A Result on State Estimation of Nonlinear Systems with Application to Fuel Cell Stacks

M. Benallouch ∗ R. Outbib. ∗∗ M. Boutayeb ∗∗∗ E. Laroche ∗

∗ LSIT, UMR CNRS-ULP 7005, ENSPS, Bd. Sebastien Brant
67412 Illkirch Cedex France
∗∗ Laboratoire LSIS - Domaine Universitaire de Saint-Jérôme
Avenue Escadrille Normandie-Niemen-13397 Marseille
Cedex 20, France
∗∗∗ CRAN, UMR CNRS 7039, University of Henri Poincaré,
Nancy I 186, 54400 Cosnes et Romain, France

Abstract: In this paper, the observation issue of the partial pressure of oxygen and nitrogen and the mass flow rate of dry air in the cathode channel of a fuel cell stack is addressed. The proposed approach considers the mass flow rate of dry air as an unknown input and uses the voltage and the total pressure as measurements. By using the Jacobian of the nonlinear functions and the convexity principle, the observer design problem is turned into a LMI feasibility problem. Simulation results with a detailed model show the good convergence properties of the observer.

Keywords: Fuel cell stacks, nonlinear observer, state and unknown input estimation

1. INTRODUCTION

Fuel cell stacks are known since the XIXth century and have received considerable interest over the years. More recently, with the increase of the oil price, new solutions have to be developed for the energy issues. One solution could be to use hydrogen for stocking energy and to use fuel cell stacks in order to convert it into electricity. Hydrogen could be produced from electric energy produced by any source, including solar cells.

Fuel cells are complex dynamical systems that include several unknown quantities. Over the last decades, tremendous research activities have focused on observer design for these systems. (Arcak et al. (2004); McKay and Stefanopoulou (2004); Görgün et al. (2005); Benallouch et al. (2007b)). These observers can be used for different applications, such as control (Pukrushpan et al. (2002, 2003)), diagnosis (Görgün et al. (2005); Mays et al. (2001)) and communication (Boutayeb et al. (2002); Benallouch et al. (2007a); Liao and Huang (1999)).

The purpose of this paper is to investigate the problem of the observation of the partial pressure of oxygen, nitrogen and the mass flow rate of dry air treated here as an unknown input. When the input is not completely available for measurement, the existence conditions for an unknown input observer are more restrictive than the classical detectability condition (Boutayeb et al. (2002), Ha and Trinh (2004) Corless and Tu (1998)). Unknown input observers find a wide applicability in the design of robust observers, decentralized control, and for fault detection (Chen and Saif (2006))

This paper is organized as follows. Section 2 presents the models of the cathode flow (Pukrushpan et al. (2004b,a)) and of the stack voltage (Pukrushpan et al. (2004b,a); Larmine and Dicks (2000)). In section 3, a novel algebraic method is introduced for simultaneous estimation of the partial pressure of oxygen, of nitrogen and of the mass flow rate of dry air.

It consists in sufficient LMI conditions. Simulation results are presented in Section 4, that illustrate the interest of the proposed method.

List of Symbols

\[ \tilde{R} : \text{universal gas constant (J.(mol.K)\(^{-1}\))} \]
\[ T_{fc} : \text{fuel cell temperature (K)} \]
\[ V_{ca} : \text{cathode volume (m}^3\text{)} \]
\[ I_{st} : \text{current (A)} \]
\[ n_{st} : \text{the number of cells in the stack} \]
\[ F_{st} : \text{Faraday number (Coulombs)} \]
\[ A_{fc} : \text{fuel cell active area} \]
\[ t_{mem} : \text{membrane thickness} \]
\[ P_{sat}(T_{st}) : \text{vapor saturation pressure (P} \text{)} \]

2. MODEL

2.1 Cathode Flow Model

The mass continuity and the ideal gas law are used to balance the pressure of the oxygen and nitrogen inside the cathode volume:

\[
\begin{cases}
\frac{dP_{O_2,ca}}{dt} = \frac{\tilde{R}I_{st}}{M_{O_2}V_{ca}} (W_{O_2,ca,in} - W_{O_2,ca,out}) - W_{O_2,reacted} \\
\frac{dP_{N_2,ca}}{dt} = \frac{\tilde{R}I_{st}}{M_{N_2}V_{ca}} (W_{N_2,ca,in} - W_{N_2,ca,out})
\end{cases}
\]

