Method for Analysis of Synchronization
Applied to Supermarket Refrigeration System

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Abstract: Synchronisation arises in dynamical systems that are composed of a number of interconnected subsystems. In the paper synchronization in a particular application - a supermarket refrigeration system - is studied. The temperature control in supermarket display cases is typically maintained by a number of distributed hysteresis controllers. Synchronization is then manifested by the opening and closing actions of expansion valves at the same time. Synchronization is interpreted in this paper as a limit cycle in a state space created by transitions among piecewise-affine dynamical systems. Stability of the resultant limit cycle is examined by a Poincaré like map. We show that the synchronisation takes place if the corresponding Poincaré map is stable.

1. INTRODUCTION

The temperature control in a supermarket refrigeration system is typically maintained by a number of distributed hysteresis controllers. A problem that often arises in this control setup is synchronization. It is manifested by the opening and closing actions of all the valves at almost the same time. Consequently, the compressors periodically have to work more intensely, which results in low efficiency and increased wear.

Synchronization - or more generally - state agreement, arises in dynamical systems that are composed of a number of interconnected subsystems Lin et al. (2005b). State agreement means that the states of the subsystems are all equal. It is known in a variety of applications, e.g. biochemical systems Gunawardena (2003), rendezvous in multi-agent systems Lin et al. (2005a), consensus in computer science Moases and Rajsbaum (2002) and Tabuada and Pappas (2002).

Synchronization in a supermarket refrigeration system will be interpreted as a stable limit cycle in a state space created by transitions among piecewise-affine dynamical systems. It will be shown how piecewise-affine systems living on polyhedra are “glued” together to form a single dynamical system defined on a coherent state space. Further by enforcing a certain transversality condition on the constituent systems it will be shown that the stability of the resultant limit cycle can be examined by a Poincaré map. The Poincaré map has been used before in Hiskens (2001) and Hiskens and Reddy (2007) for stability of limit cycles arising in switched systems. The method developed in this paper will be applied for analysis of synchronization in a refrigeration system consisting of two display cases and a compressor unit.

The paper is organized in the following way. Section 2 describes the control system architecture in a supermarket refrigeration system and explains the origin of synchronization. In Section 3 a non-linear model is presented and subsequently a simplified piecewise-affine model is derived. Section 4 puts forward a method for gluing state spaces of a switched system together. The method is then applied to the supermarket refrigeration system. In Section 5 the synchronization i.e. the resultant limit cycle on the glued space is analyzed. Section 7 concludes the paper.

2. SYSTEM DESCRIPTION

In a supermarket many of the goods need to be refrigerated to ensure preservation for consumption. These goods are normally placed in open refrigerated display cases that are located in the sales area for self service.

A simplified supermarket refrigeration circuit is shown in Fig. 1. Compressors comprises the heart of the system. In majority of supermarkets, the compressors are connected in parallel. The compressors supply the flow of refrigerant by compressing the low pressure refrigerant, which is drained from the display cases through the suction manifold. The compressors keep a certain constant pressure in the suction manifold, thus ensure the desired evaporation temperature. From the compressors, the refrigerant flows...
controls the suction pressure. To avoid excessive compressor switching, a dead band around the reference of the suction pressure is commonly used. When the pressure exceeds the upper bound, one or more additional compressors are turned on to reduce the pressure, and vice versa when the pressure falls below the lower bound. In this way, moderate changes in the suction pressure do not initiate a compressor switching. Nevertheless, pronounced synchronization effects lead to frequent compressor switchings causing large fluctuations in the suction pressure and a high wear of the compressors. Fig. 3 shows a simulation illustrating the effects of synchronization.

3. MODELING

The model of the supermarket refrigeration system is composed of individual models of the display cases, the suction manifold, the compressor rack, the condensing unit. The main emphasis in this paper is laid on the suction manifold and the display cases such that the dynamics relevant for controlling the compressors and display cases are captured. The compressor dynamic is typically much faster than the rest of system and hence not included.

3.1 Non-linear hybrid model

The nonlinear hybrid model presented here is a summary of the model presented in Larsen et al. (2007).

