the state matrices, the model is reformulated into a robust control framework, with the growth rate of the RWM modeled as an uncertain parameter. A robust controller that stabilizes the system for a range of practical growth rates is proposed and tested through simulations.

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2. PLASMA MODEL & PARAMETERIZATION

2.1 System Model

Stated below is the GA/Far-Tech DIII-D RWM model, a plasma response model for the resistive wall mode using a toroidal current sheet to represent the plasma surface (Edgell [2002]). Most of the matrices and variables presented are characteristics of the tokamak and are well known. The uncertainty is introduced through the variable \( c_{pp} \), which corresponds to a certain growth rate \( \gamma \) of the resistive wall mode. The relationship between these variables is shown empirically in Fig. 2 for a particular plasma equilibrium and is further explained in (In [2006]).

The model is represented in terms of the couplings between the plasma \((p)\), vessel wall \((w)\), and coils \((c)\). The model derived from Faraday’s law of induction results in the system dynamics that reduce to:

\[
\begin{align*}
(M_{ss} - M_{sp} c_{pp} M_{ps}) \dot{I}_s + R_s I_s &= V_s \\
(M_{ps} c_{sp}) \dot{V}_p &= M_{ws} w_c \end{align*}
\]

where \( M_{ss} \) is the mutual inductance between external conductors, including the vessel wall and the coils, \( M_{sp} \) is the mutual inductance between either the external conductors and the plasma, \( R_s \) is the resistance matrix, \( I_s \) is the current flowing in the conductors, and \( V_p \) is the externally applied voltage to the conductors. The mutual inductances are given by

\[
M_{ss} = \begin{bmatrix} M_{sw} & M_{wc} \\ M_{cw} & M_{cc} \end{bmatrix}, \quad M_{sp} = \begin{bmatrix} M_{wp} \\ M_{pc} \end{bmatrix}, \quad M_{ps} = \begin{bmatrix} M_{pw} & M_{pc} \end{bmatrix},
\]

where \( M_{ps} \) and \( M_{sp} \) satisfy the following condition:

\[
M_{ps} = \begin{bmatrix} M_{wp} & M_{cp} \end{bmatrix}, \quad M_{ps}^T = \begin{bmatrix} M_{pw} & M_{pc} \end{bmatrix}, \quad M_{ps}^T = \begin{bmatrix} M_{pw} & M_{pc} \end{bmatrix},
\]

The resistance matrix is given by

\[
R_s = \begin{bmatrix} \lambda_w & 0 \\ 0 & R_c \end{bmatrix},
\]

where \( \lambda_w \) characterizes the couplings of a wall surface eigenmode to other states by the time-varying perpendicular magnetic fields contributed by those states and \( R_c \) is the coil resistance. The current and externally applied voltage to the conductors can be written as

\[
I_s = \begin{bmatrix} I_w \\ I_c \end{bmatrix}, \quad V_p = \begin{bmatrix} 0 \\ V_c \end{bmatrix},
\]

where \( I_w \) is the wall current, \( I_c \) is the coil current, and \( V_c \) is the externally applied voltage to the coil.

This model can be represented in a state space formulation using the current in the conductors as the states \((x = I_s)\) and the applied voltage as the inputs \((u = V_p)\). This results in the following state space equation

\[
\dot{x} = Ax + Bu
\]

where

\[
A = -L_{ss}^{-1} R_s, \quad B = L_{ss}^{-1},
\]

with \( L_{ss} = M_{ss} - M_{sp} c_{pp} M_{ps} \). The output equation of the state space representation is based on sensor measurements that relate to the conductor currents through the dynamics

\[
y = (C_{sp} - C_{pp} c_{pp} M_{ps}) I_s
\]

where \( C_{sp} \) is the coupling matrix between the sensor and plasma current and

\[
C_{ss} = [C_{sw} \ C_{wc}]
\]

given by the coupling matrix between the sensor and wall current \( C_{sw} \) and the coupling matrix between the sensor and coil current \( C_{wc} \). This results in the state space output equation

\[
y = C x
\]

where \( C = C_{ss} - C_{sp} c_{pp} M_{ps} \).
2.2 Parameterization of the $L_{ss}^{-1}$ Matrix