(1)

where \(P_{O_2}\) and \(P_{N_2}\) are the oxygen and nitrogen partial pressure, \(M_{O_2}\) (kg.mol\(^{-1}\)) and \(M_{N_2}\) (kg.mol\(^{-1}\)) are the molar masses of oxygen and nitrogen, respectively. \(W_{O_2,ca,in}\) and \(W_{N_2,ca,in}\) are the oxygen and nitrogen mass flow rate entering the cathode, as shown by:
where $x_{O_2,ca,in}$ is the Oxygen mass fraction and $W_{O_2,ca,in}$ represent the mass flow rate of dry air considered here in as an unknown input. Furthermore $W_{O_2,ca,\text{out}}$ and $W_{N_2,ca,\text{out}}$ are the oxygen and nitrogen mass flow rate leaving the cathode:

\begin{align}
W_{O_2,ca,\text{out}} &= \frac{\rho_{O_2} M_{O_2}}{P_{O_2} M_{O_2} + P_{N_2} M_{N_2} + P_{v,ca} M_v} W_{ca,\text{out}} \\
W_{N_2,ca,\text{out}} &= \frac{\rho_{N_2} M_{N_2}}{P_{O_2} M_{O_2} + P_{N_2} M_{N_2} + P_{v,ca} M_v} W_{ca,\text{out}}
\end{align}

Assume that matrix $B$ has full column rank. Then there exists a matrix $N$ such that:

\[ T = [N B] \] (8)

2.2 Stack voltage model

The voltage $E$ produced by one cell is affected to different voltage losses: the loss voltage responsible for the activation polarization is denoted $v_{act}$; the ohmic voltage loss is denoted $v_{ohm}$; the concentration polarization is caused by the concentration overpotential, leading to voltage loss $v_{conc}$, as given by Pukurushan et al. (2004a). Therefore, the voltage provided by the stack current $I_{st}$:

\[ W_{O_2,\text{reacted}} = M_{O_2} \frac{n_{a} I_{st}}{4F_d} \] (4)

In order to simplify the calculation, we assume that $M_{O_2} = M_{N_2} = M_v$, then system (1) can be rewritten in the following form:

\[
\frac{dP_{O_2}}{dt} = \frac{RT_f}{V_{ca}} \begin{bmatrix} 0 & -1 & P_{O_2} & \rho_{O_2} M_{O_2} n_a I_{st} \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} k_{ca,\text{out}} \lambda_{O_2,ca,in} \\ k_{ca,\text{out}} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} I_{st}
\]

where $k_{ca,\text{out}} = k_{ca} (P_{ca} - P_m)$ and $k_{ca} (P_{ca} - P_m)$ is the total flow rate is determined using the simplified orifice equation, $k_{ca,\text{out}}$ is the orifice constant, $P_{ca} = P_{O_2} + P_{N_2} + P_{v,ca}$ is the cathode total pressure, $P_m$ is the return manifold pressure, $P_{v,ca}$ vapor partial pressure and $M_v$ (kg mol\(^{-1}\)) is the vapor molar mass.

The rate of oxygen consumed in the reaction is a function of the stack current $I_{st}$:

\[ x_{O_2,\text{in}} W_{O_2,\text{in}} = \lambda_{O_2} n_{a} I_{st} \]

The voltage

\[ v_{act} = v_0 + v_{a} (1 - e^{-c_1 \frac{I_{st}}{I_{max}}}) \]

where $v_0 = V_{st}(T_f, P_{O_2}, P_{N_2})$ is the voltage drop at zero current density and $v_a = V_{st}(T_f, P_{O_2}, P_{N_2})$ and $c_1$ are constants.

