Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

State-Dependent Riccati Equation (SDRE) Control: A Survey

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Abstract: Since the mid-90's, State-Dependent Riccati Equation (SDRE) strategies have emerged as general design methods that provide a systematic and effective means of designing nonlinear controllers, observers, and filters. These methods overcome many of the difficulties and shortcomings of existing methodologies, and deliver computationally simple algorithms that have been highly effective in a variety of practical and meaningful applications. In a special session at the 17th IFAC Symposium on Automatic Control in Aerospace 2007, theoreticians and practitioners in this area of research were brought together to discuss and present SDRE-based design methodologies as well as review the supporting theory. It became evident that the number of successful simulation, experimental and practical real-world applications of SDRE control have outpaced the available theoretical results. This paper reviews the theory developed to date on SDRE nonlinear regulation for solving nonlinear optimal control problems, and discusses issues that are still open for investigation. Existence of solutions as well as stability and optimality properties associated with SDRE controllers are the main contribution in the paper. The capabilities, design flexibility and art of systematically carrying out an effective SDRE design are also emphasized.

1. INTRODUCTION

During the 1950's and 1960's, aerospace engineering applications greatly stimulated the development of optimal control theory, where the objective was to drive the system states in such a way that some defined cost function is minimized. This turned out to have very useful applications in the design of *regulators* (where some steady state is to be maintained) and in *tracking* control strategies (where some predetermined state trajectory is to be followed). Among such applications was the problem of optimal flight trajectories for aircraft and space vehicles. Linear optimal control theory, in particular, has been very well documented and widely applied, where the plant that is controlled is assumed linear and the feedback controller is constrained to be linear with respect to its input. In recent years, however, the availability of powerful low-cost microprocessors has spurred great advantages in the theory and applications of nonlinear control. The competitive era of rapid technological change and aerospace exploration now demands stringent accuracy and cost requirements in nonlinear control systems. This has motivated the rapid development of nonlinear control theory for application to challenging complex dynamical real-world problems, particularly those that bear major practical significance in the aerospace, marine and defense industries. Despite recent advances, however, there remain many unsolved problems, so much so that practitioners often complain about the inapplicability of contemporary theories. For example, most of the techniques developed have very limited applicability because of the strong conditions imposed on the system. Control system designers continue to strive for control algorithms that are systematic, simple, and yet optimize performance, providing tradeoffs between control effort and state errors.

The ***State-Dependent Riccati Equation*** (***SDRE***) strategy is well-known and has become very popular within the control community over the last decade, providing a very effective algorithm for synthesizing nonlinear feedback controls by allowing nonlinearities in the system states while additionally offering great design flexibility through state-dependent weighting matrices. This method, first proposed by Pearson (1962) and later expanded by Wernli & Cook (1975), was independently studied by Mracek & Cloutier (1998) and alluded to by Friedland (1996). The method entails factorization (that is, parameterization) of the nonlinear dynamics into the state vector and the product of a matrix-valued function that depends on the state itself. In doing so, the SDRE algorithm fully captures the nonlinearities of the system, bringing the nonlinear system to a (nonunique) linear structure having *state-dependent coefficient* (SDC) matrices, and minimizing a nonlinear performance index having a quadratic-like structure. An algebraic Riccati equation (ARE) using the SDC matrices is then solved on-line to give the suboptimum control law. The coefficients of this equation vary with the given point in state space. The algorithm thus involves solving, at a given point in state space, an algebraic *state-dependent Riccati equation*, or *SDRE*. The nonuniqueness of the parameterization creates extra degrees of freedom, which can be used to enhance controller performance. In Cloutier, D'Souza & Mracek (1996) and Mracek & Cloutier (1998) it is shown that the SDRE feedback scheme for the infinite-time nonlinear optimal control problem (with control terms that appear affine in the dynamics and quadratically in the cost) in the *multivariable* case is *locally asymptotically stable* and *locally asymptotically optimal*, and in the *scalar* case is *optimal*. It is also shown in the general *multivariable* case that the Pontryagin necessary conditions for optimality are satisfied *asymptotically* by the algorithm.

The theoretical contribution in Cloutier, D'Souza & Mracek (1996) and Mracek & Cloutier (1998) has initiated an increasing use of SDRE techniques in a wide variety of nonlinear control applications. These include *advanced guidance law development* (Cloutier & Stansbery, 1999a; Cloutier & Zipfel, 1999), *autopilot design* (Mracek & Cloutier, 1996, 1997; Cloutier & Stansbery, 2001; Menon & Ohlmeyer, 2004; Mracek, 2007), *integrated guidance and control design* (Palumbo & Jackson, 1999; Vaddi, Menon & Ohlmeyer, 2007), *satellite and spacecraft control* (Parrish & Ridgely, 1997a; Hammett, Hall & Ridgely, 1998; Stansbery & Cloutier, 2000), *control of aeroelastic systems* (Singh & Yim, 2003; Tadi, 2003), *control of oil tanker motion* (Çimen, *to appear*), *process control* (Cloutier & Stansbery, 1999b; Banks *et al.*, 2002), *robotics* (Erdem & Alleyne, 2001), *magnetic levitation* (Erdem & Alleyne, 2004), *control of systems with parasitic effects* (Friedland, 1997), *control of artificial human pancreas* (Parrish & Ridgely, 1997b), *ducted fan control* (Sznaier *et al.*, 2000; Yu *et al.*, 2001), and *various benchmark problems* (Doyle *et al.*, 1997; Mracek & Cloutier, 1998).

In a special session at the *17th IFAC Symposium on Automatic Control in Aerospace 2007*, theoreticians and practitioners in this area of research were brought together to discuss and present SDRE-based design methodologies as well as review the supporting theory developed to date (Mracek, 2007; Friedland, 2007; Çimen, McCaffrey, Harrison & Banks, 2007; Salamci & Gökbilen, 2007; Merttopçuoğlu, Kahvecioğlu & Çimen, 2007). It became evident that the number of successful simulation, experimental and practical real-world applications of SDRE-based designs have outpaced the available theoretical results.

This paper focuses on the SDRE nonlinear regulator for solving nonlinear optimal control problems, and reviews the theory developed to date. Existence of solutions as well as optimality and stability properties associated with SDRE controllers are the main contribution in the paper, discussing issues that are still open for investigation.

The rest of the paper is organized as follows. In Section 2, the *formulation* of the nonlinear optimal control problem, the concept of *extended linearization* and the *SDRE controller* for nonlinear optimal regulation are presented, reviewing the *dditional degrees of freedom* provided by the nonuniqueness of the SDC parameterization. In Section 3, the necessary and sufficient conditions for the *existence of solutions* to the nonlinear optimal control problem, in particular by SDRE feedback control, are reviewed. A theoretical study of the *stability* and *optimality* properties of SDRE feedback controls is pursued in Section 4 and Section 5, respectively. An overview of the *capabilities*, *design flexibility* and *art* of SDRE control is presented in Section 6, demonstrating how numerous systems that do not meet the basic structure and conditions required for the direct application of the SDRE technique can be systematically converted to systems having the proper structure and conditions. Finally, the survey is concluded with a discussion of *issues for investigation* in Section 7. Due to space limitations, the practical use of the SDRE methodology is left for congress presentation.

2. SDRE NONLINEAR REGULATION

2.1 Problem Formulation

Consider the deterministic, infinite-horizon nonlinear optimal regulation (stabilization) problem, where the system is full-state observable, autonomous, nonlinear in the state, and affine in the input, represented in the form

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{u} \in \mathbb{R}^m$ is the input vector, and $t \in [0, \infty)$, with $C^1(\mathbb{R}^n)$ functions $\mathbf{f}: \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{B}: \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $\mathbf{B}(\mathbf{x}) \neq \mathbf{0} \quad \forall \mathbf{x}$. Without any loss of generality, the origin $\mathbf{x} = \mathbf{0}$ is assumed to be an *equilibrium point*, such that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. In this context, the minimization of the *infinite-time* performance criterion

$$J(\mathbf{x}_0, \mathbf{u}(\cdot)) = \tfrac{1}{2}\int_0^\infty \left\{ \mathbf{x}^T(t)\mathbf{Q}(\mathbf{x})\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}(\mathbf{x})\mathbf{u}(t) \right\} dt \tag{2}$$

is considered, which is *nonquadratic* in $\mathbf{x}$ but *quadratic* in $\mathbf{u}$. The *state* and *input* weighting matrices are assumed *state-dependent* such that $\mathbf{Q}: \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $\mathbf{R}: \mathbb{R}^n \to \mathbb{R}^{m\times m}$. These design parameters satisfy $\mathbf{Q}(\mathbf{x}) \geq \mathbf{0}$ and $\mathbf{R}(\mathbf{x}) > \mathbf{0}$ for all $\mathbf{x}$. Under the specified conditions, a control law

$$\mathbf{u}(\mathbf{x}) = \mathbf{k}(\mathbf{x}) = -\mathbf{K}(\mathbf{x})\mathbf{x}, \quad \mathbf{k}(\mathbf{0}) = \mathbf{0}\,, \tag{3}$$

where $\mathbf{k}(\cdot) \in C^1(\mathbb{R}^n)$, is then sought that will (approximately) minimize the cost (2) subject to the input-affine nonlinear differential constraint (1) while regulating the system to the origin $\forall \mathbf{x}$, such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$. This problem forms the basis of the SDRE method for *nonlinear regulation*.

2.2 Extended Linearization

Extended linearization (Friedland, 1996), also known as *apparent linearization* (Wernli & Cook, 1975) or *SDC parameterization* (Cloutier, D'Souza & Mracek, 1996; Mracek & Cloutier, 1998), is the process of *factorizing* a nonlinear system into a linear-like structure which contains SDC matrices. Under the assumptions $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{f}(\cdot) \in C^1(\mathbb{R}^n)$, a continuous nonlinear matrix-valued function $\mathbf{A}(\mathbf{x})$ always exists such that

$$\mathbf{f}(\mathbf{x}) = \mathbf{A}(\mathbf{x})\mathbf{x}\,, \tag{4}$$

where $\mathbf{A}: \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is found by *mathematical factorization* and is, clearly, *nonunique* when $n > 1$. Hence, extended linearization of the input-affine nonlinear system (1) becomes

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x})\mathbf{x}(t) + \mathbf{B}(\mathbf{x})\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0\,, \tag{5}$$

which has a *linear* structure with SDC matrices $\mathbf{A}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$. The application of any *linear control synthesis* method to the linear-like SDC structure (5), where $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are treated as constant matrices, forms an *extended linearization control method*. These represent a rather broad class of control design methods, leading to nonlinear control laws of form (3) that render the closed-loop dynamics (SDC) matrix

$$\mathbf{A}_{CL}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{K}(\mathbf{x}) \tag{6}$$

pointwise Hurwitz.

The recoverability of nonlinear state feedback laws using extended linearization control techniques has been investigated by Cloutier, Stansbery & Sznaier (1999). By recoverable it is meant that a given nonlinear state feedback law of form (3) can be obtained (or *recovered*) from a given control design method. Necessary and sufficient conditions for the recoverability of a given nonlinear state feedback control law by some extended linearization control technique, and in particular, by the SDRE method, have been provided by Cloutier, Stansbery & Sznaier (1999). These will be reviewed in Section 3.2.

2.3 SDRE Controller Structure

The SDRE methodology uses *extended linearization* as the key design concept in formulating the nonlinear optimal control problem. The underlying linear control synthesis method in this case is the LQR synthesis method. Motivated by the LQR problem, which is characterized by an ARE, SDRE feedback control is an "extended linearization control method" that provides a similar approach to the nonlinear regulation problem for the input-affine system (1) with cost functional (2). By mimicking the LQR formulation, the state-feedback controller is obtained in the form

$$\mathbf{u}(\mathbf{x}) = -\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x})\mathbf{x}, \tag{7}$$

where $\mathbf{P}(\mathbf{x})$ is the unique, symmetric, positive-definite solution of the algebraic *State-Dependent Riccati Equation*

$$\begin{aligned}&\mathbf{P}(\mathbf{x})\mathbf{A}(\mathbf{x}) + \mathbf{A}^T(\mathbf{x})\mathbf{P}(\mathbf{x})\\ &\quad -\mathbf{P}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) + \mathbf{Q}(\mathbf{x}) = \mathbf{0},\end{aligned} \tag{8}$$

hence the name *SDRE control*. The resulting SDRE-controlled trajectory becomes the solution of the *quasilinear* closed-loop dynamics

$$\dot{\mathbf{x}}(t) = \left[\mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x})\right]\mathbf{x}(t), \tag{9}$$

such that the state-feedback gain in (6) for minimizing (2) is

$$\mathbf{K}(\mathbf{x}) = \mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x}).$$

The SDRE solution to the infinite-horizon autonomous nonlinear regulator problem (1) and (2) is, therefore, a true generalization of the infinite-horizon time-invariant LQR problem, where all of the coefficient matrices are state-dependent. At each instant, the method treats the state-dependent coefficients matrices as being constant, and computes a control action by solving an LQ optimal control problem. As is evident from (8), the resulting controller relies on a solution, pointwise in $\mathbb{R}^n$, of an ARE thereby leading to the SDRE terminology. There is no attempt to solve the HJB equation as outlined, for example, in Lukes (1969). The clearest benefit of the SDRE algorithm is its simplicity and its apparent effectiveness. When the coefficient and weighting matrices are constant, the nonlinear regulator problem collapses to the LQR problem and the SDRE control method collapses to the steady-state linear regulator.