The goal of this section is to extract the uncertain parameter $c_{pp}$ from the uncertain state space system and introduce it as an uncertainty block that perturbs a nominal state space system. The initial step to obtaining the nominal state space system is to express each state matrix as a general affine state space representation using nonlinear functions of the uncertainty $c_{pp}$. As seen in (1), the majority of the complexity is introduced in the $A$ and $B$ state matrices, where the uncertainty $c_{pp}$ is introduced through $L_{ss}^{-1}$, and where $L_{ss} = (M_{ss} - M_{ps}c_{pp}M_{ps})$. Since the instability is two-dimensional, the matrix product $M_{sp}M_{ps}$ is rank 2 and the $2 \times 2$ diagonal $c_{pp}$ matrix is treated as a scalar $c_{pp}$. Thus the $L_{ss}$ matrix can be expressed as

$$L_{ss} = M_{ss} - M_{ps}c_{pp}M_{ps} = M_{ss} - c_{pp}\sum_{i=1}^{2} u_iu_j$$

where $n$ is the number of states in the RWM state space model. To obtain a parameterized form for the $L_{ss}^{-1}$ term, we must first compute the inverse of a matrix sum. Given the matrix $A_{T}$, the scalar $b_{T}$, and the vectors $C_{T}$ and $D_{T}$, the inverse of a matrix sum is given by the Sherman-Morrison formula as

$$(A_{T} - b_{T}C_{T}D_{T})^{-1} = A_{T}^{-1} + \frac{b_{T}(A_{T}^{-1}C_{T})(D_{T}A_{T}^{-1})}{1 - b_{T}D_{T}A_{T}^{-1}C_{T}}.$$  (3)

Using (2), the inverse of $L_{ss}$ can be written as

$$L_{ss}^{-1} = (M_{ss} - M_{ps}c_{pp}M_{ps})^{-1} = (M_{ss} - c_{pp}u_1u_1' - c_{pp}u_2u_2')^{-1}.$$  (4)

Now, using the matrix $A_{T} = M_{ss} - c_{pp}u_1u_1'$, the above equation can be written as $L_{ss}^{-1} = (A_{T} - c_{pp}u_2u_2')^{-1}$. This is now in the form given by (3) and thus the formula can be applied, resulting in

$$L_{ss}^{-1} = A_{T}^{-1} + c_{pp}(A_{T}^{-1}u_2')(u_2A_{T}^{-1})^{-1} 1 - c_{pp}u_2'A_{T}^{-1}u_2.$$  (4)

Now the matrix $L_{ss}^{-1}$ is expressed in terms of $A_{T}^{-1}$, which is equivalent to $(M_{ss} - c_{pp}u_1u_1')^{-1}$, and once again applying (3) results in

$$A_{T}^{-1} = (M_{ss} - c_{pp}u_1u_1')^{-1} = M_{ss}^{-1} + c_{pp}u_1'(M_{ss}^{-1}u_1)(u_1M_{ss}^{-1})^{-1} = c_{pp}u_1'M_{ss}^{-1}u_1.$$  (5)

This expression can now be substituted back into (4). The terms can be collected and rewritten in the form

$$B = L_{ss}^{-1} = \sum_{i=0}^{n} a_iB_i,$$

where $a_i$‘s are nonlinear functions of $c_{pp}$, and $B_i$‘s are constant matrices. The individual terms are given by

$$a_0 = 1, \quad a_1 = \frac{c_{pp}}{1 - c_{pp}u_1'M_{ss}^{-1}u_1}$$

$$a_2 = \frac{c_{pp}u_2'M_{ss}^{-1}u_2 - c_{pp}u_1'M_{ss}^{-1}u_1u_2'}{1 - c_{pp}u_1'M_{ss}^{-1}u_1u_2'}$$

