Abstract: In this paper we study H∞ tracking problems with preview by output feedback for linear systems with impulsive effects and the sampled-observation on the finite and infinite time interval. We consider the problems that the reference signals are previewed in a fixed time interval and known a priori over a whole time interval, and present feedback control laws for the H∞ tracking problems. Our theory can be also applied into the sampled-control system with the control input realized through a zero-order hold and the sampled-observation.

1. INTRODUCTION

It is well known that, for the design of tracking control systems, the preview information of reference signals is very useful for improving the performance of the closed-loop systems, and recently much work has been done for preview control systems. Considering the effect of modelling uncertainties or disturbance is also very important on preview control theory. U. Shaked et al. have studied the H∞ tracking theory with preview for continuous- and discrete-time systems by the game theoretic approach ([1][2][3][4][8][9]).

Control theory for linear systems with impulsive effects (or linear jump systems), which contain linear continuous and discrete time systems, can be widely applied, for example, to mechanical systems, ecosystems, chemical processes, financial engineering and so on. It has been researched in detail by A. Ichikawa and H. Katayama ([6]). Their theory can be also applied into the sampled-data control systems with the control input realized through a zero-order hold and the sampled-observation ([5]).

In this paper we study the H∞ tracking problems with preview by output feedback for linear systems with impulsive effects (or linear jump systems). Our systems are described by the ordinary differential equations with impulsive effects and the sampled-observation. We consider two different tracking problems according to the preview lengths and give the control strategies for them respectively. Our theory can be applied into the control systems with the control input realized through a zero-order hold and the sampled-observation. Our theory can be also easily reduced to the case that only the preview information of discrete reference signals are available.

2. PROBLEM FORMULATION

Consider the following linear system with impulsive effects.

\[ x(t^+) = A_d(k)x(t^-) + B_{2d}(k)u(k) + B_{3d}(k)r_d(k) \]
\[ z_c(t) = C_1(t)x(t^-) + D_{13}(l)r_c(l), \quad t \neq k\tau \]
\[ z_d(k) = C_{1d}(k)x(k\tau^-) + D_{1d}(k)u(k) + D_{3d}(k)r_d(k) \]
\[ y(k) = C_2(k)x(k\tau^-) + D_{21}(k)w_d(k) + v(k) \]

where \( x \in \mathbb{R}^n \) is the state, \( w \in \mathbb{R}^p \) and \( w_d \in \mathbb{R}^{p_d} \) are the exogenous disturbances, \( v \in \mathbb{R}^k \) is the measurement noise, \( u \in \mathbb{R}^m \) is the control input, \( y \in \mathbb{R}^c \) is the measured output, \( z_c \in \mathbb{R}^{z_c} \) and \( z_d \in \mathbb{R}^{z_d} \) are the controlled outputs, \( r_c(t) \in \mathbb{R}^{r_c} \) and \( r_d(k) \in \mathbb{R}^{r_d} \) are known or measurable reference signals, \( x_0 \) is an unknown initial state. We assume that all matrices are of compatible dimensions. Throughout this paper the dependence of the system matrices on \( t \) or \( k \) will be omitted for the sake of notation simplification.

The H∞ tracking problems we address in this paper for the system (1) are to design control law \( w(-) \in \mathbb{L}_2[0,N] \) over the finite horizon \( [0,T] \), \( N \tau < T < (N + 1)\tau \) using the information available on the known parts of the reference signals \( r_c(t) \) and \( r_d(k) \) and minimizing the sum of the energy of \( z_c(t) \) and \( z_d(k) \), for the worst case of the initial condition \( x_0 \), the disturbances \( w(t) \in \mathbb{L}_2[0,T]; \mathbb{R}^p \) and \( w_d(k) \in \mathbb{L}_2[0,N]; \mathbb{R}^{p_d} \). We denote by \( \mathbb{L}_2([0,T]; \mathbb{R}^n) \) and \( \mathbb{L}_2([0,N]; \mathbb{R}^{z_d}) \) the space of nonanticipative signals. Considering the average of the performance indices over the statistics of the unknown parts of \( r_c \) and \( r_d \), we define the following two performance indices.