The state of the combined model of a supermarket refrigeration system consists of the suction pressure $P_{suc}$, and for each display case four states - the goods temperature $T_{good}$, the air temperature $T_{air}$, the wall temperature (of the evaporator) $T_{wall}$ and the mass of refrigerant in the evaporator $M_r$. The input to the model is the volume flow produced by the compressors $V_{comp}$, and the state of the inlet valve (closed or opened, $\delta \in \{0,1\}$). The system is affected by two disturbances - the heat load from the surroundings $Q_{load}$, and $m_{r, const}$ which is a constant mass flow into the manifold giving rise to un-modeled refrigerated entities.
\[
\frac{dT_{\text{goods},i}}{dt} = -\frac{\dot{Q}_{\text{goods} \to \text{air},i}}{M_{\text{goods},i} C_{p\text{goods},i}} \tag{1}
\]
\[
\frac{dT_{\text{wall},i}}{dt} = \frac{\dot{Q}_{\text{air} \to \text{wall},i} - \dot{Q}_{e,i}}{M_{\text{wall},i} C_{p\text{air},i}} \tag{2}
\]
\[
\frac{dT_{\text{air},i}}{dt} = \frac{\dot{Q}_{\text{goods} \to \text{air},i} + \dot{Q}_{\text{load},i} - \dot{Q}_{\text{air} \to \text{wall},i}}{M_{\text{air},i} C_{p\text{air},i}} \tag{3}
\]
\[
\frac{d\dot{M}_r}{dt} = \begin{cases} 
M_{r,max,i} - \dot{M}_r & \text{if } \delta_i = 1 \\
-\frac{\dot{Q}_{e,i}}{\Delta h_{tg}} & \text{if } \delta_i = 0 \land M_{r,i} > 0 \\
0 & \text{if } \delta_i = 0 \land M_{r,i} = 0
\end{cases} \tag{4}
\]
\[
\frac{dP_{\text{suc}}}{dt} = m_{\text{suc,in}} + \dot{m}_{r,\text{const}} - \dot{V}_{\text{comp}} \cdot \rho_{\text{suc}} \tag{5}
\]

where \(i\) denotes the \(i\)th display case and \(\dot{Q}\) denotes a heat flux and the subscript the media in between which the heat is flowing. Furthermore

\[
\dot{Q}_{\text{goods} \to \text{air},i} = UA_{\text{goods} \to \text{air},i}(T_{\text{goods},i} - T_{\text{air},i}) \tag{6}
\]
\[
\dot{Q}_{\text{air} \to \text{wall},i} = UA_{\text{air} \to \text{wall},i}(T_{\text{air},i} - T_{\text{wall},i}) \tag{7}
\]
\[
\dot{Q}_{e,i} = UA_{\text{wall} \to \text{ref},i}(M_{r,i})(T_{\text{wall},i} - T_e) \tag{8}
\]
\[
UA_{\text{wall} \to \text{ref},i}(M_{r,i}) = UA_{\text{wall} \to \text{ref},i}(M_{r,i}) \frac{M_{r,i}}{M_{r,max,i}} \tag{9}
\]
\[
m_{\text{suc,in}} = \sum_{i=1}^{n} \frac{\dot{Q}_{e,i}}{\Delta h_{tg}} \tag{10}
\]
\[
\dot{V}_{\text{comp}} = \sum_{i=1}^{n} \dot{m}_r \cdot \frac{1}{100} \cdot \eta_{\text{vol}} \cdot V_{st} \tag{11}
\]

where \(n\) is the number of display cases and \(q\) is the number of discrete compressor entities. \(UA\) is the overall heat transfer coefficient with the subscript denoting the media between which the heat is transferred. \(M\) denotes the mass, and \(Cp\) the heat capacity, where the subscript indicates the media.