$$a_3 = a_2a_1, \quad a_4 = a_2a_1', \quad B_0 = M_{ss}^{-1},$$

$$B_1 = [(M_{ss}^{-1}u_1)(u_1'M_{ss}^{-1})], \quad B_2 = [(M_{ss}^{-1}u_2)(u_2'M_{ss}^{-1})]$$

$$B_3 = [(M_{ss}^{-1}u_2')(u_2'M_{ss}^{-1}u_1)(u_1'M_{ss}^{-1})]$$

$$+ [(M_{ss}^{-1}u_1)(u_1'M_{ss}^{-1})u_2')(u_2'M_{ss}^{-1})]$$

$$B_4 = [(M_{ss}^{-1}u_1)(u_1'M_{ss}^{-1})u_2u_2'(M_{ss}^{-1}u_1)(u_1'M_{ss}^{-1})].$$

2.3 Expressing the Parameterized State Space Matrices

The last section allowed us to express the $L_{ss}^{-1}$ matrix in a parameterized form, which allows the parameterization of the state and input matrices $A$ and $B$ respectively. In a similar way, the output matrix $C$ can also be parameterized. Using the fact that $c_{pp}$ is a scalar, the $C$ matrix can be written as

$$C = C_{ss} - c_{pp}e_{pp}M_{ps} = C_{ss} - c_{pp}e_{pp}M_{ps} = C_0 + a_5C_5,$$

where

$$C_0 = C_{ss}, \quad C_5 = -C_{pp}M_{ps}, \quad a_5 = c_{pp}.$$  (6)

Defining $A_T = -B_RM_{ss}$, we can finally summarize the parameterized expressions for the state matrices $A$, $B$, and $C$ in terms of $a_i$‘s, given as

$$A = A_0 + a_0A_1 + a_0A_2 + a_3A_3 + a_4A_4,$$

$$B = B_0 + a_0B_1 + a_0B_2 + a_3B_3 + a_4B_4,$$

$$C = C_0 + a_5C_5.$$  (7)

3. GROWTH RATE PARAMETERIZATION

3.1 Linear Fractional Transformation (LFT) of RWM

A system with state space representation $A, B, C, D$ has a transfer function $G(s) = D + C(sI_A - A)^{-1}B$, where $n$ is the number of states (or eigenvalues) in the system and $I_A$ is the convention used to describe an $n \times n$ identity matrix. Defining the matrix $M_{ss} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, we can write the transfer function as the linear fractional transformation of $M_{ss}$ as (Packard 1988))

$$G(s) = F_{u}(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \frac{1}{s}I_n) = F_u(M_{ss}, \frac{1}{s}I_n) = M_{ss} + M_{ss}^{-1}I_n(I_n - M_{ss}^{-1}I_n)^{-1}M_{ss}$$

$$= D + C\frac{1}{s}I_n(I_n - A^{-1}I_n)^{-1}B = D + C(sI_n - A)^{-1}B.$$  (7)

The graphical representation of $G(s)$ is shown in Fig. 3, with equivalent equations

$$\begin{align*}
\begin{bmatrix} z_1 \\
y 
\end{bmatrix} &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} w_1 \\
u \n\end{bmatrix} \\
w_1 &= \frac{1}{s}z_1, \quad y = F_u(M_{ss}, \frac{1}{s}I_n)u = G(s)u.
\end{align*}$$

To introduce the uncertainty given by the parameterized state space system (5)-(7), the $M_{ss}$ matrix can be written in the form of a general affine state space uncertainty.
Mα =
\begin{bmatrix} A_0 + \sum_{i=1}^{k} \alpha_i A_i \\ B_0 + \sum_{i=1}^{k} \alpha_i B_i \\ C_0 + \sum_{i=1}^{k} \alpha_i C_i \\ D_0 + \sum_{i=1}^{k} \alpha_i D_i \end{bmatrix}
with \( k = 5, A_5 = 0, B_5 = 0, C_1 = 0 \) for \( i = 1, \ldots, 4 \), and \( D_i = 0 \) for \( i \).