2.4 Additional Degrees of Freedom

For *scalar* systems, the SDC parameterization is unique for all $x \neq 0$, given by (Cloutier, D'Souza & Mracek, 1996)

$$a(x) = \tfrac{f(x)}{x}.$$

For *multivariable* problems, however, $\mathbf{x}$ has at least two components, x_1 and x_2. Assuming that there is a single scalar nonlinear term $f_i(\mathbf{x})$ appearing in one of the state equations, then in that state equation one parameterization has the nonlinearity $f_i(\mathbf{x})/x_1$ appearing as a coefficient of x_1 while a second parameterization has the nonlinearity $f_i(\mathbf{x})/x_2$ appearing as a coefficient of x_2. Thus, there always exists at least two parameterizations. Suppose $\mathbf{A}_1(\mathbf{x})$ and $\mathbf{A}_2(\mathbf{x})$ are two distinct SDC parameterizations, such that $\mathbf{f}(\mathbf{x}) = \mathbf{A}_1(\mathbf{x})\mathbf{x} = \mathbf{A}_2(\mathbf{x})\mathbf{x}$. Then

$$\mathbf{A}(\mathbf{x}, \alpha) = \alpha\mathbf{A}_1(\mathbf{x}) + (1-\alpha)\mathbf{A}_2(\mathbf{x})$$

is also an SDC parameterization for any α, which is easily verified by multiplying both sides with $\mathbf{x}$. Therefore, $\mathbf{A}(\mathbf{x}, \alpha)$ represents an *infinite* family of SDC parameterizations contained in a line.

The nonuniqueness of the SDC parameterization for multivariable systems creates additional degrees of freedom. In general, an SDC parameterization $\mathbf{A}(\mathbf{x}, \boldsymbol{\alpha})$ can be constructed which is the parametric representation of a hypersurface containing $k+1$ distinct parameterizations (if they exist), where $\boldsymbol{\alpha}$ is a vector of dimension k, and $\mathbf{A}(\mathbf{x}, \boldsymbol{\alpha})$ will be of the form

$$\mathbf{A}(\mathbf{x}, \boldsymbol{\alpha}) = (1-\alpha_k)\mathbf{A}_{k+1}(\mathbf{x}) + \sum_{i=1}^{k}\Big(\prod_{j=i}^{k}\alpha_j\Big)(1-\alpha_{i-1})\mathbf{A}_i(\mathbf{x}),$$

where $\alpha_0 \triangleq 0$ (Cloutier, D'Souza & Mracek, 1996). Note that if a hypersurface of parameterizations is formed to obtain $\mathbf{A}(\mathbf{x}, \boldsymbol{\alpha})$, the solution of the SDRE will be of the form $\mathbf{P}(\mathbf{x}, \boldsymbol{\alpha})$. This results in the nonlinear feedback controller being parameterized by $\boldsymbol{\alpha}$. The additional degrees of freedom available through $\boldsymbol{\alpha}$ provides *design flexibility* that can be used to enhance performance or effect tradeoffs between performance, optimality, stability, robustness, and disturbance rejection. These will be discussed in Section 6.

The implications of the nonuniqueness of the state-dependent quasilinear representation (5) have been considered by Huang & Lu (1996), Cloutier, Stansbery & Sznaier (1999) and Shamma & Cloutier (2003), and will be discussed in the paper. It is important to realize at this point that the issue of nonuniqueness plays a major role in not only recovering the *global optimal control*, but also achieving *global asymptotic stability*. In general, the solution provided by SDRE control (7) and (8) does not recover global optimality with respect to the performance index (2) for some arbitrary choice of the SDC matrix $\mathbf{A}(\mathbf{x})$. Moreover, a proper choice of $\mathbf{A}(\mathbf{x})$ also plays a significant role in affecting the controllability of the resulting parameterized pair $\{\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x})\}$. Note that the presence or lack of controllability of this pair need not have any implication on the controllability of the original dynamics given in (1). These issues have been considered in Hammett, Hall & Ridgely (1998).

3. EXISTENCE OF SOLUTIONS

3.1 HJB Eq., Lagrangian Manifolds and Viscosity Solutions

Let us first make some comments about the *existence* of the dynamic programming solution to the infinite-time horizon nonlinear optimal control problem (1), (2) in order to justify the later hypotheses. Suppose that $\mathbf{f}$, $\mathbf{B}$, $\mathbf{Q}$, and $\mathbf{R}$ are sufficiently smooth functions so that the *value function* defined by (Anderson & Moore, 1990)

$$V(\mathbf{x}) \triangleq \inf_{\mathbf{u}(\cdot)\in U} J(\mathbf{x}, \mathbf{u}(\cdot)) \tag{10}$$

is continuously differentiable, the inf being over the given set of *admissible controls* $U \in L_2(0, \infty)$. Ideally, the desired value function V is a stationary solution to the Cauchy problem for the associated HJB partial differential equation

$$\tfrac{\partial}{\partial t}V(\mathbf{x}) + \inf_{\mathbf{u}(\cdot)\in U} H(\mathbf{x}, \mathbf{u}, \tfrac{\partial}{\partial \mathbf{x}}V(\mathbf{x})) = 0,$$

where H is the Hamiltonian function. For the given *infinite-time formulation* (1) and (2), the Hamiltonian is

$$H = \tfrac{\partial V^T(\mathbf{x})}{\partial \mathbf{x}}\left[\mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}\right] + \tfrac{1}{2}\left[\mathbf{x}^T\mathbf{Q}(\mathbf{x})\mathbf{x} + \mathbf{u}^T\mathbf{R}(\mathbf{x})\mathbf{u}\right], \tag{11}$$

and the HJB equation becomes

$$\tfrac{\partial V^T(\mathbf{x})}{\partial \mathbf{x}}\left[\mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}\right] + \tfrac{1}{2}\left[\mathbf{x}^T\mathbf{Q}(\mathbf{x})\mathbf{x} + \mathbf{u}^T\mathbf{R}(\mathbf{x})\mathbf{u}\right] = 0 \tag{12}$$

with boundary condition $V(\mathbf{0}) = 0$, since $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$. One approach is to consider a family of finite-time problems, which is the standard approach to solving the infinite-horizon LQR problem, and requires *stabilizability* and *detectability*. This has been extended to nonlinear systems of the type considered in (1) by Lukes (1969) for optimal control, and by van der Schaft (1991) for H_∞ control. The key to their analysis is the link between stationary solutions to (12) and stable Lagrangian manifolds for the corresponding Hamiltonian dynamics

$$\dot{\mathbf{x}} = \tfrac{\partial H}{\partial \boldsymbol{\lambda}}, \quad \dot{\boldsymbol{\lambda}} = -\tfrac{\partial H}{\partial \mathbf{x}} \tag{13}$$

for state $\mathbf{x}$ and adjoint variable $\boldsymbol{\lambda}$ arising from the maximum principle, which have a hyperbolic equilibrium at the origin.

Hypotheses 1. *The linearization of (1), (2) at the equilibrium is stabilizable and detectable, that is, the triple* $\left\{\tfrac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{0}), \mathbf{B}(\mathbf{0}), \mathbf{Q}^{1/2}(\mathbf{0})\right\}$ *is stabilizable and detectable.*

Lemma 1 (Lemma 3, Doyle *et al.*, 1989**).** *Under Hypothesis 1, the equilibrium is hyperbolic. Thus there exists a stable Lagrangian manifold L for the Hamiltonian dynamics (13) corresponding to (1) and (2).*

Hypotheses 1 can be used to construct a smooth $V(\mathbf{x})$ geometrically in a neighborhood of the origin. The existence of a solution to the linearized problem at the origin, by this assumption, implies the existence of a stable Lagrangian manifold L through the origin. Furthermore, it implies that L locally has a well-defined projection onto state space and the corresponding stationary solution $V(\mathbf{x})$ is smooth. $V(\mathbf{x})$ is in fact the generating function for L. This means that, for $\boldsymbol{\lambda} = \partial V/\partial \mathbf{x}$, L is the set of points $(\mathbf{x}, \boldsymbol{\lambda})$ in phase space and $dV(\mathbf{x}) = \boldsymbol{\lambda}\, d\mathbf{x}$ along trajectories of the Hamiltonian flow lying on L (van der Schaft, 1991). It also follows that the optimal control is the feedback

$$\mathbf{u}^*(\mathbf{x}) = -\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\tfrac{\partial V^T(\mathbf{x})}{\partial \mathbf{x}}. \tag{14}$$

This is the nonlinear extension of the feedback that solves the linear problem.

Smoothness breaks down when the optimal trajectories start to cross (going backwards in time). At such points, singularities develop in the projection of L onto state space and $\int \boldsymbol{\lambda}\, d\mathbf{x}$ no longer gives a well-defined function of $\mathbf{x}$. However, the value function for the optimal control problem is still well-defined beyond such points and is in fact a stationary *viscosity solution* to (12), provided it is locally bounded. For the particular case considered here, where local assumptions imply the existence of a stable manifold L, Day (1998) has recently shown how to construct from L a stationary viscosity solution $V(\mathbf{x})$ to (12) beyond points at which optimal trajectories start to cross and smoothness breaks down. In addition to the conditions of local *stabilizability* and *detectability* at the origin (Hypotheses 1), the function $V(\mathbf{x})$ must be *locally Lipschitz* in order for it to be a *viscosity solution* to (12) (Theorem 3, Day, 1998). This condition is formally stated in the following Hypothesis.

Hypothesis 2. *The value function* $V(\mathbf{x})$ *defined by (10) in (12) is locally Lipschitz in a region* Ω *around the origin.*

So, to summarize, Hypotheses 1 ensures that there exists a locally smooth optimal solution $V(\mathbf{x})$. In addition, by Hypothesis 2, a larger region of the origin is assumed to exist on which $V(\mathbf{x})$ is *locally Lipschitz*.

In the region where $V(\mathbf{x})$ is a smooth nonnegative solution to (12), the minimum is achieved by (14), so that by substitution

$$\tfrac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) - \tfrac{1}{2}\tfrac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\tfrac{\partial V^T(\mathbf{x})}{\partial \mathbf{x}} + \tfrac{1}{2}\mathbf{x}^T\mathbf{Q}(\mathbf{x})\mathbf{x} = \mathbf{0} \tag{15}$$

and the corresponding optimal cost $V(\mathbf{x})$ is the solution to (15). Since $\partial V(\mathbf{0})/\partial \mathbf{x} = \mathbf{0}$ (van der Schaft, 1991), $\partial V(\mathbf{x})/\partial \mathbf{x}$ can be written in the form

$$\tfrac{\partial V(\mathbf{x})}{\partial \mathbf{x}} = \mathbf{P}(\mathbf{x})\mathbf{x} \tag{16}$$

for some matrix-valued function $\mathbf{P}(\mathbf{x})$. Also, $\mathbf{f}(\mathbf{x})$ can be given by (4) for some matrix-valued function $\mathbf{A}(\mathbf{x})$.

Remark 1. For any choice of $\mathbf{A}(\mathbf{x})$ satisfying (4), $\mathbf{A}(\mathbf{x}) \to \partial \mathbf{f}(\mathbf{0})/\partial \mathbf{x}$ as $\mathbf{x} \to \mathbf{0}$, that is, $\tfrac{\partial \mathbf{f}(\mathbf{0})}{\partial \mathbf{x}} = \mathbf{A}(\mathbf{0})$. Consequently, it is obvious that if the Jacobian linearization of (1) is unstabilizable, there is no SDC parameterization $\mathbf{A}(\mathbf{x})$ to satisfy Hypotheses 1 such that the pair $\{\mathbf{A}(\mathbf{0}), \mathbf{B}(\mathbf{0})\}$ is stabilizable.

Now, substituting for $\partial V(\mathbf{x})/\partial \mathbf{x}$ and $\mathbf{f}(\mathbf{x})$, (15) becomes

$$\begin{aligned}&\mathbf{x}^T[\mathbf{P}(\mathbf{x})\mathbf{A}(\mathbf{x}) + \mathbf{A}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) \\ &\qquad -\mathbf{P}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) + \mathbf{Q}(\mathbf{x})]\mathbf{x} = 0\end{aligned} \tag{17}$$

In the linear case, the ARE is obtained directly from (17). However, since $\mathbf{A}$ is a matrix-valued function of $\mathbf{x}$, the quantity inside the parenthesis in (17) cannot be set to zero.

Unfortunately, the complexity of the HJB equation (15) prevents any solution except in some very simple, low dimensional systems. To make real-time implementation possible, one has to avoid solving any partial differential equation or two-point boundary-value problem. This has prompted control design engineers to search for alternative, suboptimal approaches to the problem, such as the SDRE technique. The SDRE approach provides an approximation to the solution of (17) (and thus the HJB equation (15)), and yields a suboptimal feedback control law for the infinite-horizon optimization problem defined by (1) and (2). Application of the SDRE algorithm as an approximation to the solution of (17) involves ignoring the requirement that $\mathbf{P}(\mathbf{x})\mathbf{x}$ be the gradient of some function, and assumes instead that $\mathbf{P}(\mathbf{x})$ is *symmetric*. Then, at any given $\mathbf{x}$, the SDRE algorithm consists of simply finding the symmetric positive-definite solution $\mathbf{P}(\mathbf{x})$ to the algebraic SDRE (8), and applying, at that $\mathbf{x}$, the control (7). This approach is much more appealing than solving the HJB equation (15).