2.3 Measurements

We consider that the measurements available the system are: the Stack voltage $V_{st}$ and the total pressure at the cathode defined by:

\[ y = P_{ca} - P_{sat}(T_f) = P_{O_2} + P_{N_2} \]

3. MAIN RESULTS

3.1 Reformulation of the model

The equations of the system can be reformulated as following:

\[
\begin{pmatrix} \dot{x} \\ y \\ V_{st} \end{pmatrix} = \begin{pmatrix} A & BW_{a,ca,in} + f(x, y) + DL_{st} \\ 0 & 0 & 0 \end{pmatrix} \]

\[ V_{st} = h(x, y, I_{st}) \]

where $A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} \rho_{O_2} M_{O_2} \rho_{N_2} M_{N_2} \rho_{v,ca} M_v \\ P_{O_2} + P_{N_2} + P_{v,ca} \end{pmatrix}$, $C = [1 \ 1]$ and $f(x, y) = \begin{pmatrix} \frac{RT_f k_{ca,\text{out}}}{V_{ca}} P_{O_2} + P_{N_2} + P_{v,ca} \rho_{v,ca} M_v \\ \frac{RT_f k_{ca,\text{out}}}{V_{ca}} P_{O_2} + P_{N_2} + P_{v,ca} \rho_{v,ca} M_v \end{pmatrix}$.

Assume that matrix $B$ has full column rank. Then there exists a matrix $N$ such that:

\[ T = [N B] \] (8)
is non-singular. Let define $T_1$ and $T_2$ such that:

$$
T^{-1} = \begin{bmatrix} T_1 & T_2 \end{bmatrix}^T
$$

The change of state variable $x = Tz$ is introduced, leading to:

$$
\begin{align*}
\dot{z} &= T^{-1}ATz + T^{-1}BW_{a,ca,in} + T^{-1}f(Tz,y) + T^{-1}DI_a \\
y &= CTz \\
V_{fc} &= h(Tz,y,I_a)
\end{align*}
$$

where

$$
\begin{bmatrix} a_{11} & a_{12} \\
a_{21} & a_{22} \end{bmatrix}
$$

and $T^{-1}B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

In order to eliminate the component $W_{a,ca,in}$, we multiply the state equation (10) by matrix $E = \begin{bmatrix} 1 & 0 \end{bmatrix}$, yielding the following descriptor system:

$$
\begin{align*}
E\dot{z} &= \omega z + T_1f(Tz,y) + T_1DI_a \\
y &= Hz \\
V_{fc} &= h(Tz,y,I_a)
\end{align*}
$$

with $\omega = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}$ and $H = CT$.

### 3.2 Full-order observer design

We consider the nonlinear observer of the following form:

$$
\begin{align*}
\dot{\hat{z}} &= \omega z + Q(y - \hat{y}) \\
\dot{\hat{y}} &= Q^T(y - \hat{y})
\end{align*}
$$

where $\hat{z}$ denotes the state estimation vector of $z$, $L$, $R$, $K$ are matrices to be determined such that $\hat{z}$ converges asymptotically to $z$.

The main contribution of the paper consists in LMI sufficient condition for the observer synthesis problem. This result is formulated in the following theorem.

**Theorem 1.** The observer error $e(t)$ converge asymptotically towards zero if there exist matrices $P = P^T > 0$, $X$ and $Y$ of appropriate dimensions such that the following LMI is feasible:

$$
\begin{align*}
(Rz)^T P + P(Rz)^T + (RT_1J_f(\alpha)T)^T P + P(RT_1J_f(\alpha)T) - H^TX - X^TH + (J_h(\beta)T)^TY + Y^T(J_h(\beta)T) &< 0 \\
\forall \alpha, \beta \in \mathcal{H}.
\end{align*}
$$

If these LMI is feasible, matrices $F$, $K$, $N$ and $L$ are given by:

$$
\begin{align*}
F &= P^{-1}X^T, \\
K &= P^{-1}Y^T, \\
N &= Rz - FH, \\
L &= F + NQ.
\end{align*}
$$

**Proof.**

Consider the error vector:

$$
e = \hat{z} - z,
$$

Substituting (11) and (12) into (14), we obtain:

$$
e = \omega + (QH - I_2)z
$$

where $I_2$ represents the identity matrix of dimension 2. Let $R$ be a matrix such that:

$$
RE + QH = I_2
$$

Then (15) becomes $e = \omega - REz$ and the error dynamics write:

$$
\dot{e} = Ne + (LH + NRE - Rz^2)z + RT_1\{f(T\hat{z},y) - f(Tz,y)\} + K\{h(T\hat{z},y,I_a) - h(Tz,y,I_a)\}
$$