This uncertainty can be formulated into a linear fractional transform by achieving the smallest possible repeated blocks using the method outlined in (Packard [1988]). To begin this method, matrices \( J_i \)'s are formed such that
\[
J_i = \begin{bmatrix} A_i \\ B_i \\ C_i \\ D_i \end{bmatrix} \in \mathbb{R}^{(n+n_x) \times (n+n_u)}
\]
for each \( i = 1, \ldots, 5 \). Then, using singular value decomposition and grouping terms, an expression for \( J_i \) can be achieved (note: \( A^* \) is denoted as the complex conjugate transpose of \( A \))
\[
J_i = U_i \Sigma_i V_i^* = \begin{bmatrix} L_i & R_i \\ W_i & Z_i \end{bmatrix}
\]
Denoting \( q_i \) as the rank of each matrix \( J_i \), each inner matrix is given by
\[
L_i \in \mathbb{R}^{(n_x \times q_i)}, W_i \in \mathbb{R}^{(n_u \times q_i)}, R_i \in \mathbb{R}^{(n_\alpha \times q_i)}, Z_i \in \mathbb{R}^{(n_y \times q_i)}.
\]
Then, the uncertainty can be introduced as
\[
\alpha J_i = \begin{bmatrix} L_i & R_i \\ W_i & Z_i \end{bmatrix} \begin{bmatrix} \alpha_i L_{i1} \\ \alpha_i R_{i1} \\ 0 \\ \alpha_i W_{i1} \\ \alpha_i Z_{i1} \end{bmatrix},
\]
for which this particular equilibrium, \( q_1 = 1, q_2 = 1, q_3 = 2, q_4 = 1, q_5 = 2 \). Finally, the linear fractional transformed matrix can be written as
\[
M_\alpha = M_{11} + M_{12} \alpha_p M_{21},
\]
where
\[
M_{11} = \begin{bmatrix} A_0 B_0 \\ C_0 D_0 \end{bmatrix}, M_{12} = \begin{bmatrix} L_1 & \cdots & L_5 \\ W_1 & \cdots & W_5 \end{bmatrix}, M_{21} = \begin{bmatrix} R_1 Z_1 \\ \cdots \\ R_5 Z_5 \end{bmatrix}, \quad \alpha_p = \begin{bmatrix} \alpha_i L_{i1} \\ \alpha_i R_{i1} \\ 0 \\ \alpha_i W_{i1} \\ \alpha_i Z_{i1} \end{bmatrix}.
\]
This is equivalent to the lower linear fractional transformation
\[
M_\alpha = F(I(M_{11}, M_{12}, \alpha_p) = F(\text{diag}(M_{11}, M_{12}, 0), \alpha_p) = F(I(M), \alpha_p)
\]
\[
= M_{11} + M_{12} \alpha_p (I_{q_T} - M_{22} \alpha_p)^{-1} M_{21} = M_{11} + M_{12} \alpha_p M_{21}
\]
where
\[
M = \begin{bmatrix} M_{11} \\ M_{12} \\ M_{21} \end{bmatrix}, \quad q_T = \sum_{i=1}^{5} q_i = 7.
\]
Finally, the transfer function of the uncertain state space model is written as
\[
G(s) = F_u(M_{\alpha}, \frac{1}{s} I_n) = F_u(F(I(M_{11}, M_{12}, 0), \alpha_p), \frac{1}{s} I_n).
\]
The graphical representation of \( G(s) \) is shown in Fig. 4 with the equivalent equations
\[
\begin{bmatrix} z_1 \\ y \\ z_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ u \end{bmatrix},
\]
\[
w_1 = \frac{1}{s} z_1, \quad w_2 = \alpha_p z_2, \quad y = F_u(F(I(M), \alpha_p), \frac{1}{s} I_n) u = G(s) u.
\]