The model contains some non-linear refrigerant specific functions of the suction pressure:

- \(\Delta h_{tg}\), which is the enthalpy difference across the evaporator
- \(\rho_{\text{suc}}\), which is the density of the refrigerant
- \(\frac{d\rho_{\text{suc}}}{dP_{\text{suc}}}\), which is the pressure derivative of the refrigerant density
- \(T_e\), which is the evaporation temperature

In Larsen et al. (2007) a detailed description of these functions is given, and in Appendix A the values of the system parameters are provided.

As seen in Eq. (4) the system has a hybrid nature as the inputs to the system are discrete - opening/closing of inlet valves and start/stop of the compressors in the compressor rig.

### 3.2 Simplified model

In order to obtain an adequate set of equations for analyzing synchronization in accordance with Section 5, the system equations (1) to (11) are additionally simplified to a second order (for each display case) affine switched system.

The simplification relies on the following assumptions:

1. The heat capacity of the goods is large, thus the temperature of the goods in a display case is constant and equal \(T_{g0}\).
2. The heat capacity of the air is small.
3. The evaporator is instantly filled (emptied) when the inlet valve is opened (closed).
4. The mass flow out of the display case when the valve is open is constant and equal \(\dot{m}_{\text{vol}}\).
5. The evaporation temperature \(T_e\) and the density \(\rho_{\text{suc}}\) of the refrigerant in the suction manifold are affine functions of suction pressure \(P_{\text{suc}}\):

\[
T_e = a_T P_{\text{suc}} + b_T \quad \text{and} \quad \rho_{\text{suc}} = a_{\rho}P_{\text{suc}} + b_{\rho}
\]

See eq.(A.1) and (A.2) for the parameters.

6. The gradient \(\frac{d\rho_{\text{suc}}}{dP_{\text{suc}}} \equiv \frac{d\rho_{\text{suc}}}{dP_{\text{suc}}} \) is constant.
7. The compressor delivers a constant volume flow \(\dot{V}_{\text{comp}}\).
8. The heat load \(\dot{Q}_{\text{load}}\) on the display cases is constant.

Based on these assumption the dynamics of the air temperature \(T_{\text{air},i}\) in the \(i\)th display case is described by following set of equations

\[
\frac{dT_{\text{air},i}}{dt} = \frac{\dot{Q}_{\text{goods} \to \text{air},i} + \dot{Q}_{\text{load},i} - \delta_i \dot{Q}_{e,max,i}}{M_{\text{air},i} C_{p\text{air},i}} \tag{12}
\]
\[
T_{\text{wall},i} = T_{\text{air},i} - \frac{\dot{Q}_{\text{goods} \to \text{air},i} + \dot{Q}_{\text{load},i}}{UA_{\text{air} \to \text{wall},i}} \tag{13}
\]
\[
\dot{Q}_{\text{load},i} = UA_{\text{goods} \to \text{air},i}(T_{g0} - T_{\text{air},i}) \tag{14}
\]
\[
\dot{Q}_{e,max,i} = UA_{\text{wall} \to \text{ref},i}(T_{\text{wall},i} - T_e) \tag{15}
\]

where \(\delta_i \in \{0,1\}\), \(\delta_i = 1\) indicates that the inlet valve to the \(i\)th display case is open.

The suction manifold dynamics is governed by the expression

\[
\frac{dP_{\text{suc}}}{dt} = \sum_{i=1}^{n} \delta_i \dot{m}_{\text{vol},i} + \dot{m}_{r,\text{const}} - \dot{V}_{\text{comp}}(a_{\rho}P_{\text{suc}} + b_{\rho}) \tag{16}
\]

Hereby the non-linear hybrid system has been reduced to a second order affine system with discrete inputs.

### 4. SPACE GLUING

Synchronization in a supermarket refrigeration system will be interpreted as a limit cycle in a state space created by gluing certain polyhedra together. The gluing algorithm is defined by transitions among piecewise-affine dynamical systems. Stability of the resultant limit cycle is examined by the Poincaré map. The next section applies this method for examining synchronization in a refrigeration system consisting of two display cases and a compressor unit.