We assume that $f$ is differentiable on $\text{Co}(z(t),\hat{z}(t))$. From the differential mean value theorem (see in Appendix), there exist constant matrices $\eta_1$ and $\eta_2$ in $\text{Co}(z(t),\hat{z}(t))$ such that:

$$
f(T\hat{z},y) - f(Tz,y) = \frac{2}{p_{ij}}e_2(\hat{x})e_2^T(j)\frac{\partial f_i}{\partial \eta_j}(\eta_i, y)T(\hat{z} - z)
$$

with $e_2(1) = [1 \ 0]^T$ and $e_2(2) = [0 \ 1]^T$. For simplicity, we introduce the notations $J_f(\rho) = \sum_{i,j=1}^2 e_2(\hat{x})e_2^T(j)\rho_{ij}$ and $\rho_{ij} = \frac{\partial f_i}{\partial \eta_j}(\eta_i, y)$ which is equivalent to:

$$
f(T\hat{z},y) - f(Tz,y) = J_f(\rho(y))T(\hat{z} - z)
$$

A same reasoning yields $J_h(\mu) = \sum_{i,j=1}^2 e_2(\hat{x})e_2^T(j)\mu_{ij}$ with $\mu_{ij} = \frac{\partial h_i}{\partial \eta_j}(\eta_i, y)$ we obtain:

$$
h(T\hat{z},y,I_a) - h(Tz,y,I_a) = J_h(\mu(y))T(\hat{z} - z)
$$

If we set

$$
F = L - NQ
$$

and:

$$
N = Rz - FH
$$

then the error dynamics write:

$$
\dot{e} = (Rz^2 - FH + RT_1J_f(\rho)p + J_h(\mu)T)e
$$

Choose a quadratic Lyapunov function as $V = e^T Pe$. Its time-derivative writes:

$$
\dot{V} = (Rz^2 + P(Rz)^2 - H^TX - X^TH + (RT_1J_f(p))T^TP + P(RT_1J_f(p))T + J_h(\mu)p)T^TP + J_h(\mu)p)T^TY + Y^T(J_h(\mu)p) < 0
$$

Based on the Lyapunov stability theory, if $V$ is negative-definite then the convergence of the estimation error is guaranteed, which is equivalent to:

$$
\Gamma = (Rz)^T P + P(Rz)^T - H^TX - X^TH + (RT_1J_f(p)^T)T^TP + P(RT_1J_f(p)^T)T + J_h(\mu)p)T^TY + Y^T(J_h(\mu)p) < 0
$$

with $X = F^TP$ and $Y = K^TP$.

Since $\Gamma$ is affine in $p$ and $\mu$, the relationship holds for any $p$ and $\mu$ in $\mathcal{H}$ as soon as it is verified at the vertices, that is to say for $\alpha, \beta \in \mathcal{H}$ (Appendix). By using the notations $X = F^TP$ and $Y = K^TP$, condition (18) is equivalent to (13), which completes the proof.

### 3.3 Estimation of the mass flow rate of dry air

The aim of this section is to estimate the mass flow rate of dry air $W_{a,ca,in}$. For this, we need to estimate firstly the state $x$. Since $z \rightarrow \hat{z}$ when $t \rightarrow \infty$, then the estimate of $x$ is given by $\hat{x} = T\hat{z}$.

Now, let use the state estimates to reconstruct the unknown input. From (7), we have:

$$
\hat{W}_{a,ca,in} = (CB)^{-1}((\dot{y} - \hat{y}a\hat{z} - C\hat{f}(\hat{y}, y) - CD\hat{u}))
$$

### 4. SIMULATION RESULTS FOR A FUEL CELL MODEL

We apply this approach to the fuel cell model. Referring to Subsection 3.1, we choose:

$$
N = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$
We get:
\[
T = \begin{bmatrix}
1 & 2.441 \\
0 & 8.035
\end{bmatrix}
\] (20)
and
\[
\tilde{A} = \begin{bmatrix}
-22.81 & 7.451 \times 10^{-9} \\
0 & -22.81
\end{bmatrix}
\] (21)
Matrices \( R \) and \( Q \) are chosen in order to satisfy Eq. (16):
\[
R = \begin{bmatrix}
1 & 0 \\
0 & 0.5100 \times 10^{-8}
\end{bmatrix}
\] (22)
\[
Q = 10^{-7} \begin{bmatrix}
1 & 0 \\
0 & 0.9544
\end{bmatrix}
\] (23)
The initial conditions considered in the results presented in the sequel are:
\[
\begin{bmatrix}
P_{O_2}^0 \\
P_{N_2}^0
\end{bmatrix}^T = \begin{bmatrix}
1.0951 \times 10^4 \\
8.2070 \times 10^2
\end{bmatrix}^T (Pa)
\]
In the current case, a simplification is possible, allowing to obtain a formulation with 3 uncertain parameters instead of 4. The Jacobian then write \( J_f(\rho) = \sum_{i,j=1}^{n,m} H_{ij} \partial \phi_i / \partial x_j (\eta_i) \) (a − b), (A.1)
where
\[
C_o(a,b) = \{ \lambda a + (1 - \lambda) b(t), 0 \leq \lambda \leq 1 \}
\]
Fig. 2. stack current (A)
Fig. 3. Response of \( P_{O_2} \) and its estimate
Fig. 4. Response of \( P_{N_2} \) and its estimate
Appendix A. DIFFERENTIAL MEAN VALUE THEOREM (DMVT) FOR VECTOR VALUED FUNCTION (Zemouche et al. (2005))
We have proposed an efficient method for designing a nonlinear observer for fuel cells systems with unknown inputs. The proposed theorem makes use of two elementary principles: first the notion of Jacobian that allows to simplify the equations, second the convexity principle for deriving a LMI feasibility problem.

Based on this design method, a nonlinear observer was designed for the estimation of the partial pressure of oxygen and nitrogen in addition to the mass flow rate of dry air. Simulation results showed the fast convergence of the obtained estimator.
**Fig. 5. Response of $W_{a,ca,in}$ and its estimate**

and $H_{ij} = e_m(i)^T e_m(j)$, $i = 1, ..., n$, $j = 1, ..., m$,

where

$$e_m(i) = \begin{pmatrix} 0, ..., 0, 1, 0, ..., 0 \end{pmatrix}^T$$

is a vector of the canonical basis of $\mathbb{R}^n$. Then, using the notations:

$$\rho_j(t) = \frac{\partial \varphi}{\partial x_j}(\eta_i(t))$$

(A.2)

we deduce that

$$\varphi(a) - \varphi(b) = \left( \sum_{i,j=1}^{n,m} H_{ij} \rho_j(t) \right) (a - b).$$

**Proof.** We can write $\varphi(x) = \sum_{i=1}^{n} e_i(i) \varphi_i(x)$, where $\varphi_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is the $i$th component of $\varphi$. We know that for all scalar function $\varphi_i$ that's differentiable on $Co(a,b)$, there exists $\eta_i \in Co(a,b)$ such that $\varphi_i(a) - \varphi_i(b) = \frac{\partial \varphi_i}{\partial x_j}(\eta_i)(a - b)$. For $i = 1, ..., n$ we have

$$\frac{\partial \varphi}{\partial x_j}(c_i) = \sum_{j=1}^{n} e_m(i)^T e_m(j) \frac{\partial \varphi}{\partial x_j}(\eta_i),$$

we deduce that

$$\varphi(a) - \varphi(b) = \left( \sum_{i,j=1}^{n,m} e_m(i)^T e_m(j) \frac{\partial \varphi_i}{\partial x_j}(\eta_i) \right) (a - b),$$

end of prof.

Let assume that parameters $\rho_j(t)$ evolve in a bounded domain $\mathcal{K}_n$ of which $2^{n^2}$ vertices are defined by:

$$\mathcal{V}_{\mathcal{K}_n} = \{ \alpha = (\alpha_1, ..., \alpha_n) \ | \ \alpha_i \in \{ \rho_{ij} \} \}$$

(A.3)

where

$$\rho_{ij} = \max_i (\rho_{ij}(t)) \text{ and } \rho_{ij} = \min_i (\rho_{ij}(t)).$$

**ACKNOWLEDGMENT**

The authors are grateful to Anna G. Stefanopoulou, Jay T. Pukrushpan and Huei Peng for providing the details of the fuel cell stack model.

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