Remark 2. Although the heuristic derivation of the SDRE algorithm takes place in the region where $V(\mathbf{x})$ is smooth, it can clearly be applied independently of this assumption. In fact, the stability analysis presented in Section 4.3 assumes only that $V(\mathbf{x})$ is Lipschitz (Hypothesis 2).

3.2 Existence of SDRE Stabilizing Feedback Controls

Cloutier, Stansbery & Sznaier (1999) derived the necessary condition on $\mathbf{f}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ for the existence of any feedback gain matrix, $\mathbf{K}(\mathbf{x})$, that results in (6) being pointwise Hurwitz. First, let us state the following system-theoretic concept definitions, pointwise in $\mathbf{x}$, associated with the existence of SDRE stabilizing feedback controls.

Definition 1. *The SDC representation (5) is a stabilizable (controllable) parameterization of the nonlinear system (1) in a region* $\Omega \in \mathbb{R}^n$ *if the pair* $\{\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x})\}$ *is pointwise stabilizable (controllable) in the linear sense for all* $\mathbf{x} \in \Omega$.

Definition 2. *The SDC representation (5) is a detectable (observable) parameterization of the nonlinear system (1) in a region* $\Omega \in \mathbb{R}^n$ *if the pair* $\{\mathbf{A}(\mathbf{x}), \mathbf{Q}^{1/2}(\mathbf{x})\}$ *is pointwise detectable (observable) in the linear sense for all* $\mathbf{x} \in \Omega$.

Definition 3. *The SDC representation (5) is pointwise Hurwitz in a region* Ω *if the eigenvalues of* $\mathbf{A}(\mathbf{x})$ *are in the open left half plane* $\mathrm{Re}(s) < 0$ *(that is, have negative real parts) for all* $\mathbf{x} \in \Omega$.

Definition 4. *A* $C^1(\mathbb{R}^n)$ *control law (3) is said to be recoverable by SDRE control in a region* Ω *if there exists a pointwise stabilizable SDC parameterization* $\{\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x})\}$, *a pointwise positive-semidefinite state weighting matrix* $\mathbf{Q}(\mathbf{x})$, *and a pointwise positive-definite control weighting matrix* $\mathbf{R}(\mathbf{x})$ *such that the resulting state-dependent controller (7) satisfies (3) for all* $\mathbf{x}$.

Theorem 1 (Cloutier, Stansbery & Sznaier, 1999). *A* $C^1(\mathbb{R}^n)$ *control law (3) is recoverable by SDRE control in a region* Ω *if there exists a pointwise stabilizable SDC parameterization* $\{\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x})\}$ *such that the closed-loop dynamics matrix* (6) *is pointwise Hurwitz in* Ω, *and the gain* $\mathbf{K}(\mathbf{x})$ *satisfies the pointwise minimum-phase property in* Ω, *that is, the zeros of the loop gain* $\mathbf{K}(\mathbf{x})[s\mathbf{I} - \mathbf{A}(\mathbf{x})]^{-1}\mathbf{B}(\mathbf{x})$ *lie in the closed left half plane* $\mathrm{Re}(s) \le 0$, *pointwise.*

Although Theorem 1 provides the necessary and sufficient conditions for recoverability of SDRE controls, it is difficult to apply this theorem due to the fact that there are an infinite number of SDC parameterizations.

4. STABILITY ANALYSES

4.1 Local Asymptotic Stability

The following conditions are required for guaranteeing local asymptotic stability.

Hypotheses 3. $\mathbf{A}(\cdot)$, $\mathbf{B}(\cdot)$, $\mathbf{Q}(\cdot)$ *and* $\mathbf{R}(\cdot)$ *are* $C^1(\mathbb{R}^n)$ *matrix-valued functions.*

Hypotheses 4. *The respective pairs* $\{\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x})\}$ *and* $\{\mathbf{A}(\mathbf{x}), \mathbf{Q}^{1/2}(\mathbf{x})\}$ *are pointwise stabilizable and detectable SDC parameterizations of the nonlinear system (1) for all* $\mathbf{x}$.

Remark 3. A sufficient test for the stabilizability condition in Hypothesis 4 is to check that the controllability matrix

$$\mathbf{M}_C = \left[\mathbf{B}(\mathbf{x}) \mid \mathbf{A}(\mathbf{x})\mathbf{B}(\mathbf{x}) \mid \cdots \mid \mathbf{A}^{n-1}(\mathbf{x})\mathbf{B}(\mathbf{x})\right].$$

has $\mathrm{rank}(\mathbf{M}_C) = n \quad \forall \mathbf{x} \in \mathbb{R}^n$. Similarly, a sufficient test for detectability is that the observability matrix

$$\mathbf{M}_O = \left[\mathbf{Q}^{1/2}(\mathbf{x}) \mid \mathbf{Q}^{1/2}(\mathbf{x})\mathbf{A}(\mathbf{x}) \mid \ldots \mid \mathbf{Q}^{1/2}(\mathbf{x})\mathbf{A}^{n-1}(\mathbf{x})\right].$$

has $\mathrm{rank}(\mathbf{M}_O) = n \quad \forall \mathbf{x} \in \mathbb{R}^n$. This can be guaranteed by ensuring that $\mathbf{Q}(\mathbf{x})$ is positive-definite $\forall \mathbf{x} \in \mathbb{R}^n$.

Theorem 2 (Mracek & Cloutier, 1998). *Consider the nonlinear multivariable system (1) with feedback control (7) applied, where* $\mathbf{x} \in \mathbb{R}^n$ $(n > 1)$ *and* $\mathbf{P}(\mathbf{x})$ *is the unique, symmetric, positive-definite, pointwise-stabilizing solution of the SDRE (8). Then, under Hypotheses 3 and 4, the SDRE method produces a closed-loop solution which is locally asymptotically stable.*

Proof. Using SDRE control, the closed-loop solution becomes $\dot{\mathbf{x}} = \mathbf{A}_{CL}(\mathbf{x})\mathbf{x}$, where $\mathbf{A}_{CL}(\mathbf{x})$ is the closed-loop SDC matrix given by (6). From Riccati equation theory, $\mathbf{A}_{CL}(\mathbf{x})$ is guaranteed to be stable at every point $\mathbf{x}$. Under the smoothness assumptions of Hypotheses 3, $\mathbf{P}(\mathbf{x})$ is $C^1(\mathbb{R}^n)$ and hence so is $\mathbf{A}_{CL}(\mathbf{x})$. Applying the Mean Value Theorem to $\mathbf{A}_{CL}(\mathbf{x})$ gives

$$\mathbf{A}_{CL}(\mathbf{x}) = \mathbf{A}_{CL}(\mathbf{0}) + \tfrac{\partial \mathbf{A}_{CL}(\mathbf{z})}{\partial \mathbf{x}}\mathbf{x},$$

where $\partial \mathbf{A}_{CL}(\mathbf{z})/\partial \mathbf{x}$ generates a tensor, and the vector $\mathbf{z}$ is that point on the line segment joining the origin $\mathbf{0}$ and $\mathbf{x}$. By substitution,

$$\dot{\mathbf{x}} = \mathbf{A}_{CL}(\mathbf{0})\mathbf{x} + \mathbf{x}^T \frac{\partial \mathbf{A}_{CL}(\mathbf{z})}{\partial \mathbf{x}} \mathbf{x},$$

which gives

$$\dot{\mathbf{x}} = \mathbf{A}_{CL}(\mathbf{0})\mathbf{x} + \psi(\mathbf{x}, \mathbf{z}) \cdot \|\mathbf{x}\|,$$

where $\psi(\mathbf{x}, \mathbf{z}) \triangleq \frac{1}{\|\mathbf{x}\|} \mathbf{x}^T \frac{\partial \mathbf{A}_{CL}(\mathbf{z})}{\partial \mathbf{x}} \mathbf{x}$, such that $\lim_{\|\mathbf{x}\| \to 0} \psi(\mathbf{x}, \mathbf{z}) = \mathbf{0}$. Hence, in a neighborhood about the origin, the linear term which has a constant stable coefficient matrix $\mathbf{A}_{CL}(\mathbf{0})$ dominates the higher-order term, yielding local asymptotic stability. □

Theorem 2 presents the rather mild conditions that guarantee local asymptotic stability of the SDRE closed-loop solution. Since the characterization of the resulting SDRE controller has a similar structure to the LQR problem, in order that the SDRE (8) have a positive-semidefinite solution for all $\mathbf{x}$, it is sufficient that $\{\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x}), \mathbf{Q}^{1/2}(\mathbf{x})\}$ be pointwise *stabilizable* and *detectable* for all $\mathbf{x}$. The SDRE algorithm then gives a smooth feedback.

4.2 Global Asymptotic Stability

Global asymptotic stability of the closed-loop system implies that it is possible to regulate the states to the origin regardless of the initial conditions. This is obviously a very desirable property, however, it is usually difficult to achieve and/or prove. Due to the nature of LQR formulation, under Hypotheses 1, the origin of the SDRE controlled system is locally asymptotically stable, that is, all eigenvalues of the closed-loop dynamics matrix (6) have negative real parts at $\mathbf{x} = \mathbf{0}$. However, this property is not sufficient to deduce the global stability of a nonlinear system. In fact, even if all eigenvalues of $\mathbf{A}_{CL}(\mathbf{x})$ have negative real parts $\forall \mathbf{x} \in \mathbb{R}^n$, global stability of a nonlinear system still cannot be guaranteed.

Global stability results are now presented for two cases. In the first, the closed-loop coefficient matrix $\mathbf{A}_{CL}(\mathbf{x})$ is assumed to have a special structure. The second case concerns scalar systems, where $\mathbf{x} \in \mathbb{R}^n$ with $n = 1$.

Theorem 3 (Cloutier, D'Souza & Mracek, 1996)**.** *If the closed-loop coefficient matrix $\mathbf{A}_{CL}(\mathbf{x})$ is symmetric for all $\mathbf{x}$, then under the conditions given by Hypotheses 3 and 4, the SDRE closed-loop solution is globally asymptotically stable.*

Proof. Let $V(\mathbf{x}) = \mathbf{x}^T \mathbf{x}$ be the candidate Lyapunov function. Then

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \mathbf{x}^T \dot{\mathbf{x}} + \dot{\mathbf{x}}^T \mathbf{x} \\ &= \mathbf{x}^T \left[\mathbf{A}_{CL}(\mathbf{x}) + \mathbf{A}_{CL}{}^T(\mathbf{x}) \right] \mathbf{x}. \end{aligned} \tag{18}$$

Under the given assumptions, $\mathbf{A}_{CL}(\mathbf{x})$ is known to be stable for all $\mathbf{x}$. Therefore, if $\mathbf{A}_{CL}(\mathbf{x})$ is symmetric, then $\mathbf{A}_{CL}(\mathbf{x})$ is negative-definite, which implies that $\dot{V} < 0$ for all $\mathbf{x}$. □

Theorem 4 (Cloutier, D'Souza & Mracek, 1996)**.** *In the scalar case ($n = 1$), the SDRE closed-loop solution is globally asymptotically stable.*

Proof. Global stability under SDRE feedback control can easily be deduced using the Lyapunov function $V(x) = \int_0^x p(\tau)\tau \, d\tau$ or $V(x) = 0.5x^2$. □

In terms of such features as optimality, stability, and real-time implementability, Theorem 4 on SDRE control of scalar systems provides the best result. While its extension to high order systems is possible (Qu, Cloutier & Mracek, 1996; Mracek & Cloutier, 1998), optimality (suboptimality) and global stability can only be guaranteed under several conditions. For second-order systems in canonical form with single input and constant $\mathbf{B}$ matrix, Erdem & Alleyne (2004) have shown that the origin can be globally asymptotically stabilized relatively easily by arbitrary, constant, positive choices of $\mathbf{Q} = \text{diag}\{q_1, q_2\}$ and $\mathbf{R} = r$. For higher-order systems, Langson & Alleyne (2002) have shown that the SDRE technique yields globally asymptotically stabilizing controls for a class of nonlinear systems satisfying certain growth conditions. Unfortunately, the global upperbound expressed in their proposition is very conservative, and is not easy to determine or enforce by using feedback control *a priori* for global asymptotic stabilization.

4.3 A Stability Test for Estimating the Region of Attraction

As an alternative to global asymptotic stability, which is usually difficult to achieve and/or prove, it is desirable to be able to estimate the region of attraction for asymptotic stability. This is the region in state space which encloses all initial conditions such that when the system is steered from them, the origin will be reached asymptotically.

There have been very little results in the literature on the estimation of the region of attraction for SDRE regulated systems, with rather conservative results. Recently, however, McCaffrey & Banks (2001) proposed a stability test for determining the size of the region on which *large scale asymptotic stability* holds for the SDRE algorithm. The resulting test involves evaluating an inequality along trajectories of a Hamiltonian dynamical system, without the need to find the value function. The form of the inequality in question is already known and has been previously used in the literature to show that feedback controls sufficiently close to the optimal globally stabilizing feedback are themselves asymptotically stabilizing in the same domain. However, the key ingredient in such results is the proof of existence of a Lyapunov function corresponding to the optimal feedback, which has been based on the assumption that the Lyapunov function is *smooth*. These are the stable Lagrangian manifold arguments of Lukes (1969) and van der Schaft (1991). This assumption of smoothness limits the domain within which the Lyapunov function is known to exist to the largest state-space neighborhood of the equilibrium point onto which the stable manifold has a well-defined projection, that is, is single sheeted. The analysis in McCaffrey & Banks (2001) attempts to enlarge the region, within which the Lyapunov function is known to exist, to regions of state-space over which the stable manifold has multiple sheets. This has been done by applying the geometrical construction from Day (1998) along with various convexity arguments to prove the existence of a *nonsmooth* Lyapunov function at a point $\mathbf{x}$ in state-space from the fact that $\mathbf{x}$ is covered by one or more sheets of the stable manifold. The resulting function solves the associated Bellman equation in a viscosity sense. A well-known inequality is then used on this much larger region to test the stability of the SDRE feedback. The resulting estimate of the domain of attraction for the SDRE feedback is thus likely to be far closer to the true domain of attraction than conservative estimates arising from the smoothness assumptions of the existing literature.