To motivate this approach, we will start by studying the simulation of the simplified model consisting of 2 display cases, depicted in Fig. 4. The lower graph shows that the two display cases start working synchronously after...
3000 sec. It is seen in the upper part of Fig. 4 that the synchronisation is manifested by a closed orbit - a limit cycle - near the diagonal line $\varrho = \{(T_{\text{air},1}, T_{\text{air},2}) | T_{\text{air},1} = T_{\text{air},2}\}$. Perfect synchronisation of the two display cases takes place when the stable limit cycle coincides with $\varrho$.

Being less specific for a while, let us consider an autonomous switched system, consisting of a finite family $\mathcal{F}$ of dynamical systems each living on a polyhedron $P$. The transition between two systems takes place autonomously when the stable limit cycle coincides with $\varrho$. The situation is depicted in Fig. 5. We assume that the transversality condition is fulfilled, then the fixed point of a facet $F$ of a polyhedron $P$ is locally stable if all eigenvalues of differential $D\Phi(p)$ belong to the open unit disk. The mentioned transversality condition basically rules out the possibility of chattering or gazing along a face and allows the use standard methods known from smooth (non-hybrid) systems, for a detailed discussion the reader is referred to Wisniewski and Larsen (2008).

Before we discuss how to analyze the map $\Phi$, we will describe the idea of gluing the spaces on which the affine systems live on together. This defines a complex - a high dimensional mosaic of the polyhedra. We start by considering a transition between a smooth vector field $\xi_1 : P_1 \to \mathbb{R}^n$ and $\xi_2 : P_2 \to \mathbb{R}^n$ on a facet $F_1 \in P_1^{n-1}$. We assume that a reset map $R : F_1 \to F_2$, with $F_2$ a facet of $P_2$, is a diffeomorphism. Now we can glue the polyhedron $P_1$ and $P_2$ together by identifying $F_1$ with $F_2$ via the reset map $R : F_1 \to F_2$

$$P_1 \cup_R P_2 \equiv P_1 \cup P_2 / \sim,$$

where the equivalence relation $\sim$ identifies $x \in K_1$ with $R(x) \in K_2$, and $\cup$ stands for the disjoint union, $P_1 \cup P_2 = P_1 \times \{1\} \cup P_2 \times \{2\}$. In other words, a neighborhood of $F_1$ (which is now identified with $F_2$) in the set $P_1 \cup P_2$ is the union of an open neighborhood of $F_1$ in $P_1$ and an open neighborhood of $F_2$ in $P_2$. The situation is depicted in Fig. 5. We assume the transversality condition is fulfilled and define the flow map as a composition of the flow maps of the individual systems comprising the switched system. The space of a switched system is a disjoint union of polyhedra, whose facets are identified by the reset maps $X = \bigcup_{P \in \mathcal{F}} P / \sim$.

$\sim$ is the equivalence relation that for each reset map $R : F \to F' \in \mathcal{R}$ identifies $x \in F$ with $R(x) \in F'$, where $F, F'$ are facets of $P, P'$ both in $\mathcal{F}$, respectively; see Fig. 5.

We turn to the hysteresis controlled refrigeration system described in Section 3, it is an autonomous switched system consisting of four dynamical systems $\xi_i : B \to \mathbb{R}^2$, $i = 1, \ldots, 4$, one for each combination of the positions (on/off) of the inlet valves $\delta_1, \delta_2$. $B = [T_{\text{air}}, T_{\text{air}}] \times [T_{\text{air}}, T_{\text{air}}] \times \mathbb{R}$ is the polyhedron $B = P_1 = P_2 = P_3 = P_4$ that the flow lives on, and the vector fields $\xi_i$ are given by Eq. (12). The reset maps are in this case simply the identity maps on the 4 facets of $B$.

The state-space of the refrigeration system is then defined by gluing the polyhedra $P_1, P_2, P_3$, and $P_4$ together along the facets specified by the transitions. The result is a single state space homeomorphic to a band as in Fig. 6.

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1 a facet is a face of co-dimension 1
Fig. 6. The state space of the refrigeration system the product of a band with \( R \). In all Figures \( R \) is suppressed.