3.2 Normalizing \( \alpha \) Parameters

The system is now in a form where the uncertainty is given by the five \( \alpha_i \) parameters. However, as shown earlier, each of the \( \alpha_i \) parameters are nonlinear functions of the single variable \( \alpha_p \). Thus the next step is to express the linear fractional transformation in terms of the single uncertainty \( \alpha_p \). First, \( \alpha_p \) is normalized using
\[
c_{\alpha_p} = d + \delta e,
\]
where \( \alpha_p \) is the nominal value of \( \alpha_p \), and \( c_{\alpha_p} \) and \( c_{\alpha_{p\text{max}}} \) and \( c_{\alpha_{p\text{min}}} \) are its minimum and maximum values respectively. This defines a new normalized uncertainty \( \delta \) that has a range of values within \( |\delta| \leq 1 \) that corresponds to the desired \( c_{\alpha_p} \) range.

Now that each \( \alpha_i \) parameter is expressed in terms of \( \delta \), we “pull out the \( \delta \)” (Zhou et al. [1996]). This is done by drawing the block diagram for each \( \alpha_i \) system and labeling the input to each \( \delta \) block \( z_3 \), and the output of each \( \delta \) block \( w_3 \). Then the matrix \( Q \) which satisfies \( \underline{Q} = F_1(\Delta, \delta I_n) \) with \( \Delta = \delta I_{m_T} \), can be found using
\[
\begin{bmatrix} w_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ z_3 \end{bmatrix}, \quad \begin{bmatrix} w_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ z_3 \end{bmatrix}, \quad \begin{bmatrix} w_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ z_3 \end{bmatrix}, \quad \begin{bmatrix} w_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ z_3 \end{bmatrix}, \quad \begin{bmatrix} w_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ z_3 \end{bmatrix}, \quad \begin{bmatrix} w_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ z_3 \end{bmatrix},
\]
where \( w_2 \) and \( z_3 \) are vectors of length \( q_i \), based on the rank of each \( J_i \) matrix, and each \( w_3 \) and \( z_3 \) are vectors of length \( m_T \), based on the minimum number of \( \delta \)’s required to represent each \( \alpha_i \) and the value of \( q_i \). The composite \( Q \) matrix will be defined after each individual \( Q_i \) is determined, where \( Q \) is given by
\[
Q = \begin{bmatrix} Q_{11} \\ Q_{12} \\ Q_{21} \\ Q_{22} \end{bmatrix}.
\]
The total number of uncertainty elements \( m_T \) for \( \alpha_p \) is given by the total length of \( w_3 \), which is
\[
m_T = \sum_{i=1}^{5} m_i q_i.
\]
The block representation of \( \alpha_p \) is shown in Fig. 5.
Recalling that $\alpha_1 = \frac{\gamma_{pp}}{1 - \gamma_{pp}\Phi_{i}\Phi_{d}}$, and using $\alpha = u_1^T M_{yi}^{-1} u_1$ and the normalized relationship $\gamma_{pp} = d + \delta e$, we can rewrite $\alpha_1 = (1 - \delta e - \alpha d^{-1} e).$ Since there is only one uncertainty element, $m_1 = 1$. The block diagram for $\alpha_1$, shown in Fig. 6, can be directly drawn from this form, with the feedback terms in the denominator and the feedforward terms in the numerator. Thus, the governing equation for $\alpha_1$ is given by

$$Q_1 = \begin{bmatrix} d & e \alpha \end{bmatrix} \begin{bmatrix} z_{21} \\ w_{31} \end{bmatrix},$$