In addition to the conditions of local *stabilizability* and *detectability* at the origin in Hypotheses 1, recall from the outset of Section 3.1 on how to construct a stationary viscosity solution $V(\mathbf{x})$ to (12) from a stable Lagrangian manifold L, that the function $V(\mathbf{x})$ must be *locally Lipschitz* in order for it to be a viscosity solution to (12) (Hypothesis 2). The Lipschitz property follows from the topological properties of L, under the assumption that the dimension of L is ≤ 5. However, for the stability test in McCaffrey & Banks (2001), the Lipschitz property is assumed for higher dimensional case, which, at the very worst, holds on the region of the origin on which $V(\mathbf{x})$ is smooth and, in general, on a larger region. Therefore, in the worst case, the stability arguments of McCaffrey & Banks (2001) reduce to the standard smooth Lyapunov-type arguments. This viscosity solution $V(\mathbf{x})$ to (12) is the Lyapunov function.

As outlined in Section 3.1 and above, suppose Hypotheses 1 and Hypothesis 2 hold true. Under Hypotheses 1, recall the existence of a stable manifold L by Lemma 1, and note that Ω has to be covered by L, that is, for all $\mathbf{x} \in \Omega$ there exists $\boldsymbol{\lambda}$ such that $(\mathbf{x}, \boldsymbol{\lambda}) \in L$. Assuming $\mathbf{Q}(\mathbf{x})$ is positive-definite for all $\mathbf{x} \ne \mathbf{0}$, it follows that $V(\mathbf{x}) > 0$ for $\mathbf{x} \ne \mathbf{0}$ and $V(\mathbf{0}) = 0$. Note that these are basically *sufficient conditions* for *asymptotic stability* of the exact solution to (10). The proposition by McCaffrey & Banks (2001) gives a condition under which $V(\mathbf{x})$ is also a Lyapunov function for the approximate solution given by the SDRE feedback algorithm. Their result is a modification of the proof that a stationary viscosity solution to (12) gives a Lyapunov function, with the additional element of using the stable manifold and convexity to reduce the condition to one which is more easily tested.

From van der Schaft (1991), note that the positive-definite Riccati matrix $\mathbf{P}(\mathbf{0})$ solving the linearized problem at the origin is $\partial^2 V(\mathbf{0}) / \partial \mathbf{x}^2$. Since the factorization (4) satisfies $\mathbf{A}(\mathbf{0}) = \partial \mathbf{f}(\mathbf{0}) / \partial \mathbf{x}$, the solution $\mathbf{P}(\mathbf{0})$ to the SDRE (8) at $\mathbf{x} = \mathbf{0}$ also satisfies $\mathbf{P}(\mathbf{0}) = \partial^2 V(\mathbf{0}) / \partial \mathbf{x}^2$. Also note that the points $(\mathbf{x}, \boldsymbol{\lambda}) \in L$ can be generated by following trajectories of the Hamiltonian dynamics (13) corresponding to (12) backwards in time from final conditions $(\mathbf{x}_f, \boldsymbol{\lambda}_f) \in L$ lying close to the origin. For points $\mathbf{x}_f$ lying in a small ball $B_\varepsilon \setminus \{0\}$, the final conditions can be approximated arbitrarily closely by taking $(\mathbf{x}_f, \boldsymbol{\lambda}_f)$ to lie on the tangent plane to L at the origin, which is given by

$$\boldsymbol{\lambda}_f = -\tfrac{\partial^2 V(\mathbf{0})}{\partial \mathbf{x}^2} \mathbf{x}_f = -\mathbf{P}(\mathbf{0}) \mathbf{x}_f .$$

The proof of the local stable manifold theorem (for instance, see Lukes, 1969 or any standard text on differential equations) shows how to obtain higher order approximations to L, should greater accuracy be required. In the following proposition let Ω_t, for $t > 0$, be the set of all $\mathbf{x} \in \mathbb{R}^n$ which are projections of points $(\mathbf{x}, \boldsymbol{\lambda}) \in L$ that can be reached in time t or less along reverse trajectories of the Hamiltonian dynamics (13), starting from some $(\mathbf{x}_f, \boldsymbol{\lambda}_f) \in L$, $\mathbf{x}_f \in B_\varepsilon \setminus \{0\}$. The proposed *stability test* essentially involves checking that the error between the true feedback and the approximation of the SDRE feedback is small in some sense, and can be stated as follows:

Proposition 1 (McCaffrey & Banks, 2001)**.** *For any* $t > 0$ *such that* $\Omega_t \subset \Omega$*,* $V(\mathbf{x})$ *is strictly decreasing along trajectories of the SDRE feedback algorithm (7)-(9) for all* $\mathbf{x}_0 \in \Omega_t \setminus \{0\}$ *provided*

$$\begin{aligned} &\tfrac{1}{2}\left[\boldsymbol{\lambda} - \mathbf{P}(\mathbf{x})\mathbf{x}\right]^T \mathbf{B}(\mathbf{x}) \mathbf{R}^{-1}(\mathbf{x}) \mathbf{B}^T(\mathbf{x}) \left[\boldsymbol{\lambda} - \mathbf{P}(\mathbf{x})\mathbf{x}\right] \\ &\quad - \tfrac{1}{2} \mathbf{x}^T \mathbf{P}(\mathbf{x}) \mathbf{B}(\mathbf{x}) \mathbf{R}^{-1}(\mathbf{x}) \mathbf{B}^T(\mathbf{x}) \mathbf{P}(\mathbf{x}) \mathbf{x} \le 0 \end{aligned} \tag{19}$$

for all $(\mathbf{x}, \boldsymbol{\lambda}) \in L$ *such that* $\mathbf{x} \in \Omega_t \setminus \{0\}$.

Since $\mathbf{P}(\mathbf{0}) = \frac{\partial^2 V(\mathbf{0})}{\partial \mathbf{x}^2}$, as $\mathbf{x} \to \mathbf{0}$, $\mathbf{P}(\mathbf{x})\mathbf{x} \to \frac{\partial^2 V(\mathbf{0})}{\partial \mathbf{x}^2} \mathbf{x}$ and $\boldsymbol{\lambda} \to \frac{\partial^2 V(\mathbf{0})}{\partial \mathbf{x}^2} \mathbf{x}$. Thus $\mathbf{P}(\mathbf{x})\mathbf{x} \to \boldsymbol{\lambda}$ as $\mathbf{x} \to \mathbf{0}$, and so (19) will hold in a sufficiently small ball B_ε centered on the origin. The proposed stability test then involves following trajectories of the Hamiltonian dynamics (13) backwards in time from points $\mathbf{x}_f \in \partial B_\varepsilon$, $\boldsymbol{\lambda}_f = \frac{\partial^2 V(\mathbf{0})}{\partial \mathbf{x}^2} \mathbf{x}_f$ and estimating the largest t for which (19) holds in Ω_t. The SDRE feedback algorithm will then be asymptotically stable in the sublevel set $\{\mathbf{x} \in \mathbb{R}^n : V(\mathbf{x}) \le c\}$, where $c = \min\{V(\mathbf{x}) : \mathbf{x} \in \partial \Omega_t\}$. Results in the literature indicate that *asymptotic optimality* of the SDRE feedback will then hold on the same region. This is reviewed in the next section.

5. OPTIMALITY ANALYSES

5.1 Local Asymptotic Optimality

Since $\mathbf{A}(\mathbf{x}) \to \partial \mathbf{f}(\mathbf{0}) / \partial \mathbf{x}$ as $\mathbf{x} \to \mathbf{0}$, $\mathbf{P}(\mathbf{x})$ tends to the solution of the ARE for the linearized problem at the origin. Hence, in a sufficiently small neighborhood of the origin, the feedback from the SDRE algorithm is arbitrarily close to the optimal feedback. This approximation is *asymptotically optimal*, in that it converges to the optimal control close to the origin as $\mathbf{x} \to \mathbf{0}$. Mracek & Cloutier (1998) have addressed the optimality of the SDRE method by considering the *necessary conditions* for optimality of the nonlinear regulator (1) and (2). This section presents the main optimality results. Throughout the analyses, the conditions set by Hypotheses 4 are assumed to hold, so that $\mathbf{P}(\mathbf{x})$ exists for all $\mathbf{x}$. In addition, the following boundedness conditions are required.

Hypotheses 5. *$\mathbf{A}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$, $\mathbf{P}(\mathbf{x})$, $\mathbf{Q}(\mathbf{x})$ and $\mathbf{R}(\mathbf{x})$ along with their gradients $\frac{\partial \mathbf{A}(\mathbf{x})}{\partial \mathbf{x}}$, $\frac{\partial \mathbf{B}(\mathbf{x})}{\partial \mathbf{x}}$, $\frac{\partial \mathbf{P}(\mathbf{x})}{\partial \mathbf{x}}$, $\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial \mathbf{x}}$ and $\frac{\partial \mathbf{R}(\mathbf{x})}{\partial \mathbf{x}}$ are bounded in a neighborhood Ω about the origin.*

Theorem 5 (Mracek & Cloutier, 1998)**.** *In the general multivariable case ($n>1$), the SDRE nonlinear feedback solution and its associated state and costate trajectories satisfy the first necessary condition for optimality ($\frac{\partial H}{\partial \mathbf{u}}=\mathbf{0}$) of the nonlinear optimal regulator problem (1) and (2). Additionally, if Hypotheses 5 holds, under asymptotic stability, as the state $\mathbf{x}$ is driven to zero, the second necessary condition for optimality ($\dot{\boldsymbol{\lambda}}=-\frac{\partial H}{\partial \mathbf{x}}$) is asymptotically satisfied at a quadratic rate.*

Proof. From Pontryagin's maximum principle, the necessary conditions for optimality are

$$\frac{\partial H}{\partial \mathbf{u}}=\mathbf{0},\quad \dot{\boldsymbol{\lambda}}=-\frac{\partial H}{\partial \mathbf{x}},\quad \dot{\mathbf{x}}=-\frac{\partial H}{\partial \boldsymbol{\lambda}} \tag{20}$$

where H is the Hamiltonian function defined in (11), and $\boldsymbol{\lambda}=\partial V/\partial \mathbf{x}$. Hence, using (7) and (11),

$$\begin{aligned}\frac{\partial H}{\partial \mathbf{u}}&=\mathbf{B}^T(\mathbf{x})\boldsymbol{\lambda}+\mathbf{R}(\mathbf{x})\mathbf{u}\\&=\mathbf{B}^T(\mathbf{x})\boldsymbol{\lambda}-\mathbf{R}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x})\mathbf{x}\\&=\mathbf{B}^T(\mathbf{x})\left[\boldsymbol{\lambda}-\mathbf{P}(\mathbf{x})\mathbf{x}\right]\end{aligned} \tag{21}$$

Since $\boldsymbol{\lambda}=\partial V/\partial \mathbf{x}$, from (16) the costate vector satisfies

$$\boldsymbol{\lambda}=\mathbf{P}(\mathbf{x})\mathbf{x}, \tag{22}$$

and so substituting $\boldsymbol{\lambda}$ into (21) gives $\frac{\partial H}{\partial \mathbf{u}}=\mathbf{0}$. Therefore, the SDRE feedback solution satisfies the first necessary condition of the nonlinear regulator problem (1) and (2).