\[ J_T(x_0, u, w, r_c, r_d) = -\gamma^2 x_0' R^{-1} x_0 \]
\[ + \sum_{k=0}^{N-1} \left\{ \int_{k\tau^-}^{(k+1)\tau^-} \mathbb{E}_{R_c} \{ \| z_c(s) \|^2 \} ds \right\} \]
\[ + \sum_{k=0}^{N} \mathbb{E}_{R_d} \{ \| z_d(k) \|^2 \} - \gamma^2 \| w \|^2 \]

and

\[ J_{\epsilon T}(x_0, u, w, w_d, v, r_c, r_d) \]
where \( \gamma > 0 \) is a given weighting matrix for the initial state, \( E_{R_a} \) and \( E_{R_b} \) mean expectations over \( R_{a+h} \) and \( R_{b+h} \), \( h \) is the preview length of \( r_c(t) \) and \( r_d(k) \), and \( R_a \) and \( R_b \) denote the future information on \( r_c \) and \( r_d \) at time \( s \) and \( j \tau \) respectively, i.e., \( R_a := \{ r_c(l); s < l \leq T \} \) and \( R_b := \{ r_d(j); j < i \leq N \} \).

We consider two different tracking problems according to the information structures (preview lengths) of \( r_c \) and \( r_d \) as follows.

**Case a) \( H_\infty \) Fixed-Preview Tracking:**

In this case, it is assumed that at the current time \( t (k \tau + s < t 
( k + 1) \tau) \), \( r_c(s) \) is known for \( s \leq \min(T, s + h \tau) \) and at the time \( k \tau \), \( r_d(i) \) is known for \( i \leq \min(N, k + h) \), where \( h \) is the preview length.

**Case b) \( H_\infty \) Tracking of Noncausal \( \{ r_c(t) \) and \( r_d(k) \):**

In this case, the signals \( \{ r_c(t) \} \) and \( \{ r_d(k) \} \) are assumed to be known a priori for the whole time intervals \( t \in [0^+, T] \) and \( k \in [0, N] \).

In order to solve these problems, we formulate the following differential game problems for the system (1), the performance indices (2) and (3).

**The \( H_\infty \) Tracking Problem by State Feedback:**

Find \( \{ u^* \}, \{ w^* \} \) and \( x_0^* \) satisfying the following (saddle point) condition:

\[
J_T(x_0^*, u^*, w, r_c, r_d) \leq J_T(x_0^*, u^*, w^*, r_c, r_d)
\]

where the control strategies \( u^*(k), 0 \leq k \leq N \), are based on the current state \( x(k) \) and the information \( R_{a+h \tau} := \{ r_c(l); 0 < l \leq s + h \tau \} \) and \( R_{b+h} := \{ r_d(i); 0 < i \leq k + h \} \), \( 0 \leq h \leq N \).

**The \( H_\infty \) Tracking Problem by Output Feedback:**

Find \( \{ u^* \}, \{ w^* \}, \{ w^*_d \}, \{ v^* \} \) and \( x_0^* \) satisfying the following (saddle point) condition:

\[
J_{eT}(x_0^*, u^*, w, w_d, v, r_c, r_d) \leq J_{eT}(x_0^*, u^*, w^*, v^*, r_c, r_d)
\]

where the control strategies \( u^*(k), 0 \leq k \leq N \), are based on the observable output \( y(k) \) and the information \( R_{a+h \tau} \) and \( R_{b+h} \) with \( 0 \leq h \leq N \).

3. \( H_\infty \) Tracking Controllers by State Feedback

In this section we present the theory of \( H_\infty \) tracking with preview by state feedback. We consider the system (1) and assume the following standard conditions.

**A1:** \( D_{12d}^T D_{12d} = O \), \( D_{12d}^T C_{12d} = O \), \( D_{12d}^T D_{13d} = O \)

Now we consider the following Riccati equation with jump parts.

\[
\dot{X} + A'X + XA + C_1' C_1 \quad + \frac{1}{\gamma^2} XB_1 B_1'X = O, \quad t \neq k \tau
\]

(4)

\[
X(\kappa \tau) = [A_d'X(\kappa \tau)A_d + C_1' C_1 - A_d'X(\kappa \tau)B_d]
\]

\[
\times T_{2}^{-1}(k) B_d' X(\kappa \tau) A_d = O, \quad k = 0, 1, \cdots
\]

(5)

where \( T_{2}(k) = D_{12d}^T D_{12d} + B_d' X(\kappa \tau) B_d \).

We obtain the following saddle point strategy for our game problem.