5. SYNCHRONIZATION ANALYSIS

In the previous section we have motivated the approach of creating a simple coherent stable space by gluing the polyhedra \( P_i \), \( i = 1, \ldots, 4 \) together. We have stated without proof that the glued space is homeomorphic to a band. In this section we will utilize this property to study a Poincaré map. The basic idea is to analyze stability of a candidate for a limit cycle found by e.g. simulation. A possible candidate could be perceived if simulation shows that the trajectory \( \gamma \) of the switched system after several transitions returns to a neighborhood of the starting point \( p \) at the face of a polyhedron in \( \mathcal{F} \). We study a composition \( \Phi \) of the flow maps comprising the trajectory \( \gamma \). If all the eigenvalues of differential \( D\Phi(p) \) belong to an open unit disk the limit cycle is stable. We will here describe a method for finding differential \( D\Phi(p) \) without explicitly computing the Poincaré map.

We start by stating following proposition which gives us an important intermediate result to eventually compute differential \( D\Phi(p) \).

**Proposition 1.** Let \( P \) be a polyhedron, \( \xi : P \to \mathbb{R}^n \) be a vector field. Let \( K_1, K_2 \in \mathbb{R}^{n-1} \) and their supporting hyperplanes \( H_i = \{ x \in \mathbb{R}^n | \langle x, N_i \rangle = \alpha_i \} \) (\( i = 1, 2 \)). Suppose there are \( \tau > 0 \) and \( p \in K_1 \) such that \( \xi \equiv \phi^\xi (p, \tau) \in K_2 \) (i.e. \( \tau \) is the time between 2 successive transitions). Then there are an open neighborhood \( U \) of \( p \) in \( H_1 \) and a smooth map \( h : U \to \mathbb{R}^n \) such that

1. \( h(p) = \tau \),
2. \( \phi^\xi(x, h(x)) \in H_2 \) for any \( x \in U \),
3. and \( \phi^\xi \cdot (\text{id}, h) \) is a diffeomorphism from \( U \) onto an open set of \( H_2 \).

Furthermore, the differential of \( Dh \) at the point \( p \) is

\[
Dh(p) = -\frac{1}{\langle \xi(q), N_2 \rangle} N_2^T D\phi^x_\tau |_{H_1}(p),
\]

(17)

where \( \phi^x_\tau \big|_{H_1} \) is the restriction of the flow map \( \phi^x_\tau \) to \( H_1 \).

\[ \square \]

Proof of the proposition follows directly form Sec. 3.1 in Palis and de Melo (1982). Note that the differential of the flow map \( \phi^\xi \) in Eq. (17) is to be calculated with respect to the coordinates of the hyperplane \( H_1 \).

In particular if \( \xi \) is affine, \( \xi(x) = Ax + b \) then the differential according to proposition 2 becomes

\[
Dh(p) = -\frac{1}{N_2^T (Ac^\tau p + b)} N_2^T e^\tau B,
\]

(18)

\[ \text{where the isomorphism } L : \mathbb{R}^{n-1} \to H_1 \text{ is given by } L(x) = Bx + p. \]

We study the map

\[
\Psi \equiv \phi^\xi \circ (j, h) : U \to H_2,
\]

where \( U \subset H_1 \) and \( j : U \to \mathbb{R}^n \) stands for the inclusion (for a point \( p \in U \), \( j(p) \) is the same point but seen in \( \mathbb{R}^n \), taking the starting point of the flow line on \( F_1 \) and maps it to the end point on \( F_2 \). A subsequent map takes the end point on \( F_2 \) and maps it to a point on \( F_3 \).