Taking the partial derivatives of the Hamiltonian function (11) with respect to $\mathbf{x}$, the second necessary condition $\dot{\boldsymbol{\lambda}}=-\frac{\partial H}{\partial \mathbf{x}}$ becomes

$$\dot{\boldsymbol{\lambda}}=-\frac{\partial \mathbf{f}^T(\mathbf{x})}{\partial \mathbf{x}}\boldsymbol{\lambda}-\mathbf{u}^T\frac{\partial \mathbf{B}^T(\mathbf{x})}{\partial \mathbf{x}}\boldsymbol{\lambda}-\mathbf{Q}(\mathbf{x})\mathbf{x}-\tfrac{1}{2}\mathbf{x}^T\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{x}-\tfrac{1}{2}\mathbf{u}^T\frac{\partial \mathbf{R}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{u}. \tag{23}$$

Differentiating (22) with respect to time gives

$$\dot{\boldsymbol{\lambda}}=\dot{\mathbf{P}}(\mathbf{x})\mathbf{x}+\mathbf{P}(\mathbf{x})\dot{\mathbf{x}}. \tag{24}$$

Substituting (4), (7), (9) and (24) into (23), and rearranging terms gives

$$\begin{aligned}&\dot{\mathbf{P}}\mathbf{x}+\tfrac{1}{2}\mathbf{x}^T\frac{\partial \mathbf{Q}}{\partial \mathbf{x}}\mathbf{x}+\tfrac{1}{2}\mathbf{x}^T\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\frac{\partial \mathbf{R}}{\partial \mathbf{x}}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{x}+\mathbf{x}^T\frac{\partial \mathbf{A}^T}{\partial \mathbf{x}}\mathbf{P}\mathbf{x}\\&\quad-\mathbf{x}^T\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\frac{\partial \mathbf{B}^T}{\partial \mathbf{x}}\mathbf{P}\mathbf{x}+\left[\mathbf{P}\mathbf{A}+\mathbf{A}^T\mathbf{P}+\mathbf{Q}-\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\right]\mathbf{x}=\mathbf{0},\end{aligned}$$

where the argument $\mathbf{x}$ has been dropped for notational simplicity. Using (8), therefore,

$$\begin{aligned}&\dot{\mathbf{P}}\mathbf{x}+\tfrac{1}{2}\mathbf{x}^T\frac{\partial \mathbf{Q}}{\partial \mathbf{x}}\mathbf{x}+\tfrac{1}{2}\mathbf{x}^T\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\frac{\partial \mathbf{R}}{\partial \mathbf{x}}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{x}\\&\quad+\mathbf{x}^T\frac{\partial \mathbf{A}^T}{\partial \mathbf{x}}\mathbf{P}\mathbf{x}-\mathbf{x}^T\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\frac{\partial \mathbf{B}^T}{\partial \mathbf{x}}\mathbf{P}\mathbf{x}=\mathbf{0},\end{aligned} \tag{25}$$

which is called the *SDRE Necessary Condition for Optimality* (Mracek & Cloutier, 1998). Therefore, whenever this condition is satisfied, the closed-loop solution satisfies all of the first-order necessary conditions of the maximum principle, since $\frac{\partial H}{\partial \mathbf{u}}=\mathbf{0}$ is always satisfied.

In general, the SDRE Necessary Condition for Optimality is not satisfied for a given SDC parameterization $\mathbf{A}(\mathbf{x})$ in the multivariable case. However, a suboptimality property can be revealed as follows. Expanding $\dot{\mathbf{P}}$ yields

$$\dot{\mathbf{P}}(\mathbf{x})\mathbf{x}=\left(\sum_{i=1}^{n}\frac{\partial \mathbf{P}(\mathbf{x})}{\partial x_i}\dot{x}_i\right)\mathbf{x}=\sum_{i=1}^{n}\frac{\partial \mathbf{P}(\mathbf{x})}{\partial x_i}\left[a_{CL}^i\mathbf{x}\right]\mathbf{x}, \tag{26}$$

where a_{CL}^i is the i^{th} row of the closed-loop coefficient matrix $\mathbf{A}_{CL}(\mathbf{x})$ defined by (6). Equation (26) can be rewritten as $\mathbf{x}^T\mathbf{N}_i\mathbf{x}$, where the elements of $\mathbf{N}_i$ are functions of the elements of $\frac{\partial \mathbf{P}(\mathbf{x})}{\partial x_j}$ and a_{CL}^j, $j=1,\ldots,n$. Substituting this result into the necessary condition (25) yields

$$\mathbf{x}^T\mathbf{M}_i\mathbf{x}=\mathbf{0},$$

where

$$\mathbf{M}_i\triangleq\mathbf{N}_i+\tfrac{1}{2}\frac{\partial \mathbf{Q}}{\partial \mathbf{x}}+\tfrac{1}{2}\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\frac{\partial \mathbf{R}}{\partial \mathbf{x}}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}+\frac{\partial \mathbf{A}^T}{\partial \mathbf{x}}\mathbf{P}\mathbf{x}-\mathbf{x}^T\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\frac{\partial \mathbf{B}^T}{\partial \mathbf{x}}\mathbf{P}$$

whose elements are functions of the elements of $\mathbf{A}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$, $\mathbf{P}(\mathbf{x})$, $\mathbf{Q}(\mathbf{x})$, and $\mathbf{R}(\mathbf{x})$ as well as their gradients. Under asymptotic stability, the state trajectories will eventually enter and remain in Ω. From the boundedness assumption on the functions in Hypotheses 5, there exists a constant positive-definite matrix such that

$$\max_i\left|\mathbf{x}^T\mathbf{M}_i\mathbf{x}\right|\le\mathbf{x}^T\mathbf{U}\mathbf{x}\quad\text{for }\mathbf{x}\in\Omega.$$

Thus, the ∞-norm of the left hand side of the necessary condition (25) is bounded above by a quadratic function of $\mathbf{x}$. This completes the proof. □

Theorem 5 represents a suboptimality property of the SDRE method. Since the second necessary condition for optimality is only satisfied asymptotically, the theorem relates only to the local near-optimal performance of the SDRE controller in the case of sufficiently small initial conditions.

5.2 Global Optimality

Using the following standard result reveals when the SDRE (8) gives the global optimal solution and the optimal cost for a given SDC parameterization.

Lemma 2. *Suppose a vector-valued function $\mathbf{p}:\mathrm{X}\to\mathbb{R}^n$ is of class $C^1(\mathbb{R}^n)$, and let $\mathbf{p}(\mathbf{x})=[p_1(\mathbf{x}),\ldots,p_n(\mathbf{x})]^T$ for $\mathbf{x}\in\mathrm{X}$. Then there exists $V:\mathrm{X}\to\mathbb{R}$ such that $\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}=\mathbf{p}(\mathbf{x})$ iff*

$$\frac{\partial p_i(\mathbf{x})}{\partial x_j}=\frac{\partial p_j(\mathbf{x})}{\partial x_i}\quad\forall\mathbf{x}\in\mathrm{X},\quad i,j=1,2,\ldots,n. \tag{27}$$

Moreover, if (27) holds, then V with $V(\mathbf{0})=0$ is given by

$$V(\mathbf{x})=\mathbf{x}^T\int_0^1\mathbf{p}(t\mathbf{x})\,dt.$$

The optimality result now follows directly from Lemma 2.

Theorem 6 (Huang & Lu, 1996). *Suppose the SDRE (8) has a positive-definite matrix-valued solution* $\mathbf{P}:\mathbb{R}^n \to \mathbb{R}^{n\times n}$. *If the vector-valued function (16) satisfies (27) with* $\mathbf{p}(\mathbf{x}) = \mathbf{P}(\mathbf{x})\mathbf{x}$, *then (7) is the optimal state-feedback for the nonlinear optimal regulation problem (2) with (5). The value function is then given by*

$$V(\mathbf{x}) = \mathbf{x}^T \int_0^1 t\mathbf{P}(t\mathbf{x})\, dt\, \mathbf{x}, \quad \mathbf{x} \geq 0 .$$

Therefore, provided that $\frac{\partial}{\partial \mathbf{x}}(\mathbf{P}(\mathbf{x})\mathbf{x})$ is a symmetric matrix, the SDRE control (7) is in fact optimal with respect to (2). While this important symmetry condition is true in the scalar case for $n = 1$, it does not in general hold for higher dimensional systems. The next result uses this fact to highlight a unique property of the SDRE method.

Theorem 7 (Mracek & Cloutier, 1998). *For scalar systems* ($x \in \mathbb{R}^1$), *the globally asymptotically stabilizing SDRE feedback solution of the nonlinear optimal regulator problem (1) and* (2) *is always (globally) optimal on* $\mathbb{R}^1$.

Proof. In the case of scalar x, from Theorem 4, the SDRE feedback solution is globally asymptotically stabilizing, and the symmetry condition (27) is always satisfied. Hence the solution is the optimal. Alternatively, optimality can be deduced using the SDRE Necessary Condition for Optimality (25). However, this requires detailed algebraic manipulations, which is omitted for brevity. □

Therefore, in the scalar case, even when the performance index (2) is nonquadratic (that is, $\mathbf{Q}$ and $\mathbf{R}$ are functions of $\mathbf{x}$), the SDRE method produces the optimal solution of the regulator problem (1) and (2) in feedback form.

Corollary 1 (Mracek & Cloutier, 1998). *In the case of scalar* x, *the globally asymptotically optimal stabilizing SDRE feedback control of the nonlinear optimal regulator problem (1) and (2) is given by*

$$u(x) = -\tfrac{1}{b(x)}\left[f(x) + \operatorname{sgn}(x)\sqrt{f^2(x) + \tfrac{b^2(x)x^2 q(x)}{r(x)}} \right].$$

Proof. Using lower case notation, there exists only one SDC parameterization in the scalar case, which is $a(x) = f(x)/x$. Hence, the state-dependent Riccati equation (8) is given by

$$2\tfrac{f(x)}{x}p - \tfrac{b^2(x)}{r(x)}p^2 + q(x) = 0 ,$$

which has the positive-definite solution

$$p(x) = \tfrac{r(x)}{b^2(x)}\left[\tfrac{f(x)}{x} + \sqrt{\tfrac{f^2(x)}{x^2} + \tfrac{b^2(x)q(x)}{r(x)}} \right].$$

Substituting this into (7) gives the result. □

In Section 2.4, it is shown that the state-dependent parameterization is not unique, and there are an infinite number of representations. Huang & Lu (1996) have shown that there will *exist* a representation such that the SDRE feedback produces the optimal feedback control law. Before formally stating this property, consider the following lemma (Huang and Lu, 1996), whose proof is straightforward.

Lemma 3. *Suppose the function* $\mathbf{f}:\mathbb{R}^n \to \mathbb{R}^n$ *with* $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ *can be represented as* $\mathbf{f}(\mathbf{x}) = \mathbf{A}(\mathbf{x})\mathbf{x}$ *for some continuous matrix-valued function* $\mathbf{A}:\mathbb{R}^n \to \mathbb{R}^{n\times n}$, *then any representation* $\mathbf{f}(\mathbf{x}) = \mathbf{A}_0(\mathbf{x})\mathbf{x}$ *can be parameterized as*

$$\mathbf{A}_0(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \mathbf{E}(\mathbf{x}) , \tag{28}$$

where $\mathbf{E}:\mathbb{R}^n \to \mathbb{R}^{n\times n}$ *satisfies* $\mathbf{E}(\mathbf{x})\mathbf{x} = \mathbf{0}$.

Lemma 3 implies that

$$\dot{\mathbf{x}}(t) = \left[\mathbf{A}(\mathbf{x}) + \mathbf{E}(\mathbf{x})\right]\mathbf{x}(t) + \mathbf{B}(\mathbf{x})\mathbf{u}(t)$$

is also a representation of the original dynamics (1). Hence, the optimal feedback control law (3) can be given by (7), where $\mathbf{P}(\mathbf{x})$ is the positive-definite solution to the SDRE

$$\mathbf{P}(\mathbf{x})(\mathbf{A}(\mathbf{x}) + \mathbf{E}(\mathbf{x})) + (\mathbf{A}(\mathbf{x}) + \mathbf{E}(\mathbf{x}))^T\mathbf{P}(\mathbf{x}) - \mathbf{P}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) + \mathbf{Q}(\mathbf{x}) = \mathbf{0}$$

for some $\mathbf{E}(\mathbf{x})$ that satisfies $\mathbf{E}(\mathbf{x})\mathbf{x} = \mathbf{0}$.

Theorem 8 (Huang & Lu, 1996). *Under Hypotheses 3, if the value function* $V(\mathbf{x})$ *has the gradient of the form (16) for some positive-definite matrix valued function* $\mathbf{P}:\mathbb{R}^n \to \mathbb{R}^{n\times n}$, *then there always exists a parameterization (4) such that* $\mathbf{P}(\mathbf{x})$ *is the solution of the SDRE (8) which gives the optimal feedback controller.*

Proof. Suppose $\mathbf{A}(\mathbf{x})$ is a matrix-valued function satisfying (4). Then, by Lemma 3, all the possible state matrices $\mathbf{A}_0(\mathbf{x})$ with $\mathbf{f}(\mathbf{x}) = \mathbf{A}_0(\mathbf{x})\mathbf{x}$ can be parameterized by (28) with $\mathbf{E}(\mathbf{x})\mathbf{x} = \mathbf{0}$. In this case, the HJB (15) becomes

$$\mathbf{x}^T\mathbf{S}(\mathbf{x})\mathbf{x} = 0 \quad \forall \mathbf{x} \in \mathbb{R}^n , \tag{29}$$

where $\mathbf{S}(\mathbf{x})$ is equal to the bracketed expression in (17) with $\mathbf{A}(\mathbf{x}) = \mathbf{A}_0(\mathbf{x}) - \mathbf{E}(\mathbf{x})$ from (28). Hence, the SDRE (8) is equivalent to the HJB equation (15) if and only if $\mathbf{E}(\mathbf{x})$ can be found such that

$$\mathbf{S}(\mathbf{x}) + \mathbf{P}(\mathbf{x})\mathbf{E}(\mathbf{x}) + \mathbf{E}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) = \mathbf{0} \quad \forall \mathbf{x} \in \mathbb{R}^n .$$

Since $\mathbf{S}(\mathbf{x})$ is symmetric, $\mathbf{E}(\mathbf{x})$ can be parameterized as

$$\mathbf{E}(\mathbf{x}) = -\tfrac{1}{2}\mathbf{P}^{-1}(\mathbf{x})\left[\mathbf{S}(\mathbf{x}) - \mathbf{T}(\mathbf{x})\right], \tag{30}$$

where $\mathbf{T}(\mathbf{x})$ is some skew-symmetric matrix, that is, $\mathbf{T}(\mathbf{x}) = -\mathbf{T}^T(\mathbf{x})$. Since $\mathbf{E}(\mathbf{x})\mathbf{x} = \mathbf{0}$, from (30), $\mathbf{T}(\mathbf{x})$ is chosen such that

$$\left[\mathbf{S}(\mathbf{x}) - \mathbf{T}(\mathbf{x})\right]\mathbf{x} = \mathbf{0} \quad \forall \mathbf{x} \in \mathbb{R}^n . \tag{31}$$