which results in a $Q_1$ given by

$$Q_1 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}.$$

For the system matrices of the DIII-D tokamak under the particular equilibrium, the behavior of $\alpha_1$ and $\alpha_2$ are approximately the same, with an error on the order of $10^{-12}$. From this very good approximation, we can take $\alpha_1 = \alpha_2$. Although the full model could be used, this is an accurate enough assumption that allows the reduction of computational complexity. As a result of this approximation, the following changes can be made to the other parameters: $\alpha_3 = \alpha_2\alpha_1 \Rightarrow \alpha_3 = \alpha_1^2$, $\alpha_4 = \alpha_2\alpha_1^2 \Rightarrow \alpha_4 = \alpha_1^3$. Since $\alpha_2 = \alpha_1$, $m_2 = m_1 = 1$ and the $Q_2$ block is simply defined by $Q_2 = Q_1$. The parameter $\alpha_3$ is given as $\alpha_3 = \alpha_1^2$, or $\alpha_3 = F_{i}(Q_1, \delta) \cdot F_{i}(Q_1, \delta)$. A reduction can be made so that $\alpha_3 = F_{i}(Q_2, \delta I_2)$, where $I_2$ is the size 2 identity matrix, thus $m_3 = 2$. Through the series connection of the linear fractional transform of $Q_1$, the $Q_2$ block is given by

$$Q_3 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}.$$  

Similarly to $Q_3$, the parameter $\alpha_4$ is given as $\alpha_4 = \alpha_1^3$, or $\alpha_4 = F_{i}(Q_1, \delta) \cdot F_{i}(Q_1, \delta) \cdot F_{i}(Q_1, \delta)$. A reduction can be made so that $\alpha_4 = F_{i}(Q_3, \delta I_2)$, where $I_3$ is the size 3 identity matrix, $m_4 = 3$, and $Q_4$ is given by the series connection of the linear fractional transform such that

$$Q_4 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

Also, $Q_5$ can be directly written as

$$Q_5 = \begin{bmatrix} d & e \\ 0 & 1 \end{bmatrix},$$

such that $m_5 = 1$.

Now that there is an expression for each of the $\alpha_i (i = 1, \ldots, 5)$ parameters in terms of a linear fractional transformation $\alpha_i = F_{i}(Q_i, \delta I_{q_i})$, they can be combined to form one linear fractional transformation with a common uncertainty $\delta$. As shown earlier, the uncertainty in terms of $\alpha$ is given as

$$\alpha_p = \begin{bmatrix} \alpha_1 I_{q_1} & \alpha_2 I_{q_2} & \alpha_3 I_{q_3} & \alpha_4 I_{q_4} & \alpha_5 I_{q_5} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $I_{q_i}$ is the size $q_i$ identity matrix. The total number of uncertain elements is given by $m_T = \sum_i q_i = 11$. Thus, the linear fractional transformation $\alpha_p = F_{i}(Q, \delta)$ with $\Delta = \delta I_{11}$ is given by $\alpha_p = Q_{11} + Q_{12} \Delta (I_{11} - Q_{22} \Delta)^{-1} Q_{21}$ where $Q = [Q_{11} Q_{12}; Q_{21} Q_{22}]$. Each submatrix $Q_{jk}$ is given by the block diagonal matrix

$$Q_{jk} = \begin{bmatrix} Q_{j1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q_{kj} \end{bmatrix}$$

where $j = 1, 2$ and $k = 1, 2$. The matrix $Q_{jk}$ has the same number of diagonal blocks as $\alpha_p$ based on the rank of each $J_i$ matrix denoted by $q_i$.

**3.3 Model in Robust Control Framework**

The final expanded representation of entire system is $G(s) = \frac{1}{s} F_s(F_t(M, F_t(Q, \Delta)))$, which is described by Fig. 7 and corresponding equation set
4. CONTROLLER SYNTHESIS AND SIMULATION

4.1 DK-iteration Model Based Controller

The goal is to design a controller that can robustly stabilize the RWM and meet specified controller performance criteria. The robust stability of the plant is determined by the $N_{11}$ sub-matrix, where $N = F_l(P, K)$ represents the nominal closed-loop system. The sub-system $N_{11}$ term isolates the uncertainty from the input and output of the system. The robust stability is determined by the structured singular value, which is defined as

$$
\mu(N_{11}) \triangleq \frac{1}{\min_{D \in \mathcal{D}}} \left\{ \min_{k} |\det(I - k_{m}N_{11}\Delta) = 0| \right\}
$$