Since \( H_i \) (\( i = 1, 2 \)) is a hyperplane, there is an affine map \( L_i : \mathbb{R}^n \to \mathbb{R}^n \), say \( L_i(x) = B_i x + c_i \), such that \( L_i(H_i) = \{ 0 \} \times \mathbb{R}^{n-1} \). We represent the map \( \Psi \) in the local coordinates

\[
\Psi' \equiv \tilde{\pi} \circ L_2 \circ \Psi \circ L_1^{-1} \circ \tilde{i},
\]

where \( \tilde{i} : \mathbb{R}^{n-1} \to \mathbb{R}^n \), given by \( (x_1, \ldots, x_{n-1}) \mapsto (0, x_1, \ldots, x_{n-1}) \) is the inclusion, and \( \tilde{\pi} : \mathbb{R}^n \to \mathbb{R}^{n-1} \), \( (x_1, x_2, \ldots, x_n) \mapsto (x_2, \ldots, x_n) \) is the projection.

We apply the chain rule to calculate the differential of \( \Psi' \) at a point \( p \)

\[
D\Psi'(p) = \tilde{\pi} B_2 \left( D\phi^\xi_\tau(p) + \xi(q) D\tau(h(p)) \right) B_1^{-1} \tilde{i},
\]

where \( q = \phi^\xi_\tau(p) \), which according to proposition 2 is equivalent to

\[
D\Psi'(p) = \tilde{\pi} B_2 \left( \text{id} - \xi(q) N_2^T \xi(q) N_2 \right) D\phi^\xi_\tau(p) B_1^{-1} \tilde{i},
\]

(19)

where \( \text{id} \) stands for the identity matrix.

Hereby we are able to compute \( D\Phi(p) \) by iterative use of Eq. (19).

6. APPLICATION ON SIMPLE SUPERMARKET REFRIGERATION SYSTEM

In this section we apply the method developed in Sections 4 and 5 for analysis of synchronization in a supermarket refrigeration system, i.e. we study stability of the limit cycle depicted in Fig. 4.

As mentioned in Section 4 the state space of a supermarket refrigeration system is \( B = [T_{\text{air}}, T_{\text{air}}] \times [T_{\text{air}}, T_{\text{air}}] \times \mathbb{R} \), where in the particular example studied in this section \( T_{\text{air}} = 0 \) and \( T_{\text{air}} = 5 \). The polyhedron \( B \) is defined as the intersection of the four half-spaces

\[
H_\gamma^{+ \delta_1, \delta_2} \equiv \{ x \in \mathbb{R}^3 | \langle N(\delta_1, \delta_2), x \rangle \geq \alpha(\delta_1, \delta_2) \},
\]

with \( (\delta_1, \delta_2) \in \{ 0, 1 \} \), where \( N(0, 0) = [1 \ 0 \ 0]^T, N(0, 1) = [0 \ 0 \ -1]^T, N(1, 1) = [1 \ 0 \ 0]^T, N(1, 0) = [0 \ 1 \ 0]^T, \alpha(0, 0) = \alpha(1, 0) = T_{\text{air}}, \alpha(0, 1) = \alpha(1, 1) = T_{\text{air}}. \)

The supermarket refrigeration system consists of four affine dynamical systems

\[
\xi(\delta_1, \delta_2) : B \to \mathbb{R}^3, x \mapsto A(\delta_1, \delta_2) x + \alpha(\delta_1, \delta_2),
\]

given by Eq. (12).

The limit cycle is defined by the points \( (p(\delta_1, \delta_2), T(\delta_1, \delta_2)) \in H_1(\delta_1, \delta_2) \times \mathbb{R}_+ \) with
\[ p(0,1) = \phi(\xi)_{(0,0)} \] 
\[ T(0,0) = [4.90 \ 5.00 \ 0.10]^T, \quad T(0,0) = 252.57; \]
\[ p(1,1) = \phi(\xi)_{(0,1)} \] 
\[ T(0,1) \] 
\[ T(1,1) = 438.90; \]
\[ p(0,0) = \phi(\xi)_{(0,0)} \]
\[ T(0,0) = [0.00 \ 0.16 \ 0.99]^T, \quad T(1,0) = 6.67, \]

which has been found by simulation of the refrigeration system.