From (29), this is always possible. One way to construct such $\mathbf{T}(\mathbf{x})$ is as follows. If $\mathbf{x} = \mathbf{0}$, simply choose $\mathbf{T}(\mathbf{0}) = \mathbf{0}$, and so

$$\mathbf{E}(\mathbf{0}) = -\tfrac{1}{2}\mathbf{P}^{-1}(\mathbf{0})\mathbf{S}(\mathbf{0})$$

from (30). The case $\mathbf{x} \neq \mathbf{0}$ is considered in the following. Because $\mathbf{S}(\mathbf{x})$ is symmetric, the congruence transformation

$$\mathbf{S}(\mathbf{x}) = \mathbf{U}^T(\mathbf{x})\mathbf{D}(\mathbf{x})\mathbf{U}(\mathbf{x})$$

can be used to diagonalize $\mathbf{S}(\mathbf{x})$ for some orthogonal matrix-valued function $\mathbf{U}(\mathbf{x})$ with $\mathbf{D}(\mathbf{x}) = \operatorname{diag}[\lambda_1(\mathbf{x}), \ldots, \lambda_n(\mathbf{x})]$. Now letting

$$\mathbf{y}(\mathbf{x}) = \mathbf{U}(\mathbf{x})\mathbf{x} \triangleq [y_1(\mathbf{x}), \ldots, y_n(\mathbf{x})]^T ,$$

$\mathbf{y}(\mathbf{x})$ cannot be zero for any $\mathbf{x} \neq \mathbf{0}$, since $\mathbf{U}(\mathbf{x})$ has full rank. Assuming, without loss of generality, that $y_p \neq 0$ for some $p \le n$ and $y_i = 0$ for $p < i \le n$, (29) implies $\sum_{i=1}^{p} \lambda_i y_i^2 = 0$. Let

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} 0 & 0 & \cdots & \lambda_1 y_1/y_p & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_2 y_2/y_p & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\lambda_1 y_1/y_p & -\lambda_2 y_2/y_p & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

so that $\mathbf{F}(\mathbf{x})$ is a matrix with all zero entries except the first p elements on the $(p+1)^{\text{th}}$ row and column, with $\mathbf{F}(\mathbf{x}) = -\mathbf{F}^T(\mathbf{x})$. $\mathbf{T}(\mathbf{x})$ is now defined as

$$\mathbf{T}(\mathbf{x}) = \mathbf{U}^T(\mathbf{x})\mathbf{F}(\mathbf{x})\mathbf{U}(\mathbf{x}),$$

which is obviously skew-symmetric. It is easy to check that

$$\begin{aligned} \left[\mathbf{S}(\mathbf{x}) - \mathbf{T}(\mathbf{x})\right]\mathbf{x} &= \mathbf{U}^T(\mathbf{x})\left[\mathbf{D}(\mathbf{x}) - \mathbf{F}(\mathbf{x})\right]\mathbf{U}(\mathbf{x})\mathbf{x} \\ &= \mathbf{U}^T(\mathbf{x})\left[\mathbf{D}(\mathbf{x}) - \mathbf{F}(\mathbf{x})\right]\mathbf{y}(\mathbf{x}) \\ &= \mathbf{0}, \end{aligned}$$

and thus $\mathbf{T}(\mathbf{x})$ satisfies (31). Therefore, $\mathbf{E}(\mathbf{x})$ can be obtained in (30) for all $\mathbf{x} \in \mathbb{R}^n$. □

Theorem 8 confirms that the global optimal controller can always be formed from the positive-definite solution to the SDRE (8) if the gradient of the cost function $V(\mathbf{x})$ has the form $\mathbf{P}(\mathbf{x})\mathbf{x}$ and the "right" $\mathbf{A}(\mathbf{x})$ is chosen. Though there are multiple solutions to an ARE, there is at most one solution which gives the optimal performance for both the original system (that is, the HJB equation) and the SDRE system. From the discussion in Section 3, the positive-definite solution of the SDRE (8) gives a pointwise stabilizing state-feedback solution. Therefore, if $\mathbf{P}(\mathbf{x})$ in (16) is positive-definite, then with a right choice of SDC representation (5) for system (1), the unique positive-definite solution of (8), which is thus $\mathbf{P}(\mathbf{x})$, recovers the optimality. Although the representation (16) of $\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}$ is not unique, the following theorem essentially follows from the above discussion.

Theorem 9 (Huang & Lu, 1996)**.** *If V is a positive-definite solution of the HJB equation (15), then there exists at most one positive-definite matrix valued function $\mathbf{P}(\mathbf{x})$ such that (16) is satisfied.*

While there always exists some choice of SDC parameterization in which the SDRE recovers the global optimal, finding the "right" representation in SDC form using the above results is very difficult since the value function $V(\mathbf{x})$ is assumed known *a priori* in the above discussion. Cloutier, D'Souza & Mracek (1996) have provided an approach in which the nonuniqueness of $\mathbf{A}(\mathbf{x})$ is exploited by attempting to find an SDC parameterization for which the optimality condition holds. Unfortunately, this endeavor is by no means trivial.

Typically, most SDRE controllers are simply implemented by choosing an $\mathbf{A}(\mathbf{x})$ that satisfies Hypothesis 4, and constructing a suboptimal and locally stabilizing controller. Thus far this approach has shown great promise, as evidenced by the growing number of application papers dealing with SDRE control. However, the methodology is as yet unsupported by proofs on how to systematically achieve global optimality or global asymptotic stability.

6. CAPABILITIES AND ART OF SDRE DESIGN

Some nonlinear control techniques are restricted to systems having certain structures such as cascaded systems, while others are not systematic and require mini-designs to be carried out on one equation at a time, and yet others have very limited applicability because of the strong conditions imposed on the system. In contrast, the SDRE method allows for the systematic design of a broad class of nonlinear systems, and has many capabilities that other nonlinear design methods do not have, at least collectively. While some nonlinear techniques only address stability, the SDRE method directly addresses performance through the specification of the performance index (2) in the nonlinear regulator problem. Furthermore, the state-dependent state and control weightings can be adjusted to directly affect performance with predictable results; for example, an increase in $\mathbf{Q}(\mathbf{x})$ results in faster regulation of the states at the expense of greater control effort. The extra design degrees of freedom that are available in the nonuniqueness of the SDC parameterization of $\mathbf{A}(\mathbf{x})$ can also be used to enhance controller performance. Such degrees of freedom are not available in traditional nonlinear control techniques. In addition, most practical control problems involve in one way or another hard constraints on states and inputs. Since there are very few design approaches that can handle these constraints *a priori*, the designer has to tweak the controller using ad-hoc "anti-windup" schemes *a posteriori*. On the contrary, the SDRE approach offers the capability of imposing hard bounds on the control, control rate or even control acceleration to avoid actuator saturation (see, for example, Cloutier, D'Souza & Mracek, 1996; Mracek & Cloutier, 1998). The technique also possesses the ability to satisfy state constraints (Friedland, 1998; Cloutier & Cockburn, 2001), and combined state and control constraints. The method can even be used to directly handle unstable nonminimum phase systems, a capability which has been illustrated in Mracek & Cloutier (1996, 1997) and Mracek (2007). More importantly, however, the SDRE method has the capability to preserve beneficial nonlinearities, since the method neither dynamically inverts nor feedback linearizes the nonlinear system. Because of such capabilities and the systematic nature of the SDRE technique, increasingly, control practitioners are using the method in a variety of real-world applications, in spite of the fact that stability typically has to be verified via simulation.

This section provides an overview of the capabilities of SDRE control and goes into detail concerning the art of carrying out an effective and systematic SDRE design for systems that both do and do not conform to the basic structure and conditions required by the method (Cloutier & Stansbery, 2002a). In the following, a discussion is pursued on how the additional degrees of freedom provided by the nonuniqueness of the SDC parameterization can be used to enhance controller performance. The SDRE nonlinear regulator with integral servomechanism action is then presented to show how *command following* can be performed. Several situations that prevent a straightforward application of the SDRE method to the control problem at hand are also addressed. These include the existence of state-independent terms, the existence of state-dependent terms which do not go to zero as the state vector goes to zero, the existence of nonlinearity (such as hard constraints) in the controls, the existence of state constraints, and the existence of uncontrollable and unstable but bounded state dynamics.

6.1 Design Flexibility

The weighting matrices $\mathbf{Q}(\mathbf{x})$ and $\mathbf{R}(\mathbf{x})$ are the obvious design parameters in the SDRE approach. However, performance of the controller over the domain of interest is dependent not only on the chosen state and control weightings, but also on the choice of the SDC representation $\mathbf{A}(\mathbf{x})$ in (4). In a deterministic setting, the SDC parameterization fully captures the nonlinearities of the system. It is shown in Section 2.4 that although the SDC parameterization is unique in the case of *scalar* x for all $x \neq 0$, it is not unique in the *multivariable* case and that the SDC parameterization $\mathbf{A}(\mathbf{x})$ itself can be parameterized as $\mathbf{A}(\mathbf{x}, \boldsymbol{\alpha})$, where $\boldsymbol{\alpha}$ is a vector of free design parameters. The introduction of $\boldsymbol{\alpha}$ creates extra degrees of freedom that are not available in traditional methods. These additional degrees of freedom provided by the nonuniqueness of the SDC parameterization can be used not only to enhance controller performance, but also to avoid singularities or loss of controllability, as well as effect tradeoffs between performance, optimality, stability, robustness, and disturbance rejection, thus offering a more flexible nonlinear control policy.

The flexibility in the design process for a conforming system, therefore, consists of the selection of the SDC matrix $\mathbf{A}(\mathbf{x}, \boldsymbol{\alpha})$ together with the selection of the state-dependent state and control weighting matrices $\mathbf{Q}(\mathbf{x})$ and $\mathbf{R}(\mathbf{x})$. In order to obtain a legitimate ARE solution, $\mathbf{A}(\mathbf{x}, \boldsymbol{\alpha})$ must be chosen so that the parameterized pair $\{\mathbf{A}(\mathbf{x}, \boldsymbol{\alpha}), \mathbf{B}(\mathbf{x})\}$ is pointwise stabilizable in the linear sense. This also guarantees that the SDRE controller is locally asymptotically stable. In addition to satisfying the pointwise stabilizability requirement, a rule of thumb in selecting the state-dependent factorization is to place a *nonzero* entry in the $\{i, j\}$-element of the $\mathbf{A}(\mathbf{x}, \boldsymbol{\alpha})$ matrix if the i^{th} state derivative depends on the j^{th} state. For example, if $\dot{x}_3 = x_1 x_2$, two possible factorizations for $\mathbf{A}(\mathbf{x}) = [a_{ij}]$ are $a_{31} = 0$, $a_{32} = x_1$ and $a_{31} = x_2$, $a_{32} = 0$. Neither one of these parameterizations reflect in the $\mathbf{A}(\mathbf{x})$ matrix the fact that $\dot{x}_3$ depends on both x_1 and x_2. While both these parameterizations may work, it is expected that better responses can be obtained with the parameterizations $a_{31} = \alpha x_2$ and $a_{32} = (1-\alpha) x_1$ with α being a free design parameter. Additionally, both of the previous factorizations can be tested by setting $\alpha = 0$ and $\alpha = 1$, respectively.

A complete characterization of the possible factorizations of $\mathbf{f}(\mathbf{x})$ into $\mathbf{A}(\mathbf{x}, \boldsymbol{\alpha})\mathbf{x}$ has been discussed in Section 2.4. However, to obtain the optimal feedback solution of the nonlinear regulator, $\boldsymbol{\alpha}$ may be required to vary as a function of the state $\mathbf{x}$ (Cloutier, D'Souza & Mracek, 1996). Furthermore, solving for the optimal $\boldsymbol{\alpha}(\mathbf{x})$ for given $\mathbf{Q}(\mathbf{x})$ and $\mathbf{R}(\mathbf{x})$ weightings would require solving a partial differential equation, which would be as difficult as solving the HJB equation, and would not be real-time implementable. Fortunately, the real advantage of the SDRE technique is that there is no need to do so. Since design performance specifications are typically given in terms of the desired response characteristics (as opposed to a performance index), $\boldsymbol{\alpha}$ is normally used as a constant tuning parameter to aid in the achievement of the response specifications. In fact, even with $\mathbf{Q}(\mathbf{x})$ and $\mathbf{R}(\mathbf{x})$ *a priori* specified, satisfactory performance relative to the optimal value of the performance index can usually be obtained using a constant value of $\boldsymbol{\alpha}$.