**Proposition 3.1.** ([7]) Consider the system (1) and the performance index (2), and suppose A1. Then the \( H_\infty \) Tracking Problem is solvable by State Feedback if and only if there exists a matrix \( X(t) \) greater than \( 0 \) satisfying the conditions \( X(0^+) < \gamma^2 R^{-1} \) and \( X(T) = O \) such that the Riccati equation (4)(5) holds over \([0, T] \). A saddle point strategy is given by

\[
x_0^* = \gamma^2 R^{-1} - X(0^+) \gamma \]

\[
w^* = \frac{1}{\gamma} B_1' (X + \theta)
\]

\[
u^*(k) = -T_{2}^{-1}(k) B_d'
\]

\[
x(\kappa \tau) (A_d x(\kappa \tau) + B_d r_d(k)) + \theta_c(\kappa \tau).
\]

(6)

where

\[
\theta_c = \frac{1}{\gamma^2} B_1 B_1' X + \frac{1}{\gamma^2} B_1 B_1' X + \frac{1}{\gamma^2} B_1 B_1' X + \frac{1}{\gamma^2} B_1 B_1' X.
\]

\[
\theta_c(\kappa \tau) = \frac{1}{\gamma^2} B_1 B_1' X + \frac{1}{\gamma^2} B_1 B_1' X + \frac{1}{\gamma^2} B_1 B_1' X + \frac{1}{\gamma^2} B_1 B_1' X.
\]

(7)

and \( \theta_c \) is the 'causal' part of \( \theta(\cdot) \) at time \( t \). This \( \theta_c \) is the expected value of \( \theta \) over \( R_a \) and \( R_b \) and given by

\[
\hat{\theta}_c(s) = \frac{1}{\gamma^2} B_1 B_1' X + \frac{1}{\gamma^2} B_1 B_1' X + \frac{1}{\gamma^2} B_1 B_1' X + \frac{1}{\gamma^2} B_1 B_1' X.
\]

(8)

Moreover, the value of the game is

\[
J_T(x_0^*, u^*, w^*, r_c, r_d) = \sum_{k=0}^{N} E_{R_a} \{ \| T_{2}^{-1}(k) B_d \theta_1(\kappa \tau) \|^2 \}
\]

\[
+ \hat{\theta}_c(r_c) + \hat{\theta}_d(r_d)
\]

(9)
\[ J_d(r_d) = \gamma^2 E_{R_0} \{ \| \theta(0^-) \|^2 \}
+ \sum_{k=0}^N E_{R_k} \{ -\| T_2^{-1}(k) B_1 \theta(k \tau^+) \|^2_{T_2(k)} + \| D_{3d} r_d(k) \|^2 \}
+ r_d(k) B_{3d} X(k \tau) [I_n - B_d T_2^{-1}(k) B_d' X(k \tau) B_d r_d(k)]
+ 2 \theta^T(k \tau^+) [I_n - T_2^{-1}(k) B_d' X(k \tau) B_d r_d(k) + \theta_r^T(k \tau)] \] and \( P_0 = [R^{-1} - \gamma^2 X(0^-)]^{-1}. \)

4. \( H_{\infty} \) TRACKING WITH PREVIEW BY OUTPUT FEEDBACK

In this section, utilizing the result in the previous section, we present the solution of the \( H_{\infty} \) tracking problems by output feedback for the system \( (1) \) and the design method of output feedback controllers for these problems. For the system \( (1) \), we assume the following standard condition in addition to \( A1 \).

A2: \( B_d' D_{21} = 0 \)