for $\mathcal{D}(\Delta) \leq 1$. Larger $\mu$ values mean $(I - N_{11}\Delta)$ becomes singular with small perturbations, thus the smaller $\mu$ the better. The robust stability condition is found by finding the smallest value of $k_{m}$ at the onset of instability, or $\det(I - k_{m}N_{11}\Delta) = 0$, which yields $k_{m} = \frac{\mu(N_{11})}{\|\Delta\|}$, where $k_{m}$ is a measure of the robust stability to perturbations in $\Delta$. Thus, assuming $N_{11}$ and $\Delta$ are stable, the system is robustly stable if and only if $\mu(N_{11}(\text{ja})) < 1, \forall \alpha$. Similarly, the robust performance is given by $\mu(N(\text{ja})) < 1, \forall \alpha$. Both conditions assume that $N$ is internally stable.

$DK$-iteration is one available procedure to design a controller using $\mu$-synthesis. Since there is no direct method to synthesize a $\mu$-optimal controller, this method is used by combining $\mathcal{H}_{\infty}$ synthesis and $\mu$-analysis. This method starts with the upper bound on $\mu$ in terms of the scaled singular value

$$
\mu(N) \leq \min_{D \in \mathcal{D}} \bar{\sigma}(DN^D)
$$

where $\mathcal{D}$ is the set of matrices $D$ which commute with $\Delta$, i.e., $DA = AD$. Then, the controller that minimizes the peak value over frequency of this upper bound is found, namely

$$
\min_{K} \left( \min_{D \in \mathcal{D}} \|DN(K)D^D\|_{\infty} \right).
$$

The controller is designed by alternating between the two minimization problems until reasonable performance is achieved. The $DK$-iteration can summarized as follows (Skogestad [2005]):

1. $K$-step. Synthesize an $\mathcal{H}_{\infty}$ controller for the scaled problem, $\min_{K} \|DN(K)D^D\|_{\infty}$ with fixed $D(s)$.

2. $D$-step. Find $D(\text{ja})$ to minimize $\bar{\sigma}(DN^D(\text{ja}))$ at each frequency with fixed $N$.

3. Fit the magnitude of each element of $D(\text{ja})$ to a stable and minimum-phase transfer function $D(s)$ and go to step 1.

The iteration continues until $\|DN(K)D^D\|_{\infty} < 1$ or the $\mathcal{H}_{\infty}$ norm no longer decreases.

Using the derived $P - \Delta$ formulation (Fig. 9), a controller can be designed with the $DK$-iteration method for robust stabilization. For the model being used the growth rate $\gamma$ ranges from 10 rad/s to 5,000 rad/s. This results in a range for the uncertain parameter $e_{pp}$ that goes from 71 to 0.3325. This is the range of values for which the system should be stabilized so that the robust controller can be considered a suitable design.

The complete system that is used to design the controller has an additional two time delay blocks preceding the plasma model. The time delays physically represent the plasma control system and the power supply. For design purposes, the time delays are linearized using second order Padé approximations.
Two controllers are synthesized using the dksyn command in Matlab, one using the nominal plant and the other an augmented plant with input weight. The performance weight is added to the inputs of the system to achieve desired loop-shaping results. The weight is of the form $W = \frac{(M^{-1/n_s} + \omega_c^2)^n}{(s + \omega_c A^1/n)^m}$, where $M = 10^8$, $\omega_c = 10^6$, $A = 1$, and $n = 2$. The DK controllers are synthesized using a $P - \Delta$ system constructed for $c_{pp} = 0.34125$ ($\gamma = 4.890\, \text{rad/s}$) and guarantees $\mu < 1$ for the range defined by $c_{pp\min} = 0.3325$, $c_{pp\max} = 0.35$, which is equivalent to $\gamma_{\text{max}} = 5,000\, \text{rad/s}$, $\gamma_{\text{min}} = 4,660\, \text{rad/s}$. However, these results are conservative and, as it will be shown in the next part, the stability and performance ranges for our system are indeed bigger. The conservatism is explained by the fact that the DK-iteration implicitly assumes that the uncertain parameter is complex and does not take advantage of the known phase information of the real uncertainty. The real uncertainty can be considered using a modified algorithm, the DKG-iteration (Young [1993]), however this algorithm greatly increases the numerical complexity. The controllers were designed using a 15 eigenmode model with 36 states. The designed controllers have orders of 108 and 107 for the plant without weight and with weight respectively. In both cases, the controller order is reduced to 16 before computing the effective stability and performance ranges.