6.2 SDRE Integral Servomechanism

In order to perform *tracking* (or *command following*), the SDRE controller can be implemented as an integral servomechanism as demonstrated in Cloutier & Stansbery (2001). This is accomplished as follows. First, the state $\mathbf{x}$ is decomposed as $\mathbf{x}^T = [\mathbf{x}_R^T \quad \mathbf{x}_N^T]$, where it is desired for the vector components of $\mathbf{x}_R$ to track a reference command $\mathbf{r}_C$. The state vector $\mathbf{x}$ is then augmented with $\mathbf{x}_I$, the integral states of $\mathbf{x}_R$:

$$\tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{x}_I^T & \mathbf{x}_R^T & \mathbf{x}_N^T \end{bmatrix}. \qquad (32)$$

The augmented system is given by

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}(\tilde{\mathbf{x}}, \boldsymbol{\alpha}) + \tilde{\mathbf{B}}(\tilde{\mathbf{x}})\mathbf{u}, \qquad (33)$$

where

$$\tilde{\mathbf{A}}(\tilde{\mathbf{x}}, \boldsymbol{\alpha}) = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{I} \;\vdots\; \mathbf{0} \\ \hline \mathbf{0} & \mathbf{A}(\mathbf{x}, \boldsymbol{\alpha}) \end{array}\right], \quad \tilde{\mathbf{B}}(\tilde{\mathbf{x}}) = \begin{bmatrix} \mathbf{0} \\ \mathbf{B}(\mathbf{x}) \end{bmatrix}, \qquad (34)$$

and the SDRE integral servo controller is given by

$$\mathbf{u} = -\tilde{\mathbf{R}}^{-1}(\tilde{\mathbf{x}})\tilde{\mathbf{B}}^T(\tilde{\mathbf{x}})\tilde{\mathbf{P}}(\tilde{\mathbf{x}}) \begin{bmatrix} \mathbf{x}_I - \int \mathbf{r}_C dt \\ \mathbf{x}_R - \mathbf{r}_C \\ \mathbf{x}_N \end{bmatrix}. \qquad (35)$$

Recall that in order for the SDRE to have a solution, the pointwise detectability condition must be satisfied. This is accomplished by penalizing the integral states with the corresponding nonzero diagonal elements of $\tilde{\mathbf{Q}}(\tilde{\mathbf{x}})$. An alternative, yet approximate, SDRE tracking approach has also been proposed in Çimen (*to appear*), which does not require increasing the state dimension.

6.3 Conforming to the Proper Structure and Conditions

When the system dynamics are affine in the control and $\mathbf{f}(\mathbf{x}) \in C^1(\mathbb{R}^n)$ with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, the system conforms to the basic structure and conditions required for the straightforward application of the SDRE method. In this case, the steps for carrying out an SDRE design are given by (7)-(9), and additionally (32)-(35) if the SDRE integral servomechanism is employed. However, there are many systems that do not conform to the structure or conditions specified in Section 2.1, and the SDRE technique cannot be directly applied. In these cases, the given system must be converted to a system that is conforming so that an effective SDRE design can be performed. In the sequel, several nonconforming cases are presented, showing in each case the systematic procedure for converting the system to a conforming one. The converted, conforming systems can then be used to produce effective SDRE designs for controlling the original plants.

The Presence of State-Independent Terms: In the presence of state-independent terms, referred to as *bias* terms, the condition $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ is violated. This prevents a direct $C^1(\mathbb{R}^n)$ factorization of $\mathbf{f}(\mathbf{x})$ into $\mathbf{A}(\mathbf{x},\alpha)\mathbf{x}$. There are three ways to handle a bias term (Cloutier & Stansbery, 2002a), denoted as $b(t)$. First, if the bias term is constant or slowly-varying, then it can be modeled as a stable state

$$\dot{b}(t) = -\lambda b(t),$$

where λ is a small positive number. Each time through the controller, the actual value of $b(t)$ is used in calculating the SDRE control using (7). Second, the bias term can be multiplied and divided by a state or a combination of states that it is known will not go to zero (Stansbery & Cloutier, 2000). For example, in a stirred tank problem (Cloutier & Stansbery, 1999b), if the temperature T is a state and is expressed in degrees Kelvin, then T will not go to zero, and if a bias term exists, it can be factored as

$$b(t) = \left[\tfrac{b(t)}{T}\right] T .$$

In a missile control problem, any component of the velocity vector $\mathbf{V}$ can go to zero, but the speed of the missile will not go to zero. In this case, the bias term, which may be gravity, can be multiplied and divided by the magnitude squared of the velocity vector. The bias term can then be factored as

$$b(t) = \left[\tfrac{b(t)\mathbf{V}^T}{\mathbf{V}^T\mathbf{V}}\right]\mathbf{V} .$$

A third alternative is to augment the system with a stable state z (Cloutier & Stansbery, 2001), such that

$$\dot{z}(t) = -\lambda z(t)$$

with $\lambda > 0$. The bias term can then be factored as

$$b(t) = \left[\tfrac{b(t)}{z}\right] z .$$

Each time through the controller, the initial value $z(0)$ is used in the SDC matrix in calculating the control.

The Presence of State-Dependent Terms which Exclude the Origin: Consider now the presence of state-dependent terms which exclude the origin, that is, terms which do not go to zero as the state goes to zero. This also violates the condition $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. Like biases, these terms prevent a direct $C^1(\mathbb{R}^n)$ factorization of $\mathbf{f}(\mathbf{x})$ into $\mathbf{A}(\mathbf{x},\alpha)\mathbf{x}$, and can be handled using either the second or third way discussed above for handling biases. However, it is more desirable to capture their state dependency in the proper element of the matrix $\mathbf{A}(\mathbf{x},\alpha)$. For example, suppose that $\dot{x}_2 = \cos x_1$. It is desirable to have a nonzero entry in the (2,1)-element of the $\mathbf{A}(\mathbf{x},\alpha)$ matrix that reflects the fact that $\dot{x}_2$ depends on x_1. This is accomplished by shifting the term so that it goes through the origin. This is done by adding and subtracting a bias to the term. For the given example, adding and subtracting one gives $\cos x_1 = [\cos x_1 - 1] + 1$. The function $\cos x_1 - 1$ goes through the origin and can now be factored as

$$\cos x_1 - 1 = \left[\tfrac{\cos x_1 - 1}{x_1}\right] x_1 .$$

The bias term created, which in this case is 1, can then be accounted for using one of the bias handling techniques above. This shifting procedure can be used for any state-dependent term which does not go through the origin. It is also desirable to shift state-dependent factors which exclude the origin even though they are embedded in a term which goes to zero as the state goes to zero. For example, consider $\dot{x}_2 = e^{x_1} x_3$. Obviously, this term goes to zero as x_3 goes to zero and can be factored as $a_{21} = 0$ and $a_{23} = e^{x_1}$. But this factorization does not reflect the fact that $\dot{x}_2$ depends on x_1 within the pointwise LQR structure since $a_{21} = 0$ and, during execution of the controller, a_{23} will just be a number in the $\mathbf{A}(\mathbf{x},\alpha)$ matrix. By shifting e^{x_1}, and writing

$$\dot{x}_2 = \left[(e^{x_1} - 1) + 1\right] x_3 = \left[\tfrac{e^{x_1} - 1}{x_1}\right] x_1 x_3 + x_3$$

allows the system to be parameterized as

$$a_{21} = \alpha \left[\tfrac{e^{x_1} - 1}{x_1}\right] x_3, \quad a_{23} = (\alpha - 1)(e^{x_1} - 1) + 1 ,$$

which yields the desired nonzero entry in a_{21}.

Nonlinearity and Constraints in the Controls : A system which is nonlinear in the control (such as hard bounds on the control and/or its rate and acceleration) can be represented as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x},\mathbf{u}) .$$

Such a system can be brought to the required structure given in (1) by introducing integral control (Cloutier & Stansbery, 2001; Cloutier & Stansbery, 2002b):

$$\dot{\mathbf{u}} = \mathbf{C}\mathbf{u} + \mathbf{D}\tilde{\mathbf{u}} .$$

In its simplest form, $\mathbf{C} = \mathbf{0}$ and $\mathbf{D} = \mathbf{I}$. The augmented system is then given by

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x},\mathbf{u}) \\ \mathbf{C}\mathbf{u} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{D} \end{bmatrix} \tilde{\mathbf{u}} ,$$

which conforms to the required structure, being affine in the pseudo-control $\tilde{\mathbf{u}}$. If the condition $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ is not satisfied in the augmented system, then the techniques discussed above on handling bias and shifting state-dependent terms to the origin can be employed.

State Constraints: Extension of the SDRE method to regulation of systems described by (1) with state constraints has been considered in Cloutier & Cockburn (2001). Suppose

$$\mathcal{X} = \{\mathbf{x} : \mathbf{h}(\mathbf{x}) \le \mathbf{0}, \mathbf{h}(\mathbf{x}) \in \mathbb{R}^p, \mathbf{h}(\cdot) \in C^1(\mathbb{R}^n)\} \quad (36)$$

is a set of allowable states, and the objective is to design a state feedback controller of form (3), such that the closed-loop system is stable and $\mathbf{x} \in \mathcal{X}, \forall t > 0$. Thus, any feasible trajectory of the closed-loop system must not cross the boundary of $\mathcal{X}$, $\partial\mathcal{X}$, defined as

$$\partial\mathcal{X} = \{\mathbf{x} : \mathbf{h}(\mathbf{x}) = \mathbf{0}, \mathbf{h}(\mathbf{x}) \in \mathbb{R}^p, \mathbf{h}(\cdot) \in C^1(\mathbb{R}^n)\}. \quad (37)$$

A sufficient condition to characterize state constraints (for $\mathbf{x}$ to remain in $\mathcal{X}$) is introduced by $\nabla\mathbf{h}(\mathbf{x})\dot{\mathbf{x}} = \mathbf{0}$. Equivalently,

$$\nabla\mathbf{h}(\mathbf{x})[\mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}] = \mathbf{0}. \quad (38)$$

A controller that satisfies (38) forces the closed-loop trajectories to follow level sets of $\mathcal{X}$. This condition has been exploited in Cloutier & Cockburn (2001) in the design of SDRE nonlinear regulators that render $\mathcal{X}$ invariant. The SDRE nonlinear regulator design strategy with state constraints is based on the enforcement of the sufficient condition (38) when the states are close to the boundary $\partial\mathcal{X}$, and total relaxation of condition (38) when the states are far from $\partial\mathcal{X}$. SDC representation of the left hand side of (38) is

$$\mathbf{z} \triangleq \nabla\mathbf{h}(\mathbf{x})[\mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u}] = \mathbf{C}(\mathbf{x})\mathbf{x} + \mathbf{D}(\mathbf{x})\mathbf{u} = \mathbf{0}, \quad (39)$$

where $\mathbf{z} \in \mathbb{R}^p$ is a fictitious output, with $\mathbf{C}(\mathbf{x}) \triangleq \nabla\mathbf{h}(\mathbf{x})\mathbf{A}(\mathbf{x})$ and $\mathbf{D}(\mathbf{x}) \triangleq \nabla\mathbf{h}(\mathbf{x})\mathbf{B}(\mathbf{x})$. Assuming that $\mathbf{x}$ is close to $\partial\mathcal{X}$, the state-feedback law that satisfies the algebraic equation $\mathbf{z} = \mathbf{0}$ is

$$\mathbf{u}(\mathbf{x}) = -\mathbf{D}'(\mathbf{x})\mathbf{C}(\mathbf{x})\mathbf{x}. \quad (40)$$

where $\mathbf{D}'(\mathbf{x}) \triangleq \mathbf{D}^T(\mathbf{x})[\mathbf{D}(\mathbf{x})\mathbf{D}^T(\mathbf{x})]^{-1}$ is the right inverse of $\mathbf{D}(\mathbf{x})$, such that $\mathbf{D}(\mathbf{x})\mathbf{D}'(\mathbf{x}) = \mathbf{I}$. This right inverse exists provided that $\mathbf{D}(\mathbf{x})$ has full row rank for all $\mathbf{x}$.

The control law (40) can be asymptotically recovered by solving a state-dependent nonlinear regulator problem that minimizes $J_{\mathcal{X}} = \frac{1}{2}\int_0^\infty \mathbf{z}^T\mathbf{W}(\mathbf{x})\mathbf{z}\, dt$, $\mathbf{W}(\mathbf{x}) > \mathbf{0}$, subject to (5). Thus,

$$J_{\mathcal{X}}(\mathbf{x}_0, \mathbf{u}(\cdot)) = \tfrac{1}{2}\int_0^\infty \left\{\mathbf{x}^T\mathbf{Q}_{\mathcal{X}}(\mathbf{x})\mathbf{x} + 2\mathbf{x}^T\mathbf{S}_{\mathcal{X}}(\mathbf{x})\mathbf{u} + \mathbf{u}^T\mathbf{R}_{\mathcal{X}}(\mathbf{x})\mathbf{u}\right\} dt, \quad (41)$$

where

$$\left.\begin{aligned} \mathbf{Q}_{\mathcal{X}}(\mathbf{x}) &\triangleq \mathbf{C}^T(\mathbf{x})\mathbf{W}(\mathbf{x})\mathbf{C}(\mathbf{x}) \\ \mathbf{R}_{\mathcal{X}}(\mathbf{x}) &\triangleq \mathbf{D}^T(\mathbf{x})\mathbf{W}(\mathbf{x})\mathbf{D}(\mathbf{x}) \\ \mathbf{S}_{\mathcal{X}}(\mathbf{x}) &\triangleq \mathbf{C}^T(\mathbf{x})\mathbf{W}(\mathbf{x})\mathbf{D}(\mathbf{x}) \end{aligned}\right\} \quad (42)$$

and $\mathbf{W}(\mathbf{x}) \in \mathbb{R}^{p\times p}$ is a diagonal weighting matrix, such that its i^{th} element is large when $\mathbf{x}$ is close to the boundary of the i^{th} constraint, and small otherwise.