Introducing
\[ \tilde{u}(k) = u(k) + T_2^{-1}(k) B_d' [X(k \tau) B_{3d} r_d(k) + \theta(k \tau^+)] \]
and
\[ \tilde{w} = w - \gamma^2 B_1' (X \theta + \theta), \]
using the Riccati equation \( (4) \) with the jump parts \( (5) \) and the terminal condition \( X(T) = O \), the performance index \( J_{vT} \) can be rewritten as
\[
\begin{align*}
J_{vT}(x_0, \bar{u}, \bar{w}, w_d, v, r_c, r_d)
&= \left\{ \sum_{k=0}^{N-1} \int_{k \tau^+}^{(k+1) \tau^+} + \int_{N \tau^+}^{T} \right\} E_{R_k} \{ -\gamma^2 \| \tilde{w} \|^2 \} ds \\
&+ \sum_{k=0}^{N-1} E_{R_k} \{ \| \tilde{u}(k) + T_2^{-1}(k) B_d' X(k \tau) A_d x(k \tau) \|^2_{T_2(k)} \}
+ J_c(r_c) + J_d(r_d)
- \gamma^2 \left[ \| w_d \|^2 + \| v \|^2 \right]
\end{align*}
\]
where \( P_0 = [R^{-1} - \gamma^2 X(0^-)]^{-1} \) and \( \bar{x}_0 = \gamma^{-2} P_0 \theta(0). \)
Therefore, our output feedback problem can be reduced to the problem of finding the maximizing \( x_0, \bar{w}, w_d \) and \( v \), and minimizing \( \bar{u}(k) \) for the performance index \( (11) \). Using the above \( \tilde{u}(k) \) and \( \tilde{w} \), we can rewrite the system dynamics as follows.
\[
\begin{align*}
\dot{x} &= \tilde{A} x + B_1 \bar{w} + \bar{r}_c, \ t \neq k \tau, \ x(0) = x_0
x(k \tau^+) &= A_d x(k \tau) + B_d \tilde{u}(k) + \bar{r}_d
\end{align*}
\]
where \( \tilde{A} = A + \frac{1}{\gamma^2} B_1' X, \bar{r}_c = B_3 r_c + \frac{1}{\gamma^2} B_1' \theta \) and \( \bar{r}_d = B_{3d} r_d(k) - B_d T_2^{-1}(k) B_d' X(k \tau) B_d r_d(k) + \theta(k \tau^+) \). For this system, we consider the following type of controller.
\[
\begin{align*}
\dot{x} &= \tilde{A} x + \bar{r}_c, \ t \neq k \tau, \ \hat{x}(0) = \hat{x}_0
\dot{x}(k \tau^+) &= A_d \hat{x}(k \tau) + B_d \bar{u}_s(k) + \bar{r}_d
+ L_1 [g(k) - C_2 \hat{x}], \ \hat{x}(0) = x_0^*
\bar{u}_s(k) &= -T_2^{-1}(k) B_d' X(k \tau) A_d \hat{x}(k \tau)
+ L_2 [g(k) - C_2 \hat{x}(k \tau)]
\end{align*}
\]
where \( L_1 \) and \( L_2 \) are the controller gains to decide later, using the solutions of the Riccati equations.

Let \( e := x - \hat{x} \), we get the error system
\[
\begin{align*}
\dot{e} &= \tilde{A} e + B_1 \bar{w}, \ t \neq k \tau, \ e(0) = x_0 - \hat{x}_0
eq e(k \tau^+) = (A_d - L_1 C_2) e(k \tau)
+ [-L_1 D_{21} - L_1] \begin{bmatrix} w_d \\ v \end{bmatrix}. \tag{13}
\end{align*}
\]
Now we define the estimated output as follows.
\[
\begin{align*}
f(k) := T_2^+(k) \{ \bar{u}_s(k) + T_2^{-1}(k) B_d' X(k \tau) A_d x(k \tau) \}
&= T_2^+(k) \{ (T_2^{-1}(k) B_d' X(k \tau) A_d
+ L_2 C_2) e(k \tau) + [L_2 D_{21} L_2] \begin{bmatrix} w_d \\ v \end{bmatrix} \}
\end{align*}
\]
For the error system \( (13) \) with the estimated output \( f(k) \), we consider the problem to find \( x_0, \bar{w}, v \) and \( w_d \) maximizing the performance index
\[
\begin{align*}
J_{vT}(x_0, \bar{u}, \bar{w}, w_d, v, r_c, r_d)
&= \left\{ \sum_{k=0}^{N-1} \int_{k \tau^+}^{(k+1) \tau^+} + \int_{N \tau^+}^{T} \right\} E_{R_k} \{ -\gamma^2 \| \tilde{w} \|^2 \} ds \\
&+ \sum_{k=0}^{N} E_{R_k} \{ ||f(k)||^2 \} - \gamma^2 E_{R_0} \{ \| x_0 - \hat{x}_0 \|^2_{P_0^{-1}} \}
+ J_{L}(r_c) + J_{d}(r_d) - \gamma^2 \left[ \| w_d \|^2 + \| v \|^2 \right]. \tag{14}
\end{align*}
\]
Note that this problem is equivalent to the problem to give \( x_0, \bar{w} \) and \( w_d \) maximizing
\[
\begin{align*}
&\left\{ \sum_{k=0}^{N-1} \int_{k \tau^+}^{(k+1) \tau^+} + \int_{N \tau^+}^{T} \right\} E_{R_k} \{ -\gamma^2 \| \tilde{w} \|^2 \} ds - \gamma^2 \| w_d \|^2 \\
&+ \sum_{k=0}^{N} E_{R_k} \{ ||\hat{z}(k)||^2 \} - \gamma^2 E_{R_0} \{ \| x_0 - \hat{x}_0 \|^2_{P_0^{-1}} \}
\end{align*}
\]
for the system
\[
\begin{align*}
\dot{\hat{e}} &= \tilde{A} \hat{e} + B_1 \bar{w}, \ t \neq k \tau, \ e(0) = x_0 - \hat{x}_0
e(k \tau^+) &= \hat{A}_d e(k \tau) + \hat{B}_d w_d
\hat{z}(k) &= \hat{C} e(k \tau) + \hat{D}_w w_d, \ 
\hat{A}_d = A_d - L_1 C_2, \ 
\hat{B}_d = [-L_1 D_{21} - L_1], \ 
\hat{C} = T_2^+(k) (T_2^{-1}(k) B_d' X(k \tau) A_d + L_2 C_2), \ 
\hat{D}_w = T_2^+(k) [L_2 D_{21} L_2], \ 
\hat{w}_d = \begin{bmatrix} w_d \\ v \end{bmatrix}.
\end{align*}
\]
Namely our problem can be reduced to the so-called output estimation (OE) problem on the standard \( H_{\infty} \) disturbance attenuation theory. In order to solve this problem, we consider the following Riccati equation with the jump parts and the initial condition for it.
\[
\dot{Q} + \hat{A}' Q + Q \hat{A} + \frac{1}{\gamma^2} Q B_1 B_1' Q = O, \ t \neq k \tau \tag{15}
\]
\[
J_0 \]
\[ Q(k\tau -) = \hat{A}'Q(k\tau)\hat{A} + \hat{C}'\hat{C} \] (16)
\[ + \hat{R}'(k)T^{-1}(k)\hat{R}(k), \quad Q(0^-) = \gamma^2P_0^{-1} \]
where \( \hat{R}(k) = \hat{B}'Q(k\tau)\hat{A} + \hat{D}'\hat{C} \)
and \( \hat{T}(k) = \gamma^2I_{pa+k} - \hat{D}'\hat{D} - \hat{B}'Q(k\tau)\hat{B}, \) and we assume
\[ \hat{T}(k) > 0. \] (17)