4.2 Controller Simulation and Results

In order to be able to compare the proposed model-based DK controllers with present non-model-based controllers, a proportional-derivative (PD) controller is designed (integral action is not required for this system). The PD controller is synthesized to maximize the stability range as a function of $\gamma$ and is of the form $K_{ij} = \frac{G_{ij\gamma} = G_{ij}}{1 + \tau_{pc} s}$ for $i = 1 \ldots 3$, $j = 1 \ldots 2$, and with $\tau_{pc} = 4 \times 10^{-4}$ sec. The resulting non-zero gains are $G_{13} = 3.80 \times 10^4$, $G_{14} = 76$, $G_{23} = 1.38 \times 10^3$, $G_{24} = 40$, $G_{33} = 6.62 \times 10^2$, $G_{34} = 103$. Table 1 provides the performance constraints in response to a unit step in the RWM mode amplitude.

Fig. 10 shows the time response to a unit step in the RWM mode amplitude at constant RWM growth rates of $\gamma = 10$ rad/s and $\gamma = 5,000$ rad/s, the lower and upper limits of the growth rate range of our interest. For the slower growth rate (top graph), the DK and PD controllers have similar responses with approximately 20% overshoot, and a fast rise and settling time. For the faster growth rate (bottom graph), the settling time is increased to approximately 5 ms. Another example is presented in Fig. 11, which shows the response to initial conditions of the plasma, normalized to a starting RWM mode amplitude of 1 Gauss. The DK controllers provide quick suppression of the RWM mode amplitude, out-performing the PD controller, which does not provide quick suppression at the faster growth rate. While the settling time is similar for both DK controllers, the weighted version slightly out-performs the non-weighted one. For both growth rates, the weighted DK controller design uses less applied voltage to achieve similar results.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Target Value</th>
<th>Maximum Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rise Time</td>
<td>1.0ms</td>
<td>5.0ms</td>
</tr>
<tr>
<td>Settling Time</td>
<td>5.0ms</td>
<td>10ms</td>
</tr>
<tr>
<td>Overshoot</td>
<td>15%</td>
<td>50%</td>
</tr>
<tr>
<td>Input Voltage</td>
<td>N/A</td>
<td>± 100V</td>
</tr>
</tbody>
</table>

Table 1. Performance Targets and Constraints.

4.3 Closed-loop Stability and Performance

Table 2 provides the ranges of $\gamma$ for which stability and performance conditions are satisfied. The first row "Stability Range" indicates the range of $\gamma$ for which the system remains stable when using a unit step input for the RWM model amplitude. The second row "Perf. Range (Step)" represents the range of $\gamma$ for which the performance conditions are satisfied under the same control input. The final row "Perf. Range (Initial)" indicates the range of $\gamma$ for which the performance conditions are satisfied when an initial unit excitation of the RWM mode amplitude is forced through appropriate initial conditions. Both model-based DK controllers show good stability and performance properties well beyond the desired $\gamma$ range and that of the PD controller, with the weighted DK controller design having a larger range in both stability and performance.
controller. As measured by the structured singular value, for
a predetermined range of γ, Augmenting the nominal system
with performance weight provides better loop-shaping of the
closed-loop system, which results in improved controller per-
formance. Since the plasma RWM growth rate can vary with
operating conditions, the design of a controller that can stabilize
the system over the entire physical range of γ is critical. In terms
of robust stability, this method eliminates the need of online
identification and controller scheduling.

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