In general, the minimization of (41) leads to singular regulators (for example, $\mathbf{R}_{\mathcal{X}}(\mathbf{x})$ not invertible) and makes the level sets of $\mathcal{X}$ positively invariant (Cloutier & Cockburn, 2001). However, the regulation objective is to derive the states to a desired equilibrium while remaining in the set $\mathcal{X}$. This can be achieved by minimizing the augmented cost functional

$$J(\mathbf{x}_0, \mathbf{u}(\cdot)) = J_o(\mathbf{x}_0, \mathbf{u}(\cdot)) + J_{\mathcal{X}}(\mathbf{x}_0, \mathbf{u}(\cdot)),$$

where

$$J_o(\mathbf{x}_0, \mathbf{u}(\cdot)) = \tfrac{1}{2}\int_0^\infty \left\{\mathbf{x}^T\mathbf{Q}_o(\mathbf{x})\mathbf{x} + \mathbf{u}^T\mathbf{R}_o(\mathbf{x})\mathbf{u}\right\} dt,$$

Therefore,

$$J(\mathbf{x}_0, \mathbf{u}(\cdot)) = \tfrac{1}{2}\int_0^\infty \left\{\mathbf{x}^T\mathbf{Q}(\mathbf{x})\mathbf{x} + 2\mathbf{x}^T\mathbf{S}(\mathbf{x})\mathbf{u} + \mathbf{u}^T\mathbf{R}(\mathbf{x})\mathbf{u}\right\} dt, \quad (43)$$

where

$$\begin{aligned} \mathbf{Q}(\mathbf{x}) &\triangleq \mathbf{Q}_o(\mathbf{x}) + \mathbf{Q}_{\mathcal{X}}(\mathbf{x}) = \mathbf{Q}_o(\mathbf{x}) + \mathbf{C}^T(\mathbf{x})\mathbf{W}(\mathbf{x})\mathbf{C}(\mathbf{x}) \\ \mathbf{R}(\mathbf{x}) &\triangleq \mathbf{R}_o(\mathbf{x}) + \mathbf{R}_{\mathcal{X}}(\mathbf{x}) = \mathbf{R}_o(\mathbf{x}) + \mathbf{D}^T(\mathbf{x})\mathbf{W}(\mathbf{x})\mathbf{D}(\mathbf{x}) \\ \mathbf{S}(\mathbf{x}) &\triangleq \mathbf{S}_{\mathcal{X}}(\mathbf{x}) = \mathbf{C}^T(\mathbf{x})\mathbf{W}(\mathbf{x})\mathbf{D}(\mathbf{x}). \end{aligned}$$

The state-feedback gain matrix that minimizes (43) is

$$\begin{aligned} \mathbf{K}(\mathbf{x}) &= \mathbf{R}^{-1}(\mathbf{x})[\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) + \mathbf{S}^T(\mathbf{x})] \\ &= \mathbf{K}_o(\mathbf{x}) + \mathbf{K}_{\mathcal{X}}(\mathbf{x}) \end{aligned} \quad (44)$$

with

$$\begin{aligned} \mathbf{K}_o(\mathbf{x}) &\triangleq [\mathbf{R}_o(\mathbf{x}) + \mathbf{D}^T(\mathbf{x})\mathbf{W}(\mathbf{x})\mathbf{D}(\mathbf{x})]^{-1}\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) \\ \mathbf{K}_{\mathcal{X}}(\mathbf{x}) &\triangleq [\mathbf{R}_o(\mathbf{x}) + \mathbf{D}^T(\mathbf{x})\mathbf{W}(\mathbf{x})\mathbf{D}(\mathbf{x})]^{-1}\mathbf{D}^T(\mathbf{x})\mathbf{W}(\mathbf{x})\mathbf{C}(\mathbf{x}), \end{aligned}$$

and $\mathbf{P}(\mathbf{x}) \ge \mathbf{0}$ satisfies the SDRE

$$\begin{aligned} &\mathbf{P}(\mathbf{x})\tilde{\mathbf{A}}(\mathbf{x}) + \tilde{\mathbf{A}}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) \\ &\qquad -\mathbf{P}(\mathbf{x})\mathbf{B}(\mathbf{x})\tilde{\mathbf{R}}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) + \tilde{\mathbf{Q}}(\mathbf{x}) = \mathbf{0}. \end{aligned} \quad (45)$$

Dropping the dependence on $\mathbf{x}$ for notational simplicity, the coefficients of the SDRE (45) become

$$\begin{aligned} \tilde{\mathbf{A}} &\triangleq \mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{S}^T = \mathbf{A} - \mathbf{B}[\mathbf{R}_o + \mathbf{D}^T\mathbf{W}\mathbf{D}]^{-1}\mathbf{D}^T\mathbf{W}\mathbf{C} \\ \tilde{\mathbf{Q}} &\triangleq \mathbf{Q} - \mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T = \mathbf{Q}_o + \mathbf{C}^T[\mathbf{W} - \mathbf{W}\mathbf{D}(\mathbf{R}_o + \mathbf{D}^T\mathbf{W}\mathbf{D})^{-1}\mathbf{D}^T\mathbf{W}]\mathbf{C} \\ \tilde{\mathbf{R}} &\triangleq \mathbf{R} = \mathbf{R}_o + \mathbf{D}^T\mathbf{W}\mathbf{D}. \end{aligned}$$

The control law (44) that results from solving the SDRE nonlinear regulator problem with state constraints exhibits an additive multi-objective structure of the state-dependent gain, with the interpretation that $\mathbf{K}_o(\mathbf{x})$ is designed for stabilization and performance, whereas $\mathbf{K}_{\mathcal{X}}(\mathbf{x})$ is designed to satisfy the state constraints (38). The objective then becomes choosing $\mathbf{W}(\mathbf{x}) > \mathbf{0}$ such that (38) is enforced as $\mathbf{x}$ approaches $\partial\mathcal{X}$, and $\mathbf{x} \to \mathbf{0}$ asymptotically.

A simple choice for the weighting $\mathbf{W}(\mathbf{x})$ is based on the *distance of* $\mathbf{x}$ *to the boundary* $\partial\mathcal{X}$. For each fixed $\mathbf{x}$, let

$$\phi_i(\mathbf{x}) = \frac{1}{\left(\|h_i(\mathbf{x})\|_2 + \varepsilon_i\right)^{2N_i}}, \; i = 1, \ldots, p$$

with $N_i \in \mathbb{Z}$, $N_i > 1$ and $0 < \varepsilon_i < 1$. Then, defining

$$\mathbf{W}(\mathbf{x}) = \operatorname{diag}(\phi_1(\mathbf{x}), \ldots, \phi_p(\mathbf{x})), \quad (46)$$

as $\mathbf{x} \to \partial\mathcal{X}$, $h_i(\mathbf{x}) \to 0$ and thus $\phi_i(\mathbf{x}) = \frac{1}{\varepsilon_i^{2N_i}}$. The tuning parameters, N_i and ε_i, can be selected to make $\phi_i(\mathbf{x})$ as large as necessary as $\mathbf{x}$ approaches $\partial\mathcal{X}$.

Another weighting strategy for SDRE design of nonlinear regulator systems with state constraints was considered in Friedland (1998), and is based on *state penalty*. Let $\mathcal{I}$ denote the set of all the states to be penalized, and define $\mathbf{z} = \mathbf{C}\mathbf{x}$, where $\mathbf{C} = \operatorname{diag}(\kappa_1, \ldots, \kappa_n)$ and $\kappa_i = 1 \forall i \in \mathcal{I}$ and 0 otherwise. For the particular case of symmetric state constraints $h_i(\mathbf{x}) = |x_i| - B_i, i \in \mathcal{I}$,

$$\phi_i(\mathbf{x}) = \left(\frac{x_i}{\nu_i B_i}\right)^{2N_i},$$

where $0 < \nu_i \le 1$. In this case, the weight $\mathbf{W}(\mathbf{x})$ is also given by (46) with $p = n$. Clearly, as $\mathbf{x} \to \partial\mathcal{X}$, $h_i(\mathbf{x}) \to 0$ (that is,

$|x_i| \to B_i$) and thus $\phi_i(\mathbf{x}) = (\frac{1}{\nu_i})^{2N_i}$ for some i, and as $\mathbf{x} \to \mathbf{0}$, $\phi_i(\mathbf{x}) \to 0$. The satisfaction of the state constraints can then be guaranteed by a suitable choice of N_i and ν_i.

Uncontrollable and Unstable but Bounded State Dynamics: A state of the system having uncontrollable and unstable but bounded dynamics results in the parameterized pair $\{\mathbf{A}(\mathbf{x}, \alpha), \mathbf{B}(\mathbf{x})\}$ not being pointwise stabilizable, which in turn means that a legitimate Riccati equation solution is not obtainable. This situation can be handled by simply adding a stabilizing term to the dynamics of the unstable state (Hull *et al.*, 1998; Cloutier & Stansbery, 2001). For example, if x_1 is the unstable state, the term $-\lambda x_1$ with $\lambda > 0$ is added to the dynamics.

7. ISSUES FOR INVESTIGATION

7.1 Implementation

In implementing the SDRE approach, the most desirable option is to solve the state-dependent Riccati equation (8) using a symbolic software package. This may be possible for some systems having special structures, such as sparseness or $\mathbf{Q}(\mathbf{x}) = \mathbf{0}$. In general, however, an analytical solution cannot be obtained, and the second option is to solve the SDRE on-line at a relatively high rate. The step length between successive solutions of (7)-(9) can be set by a simple Euler or by the Runge-Kutta routine.

Computational implementation is an important practical consideration. Implementing the SDRE algorithm, at least for *simulations*, is relatively straightforward and can be easily mechanized using commercially available software. On-line computation of SDRE feedback controls makes the technique ideal for real-time implementation, so that the controller must perform all operations in “real-time”. The computational simplicity of this control algorithm together with the current advances in technology make these features practically realizable (Menon & Ohlmeyer, 2004) as demonstrated in numerous papers, such as the real-time experiments on the Nonlinear Benchmark problem in Langson & Alleyne (2002), magnetic levitation experiments in Erdem & Alleyne (2004), control of the control actuation system of a guided missile in Merttopçuoğlu *et al.* (2007), control of small autonomous helicopters in Bogdanov & Wan (2007), and control of large tankers in Çimen (*to appear*).

Computational cost has been a drawback of the SDRE method, especially for on-line control of high order systems. However, a study on real-time execution of SDRE controls by Menon, Lam, Crawford & Cheng (2002) showed that an SDRE autopilot with six states and three controls could be executed at a rate of up to 2 kHz with commercial off-the-shelf processors. Since the computational complexity is only of polynomial growth rate with state dimension, with faster computers the impact of this issue is slowly diminishing as evidenced by several application papers on real-time implementation, testing the feasibility of calculating the solution to the SDRE on-line.

7.2 Stability

Although numerous examples over the past dozen or so years have demonstrated the effectiveness of the SDRE method, a number of issues remain, most notably the issue of *global asymptotic stability*. As with all suboptimum control methods, stability is an issue in extended linearization techniques. These methods only guarantee *local asymptotic stability* (Mracek & Cloutier, 1998), provided that $\mathbf{K}(\cdot) \in C^1$ as shown in the paper. Much criticism has been leveled against the SDRE method because it does not provide assurance of global asymptotic stability. For instance, feedback linearization, when applicable, does provide such a proof since it effectively converts the nonlinear dynamic system to a linear system but, in so doing, ignores the consequences on the control signals, and hence on the performance criterion. It may result in the requirement of large control signals which, if undesirable, the suboptimum control methods seek to avoid. Surprisingly, however, empirical experience shows that in many cases the domain of attraction of extended linearization techniques may be as large as the domain of interest (see, for example, Mracek & Cloutier, 1996,1997 and 1998). Nevertheless, given the lack of an *a priori* guarantee of global asymptotic stability and given the wealth of well-understood and theoretically supported nonlinear synthesis methods such as feedback linearization, extended linearization control techniques are usually not the method of choice when the *only* concern is to stabilize the system. However, the situation changes significantly when, in addition to stability, the goal involves minimizing the cost given by a performance index such as (2). In this case, a workshop on nonlinear control (Doyle *et al.*, 1997) illustrated the fact that the performance of commonly used nonlinear design techniques (such as feedback linearization, control Lyapunov functions, and recursive backstepping) is highly problem dependent, ranging, for any given method, from near optimal to very poor. The greatest advantage offered by SDRE control is the opportunity to make tradeoffs between control effort and state errors by “heuristically tuning” the corresponding weighting matrices as functions of the state. Furthermore, the system to be controlled need not satisfy a very restrictive form of the state equations, as is the case for other nonlinear control methods such as backstepping.

There are situations in which global asymptotic stability cannot be achieved (for example, systems with multiple equilibrium states). In some applications, especially in aerospace, estimating the region of attraction may be more important. In these cases, the suboptimum methods have some promise. For the SDRE method, in particular, a method for estimating the size of the region of attraction is available, and can be automated easily with software like MATLAB®.

7.3 Optimum Factorization

The factorization of the original nonlinear dynamics, that is, $\mathbf{f}(\mathbf{x}) = \mathbf{A}(\mathbf{x})\mathbf{x}$, is not unique for systems of order greater than 1. This being the case, it is apparent that the choices of parameterizing the nonlinear dynamics lead to different control laws, and hence, different performance. It has been shown that, under mild conditions, an optimum factorization exists, in the sense that the SDRE control actually achieves the minimum performance value. However, a method of determining this factorization is not yet known.

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