A Poincaré map is the following composition
\[ \Phi \equiv \psi(0,1) \circ \psi(1,1) \circ \psi(0,0), \]
where \( \psi(\delta_1, \delta_2) \equiv \phi(\xi(\delta_1, \delta_2) \circ (j(\delta_1, \delta_2), h(\delta_1, \delta_2))) \) and the maps \( j(\delta_1, \delta_2) \) and \( h(\delta_1, \delta_2) \) are defined in Proposition 2. The derivative of \( \Phi \) at \( p(0,0) \) is
\[ D\Phi(p(0,0)) = \]
\[ D\psi(1,1)(p(1,0))D\psi(1,1)(p(1,1))D\psi(0,1)(p(1,1))D\psi(0,0)(p(0,0)), \]
where \( D\psi(\delta_1, \delta_2) \) is given by Eq. (19), for instance
\[ D\psi(0,0)(0,0) = \]
\[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi(0,0)(p(1,0)) \xi(0,0)(p(1,1)) \xi(0,0)(p(0,1)) \xi(0,0)(p(0,0)) \end{bmatrix} e^{A_0(p(0,0))} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]
\]
Now evaluating each \( D\psi(\delta_1, \delta_2) \) for a proper combinations of \( \delta_1, \delta_2 \in \{0, 1\} \) and the system parameters in Appendix A the eigenvalues of \( D\Phi(p(0,0)) \) are computed,
\[ \text{eig}(D\Phi(p(0,0))) = [0.66, 0.0]. \]

Each eigenvalue belongs to the open unit disk, hence the closed orbit with transitions at the points \( p(0,1), \quad p(1,1), \quad p(1,0) \) and \( p(0,0) \) is a stable limit cycle.

7. CONCLUSION

A method for analyzing limit cycles arising in autonomous switched systems was proposed. It was shown how to glue the state spaces of the switched systems to get a single coherent space. It was demonstrated that by relatively simple computation, stability of the Poincaré map is determined. The method applies generally to switched non-linear systems as long as the transversality condition is fulfilled. However, explicit solution was provided for switched affine systems.

The method was applied for a simplified model of a supermarkets refrigeration consisting of two display cases and a compressor unit. The approach developed in this paper shows that a limit cycle generated by interaction of distributed hysteresis controllers is stable, thus synchronisation in the supermarket refrigeration system takes place.

REFERENCES


Appendix A. REFRIGERATOR PROPERTIES AND SIMULATION PARAMETERS

A number of refrigerator depended properties is used in the model described in Section 3. These properties can be either computed by using the freeware software package "RefEqns" for Matlab (Skovrup (2000)) or estimated as follows. The refrigerator R134a is employed. The first order approximations of the evaporation temperature \( T_e \) and the density \( \rho_{suc} \) in the vicinity of \( P_{wall} = 1.5 \text{bar} \) are
\[ T_e = 16.2, \quad T_{suc} = 41.9, \quad \rho_{suc} = 4.6. \]

In the model described in Section 3 the following parameters are used:

**Display cases**

| \( U_A_{wall-ref, max} \) | 500 kg\( \cdot \)K | \( M_{air} \) | 50 kg |
| \( U_A_{goods-air} \) | 300 kg\( \cdot \)K | \( C_p, air \) | 1000 kg\( \cdot \)K |
| \( C_p, goods \) | 1000 kg\( \cdot \)K | \( M_{r, max} \) | 1 kg |
| \( U_A_{wall-air} \) | 385 kg\( \cdot \)K | \( T_{f \_{air}} \) | 40 K |
| \( U_A_{wall-air} \) | 385 kg\( \cdot \)K | \( T_{S\_{bar}} \) | 3.5 °C |
| \( U_A_{wall-air} \) | 385 kg\( \cdot \)K | \( \theta_0 \) | 1.0 kg/s |
| \( U_A_{wall-air} \) | 385 kg\( \cdot \)K | \( d_{P, max} \) | 4.6 m\( \cdot \)bar |

The same parameters are used for all display cases.

**Compressor**

| \( V_{sl} \) | 0.08 m\( ^3 \) | \( \eta_{vol} \) | 0.81 |
| \( V_{suc} \) | 5.00 m\( ^3 \) |
| \( V_{suc} \) | 5.00 m\( ^3 \) |