Then, using the Riccati equation (15)(16) and completing the performance index (14) with respect to \( col(w_d', v') \), we have ([5],[6])

\[ J_{cT}(x_0, \hat{u}_*, \hat{w}, w_d, v, r_c, r_d) = -\sum_{k=0}^{N-1} \left\{ \int_{k\tau^+}^{(k+1)\tau^+} T \right\} \mathbf{E}_{R_c}(\gamma^2\|\hat{w} - \gamma^{-2}B'_1Qe\|^2)ds \]
\[ + \int_{k\tau^+}^{(k+1)\tau^+} \mathbf{E}_{R_d}(\|\hat{w}_d - \gamma^{-2}B'_1Qe\|^2)ds \]
\[ + J_c(r_c) + J_d(r_d) - c'(T)Q(T)e(T) \]
\[ - \sum_{k=0}^{N} \mathbf{E}_{R_c}(\|T^{-1/2}(k)[\begin{bmatrix} w_d' \\ v \end{bmatrix} - T^{-1}(k)R_d(k)e(k)]\|^2). \]

For \( \hat{w}, w_d, v \) and \( x_0 \) in the error system (13), let \( \hat{w} = w*, = \gamma^{-2}B'_1Qe, \)
\[ \begin{bmatrix} w_d' \\ v \end{bmatrix} = \begin{bmatrix} \bar{w}_d' \\ v \end{bmatrix} = \hat{T}_{d}(k)\hat{R}(k)e(k), \]
\[ x_0 = x_0^* = \bar{x}_0. \]
Then we get \( e(t) = 0 \) over \( [0, T] \), because \( e(t) = 0 \) is an equilibrium point of the error system (13) for \( t \in [0, T] \). Therefore, since \( \begin{bmatrix} \bar{w}_d' \\ v \end{bmatrix} \) \( (k) = 0, \ k \in [0, N], \) the estimated output
\[ f(k) = \hat{C}e(k\tau) + \hat{D} \begin{bmatrix} \bar{w}_d' \\ v \end{bmatrix} \]
for \( k \in [0, N]. \) As a result, we get
\[ J_{cT}(x_0^*, \hat{u}_*, \hat{w}, w_d, v, r_c, r_d) = J_c(r_c) + J_d(r_d). \]

Now we adopt \( \hat{u}_* = \hat{u}_* \) as the optimal input minimizing \( J_{cT} \) for the worst case disturbance and the worst case initial state and so
\[ J_{cT}(x_0^*, \hat{u}_*, \hat{w}, w_d, v, r_c, r_d) = \sum_{k=0}^{N} \mathbf{E}_{R_c}(\|\hat{w}(k) - T_{2}^{-1}(k)B'_dQe(k)\|_{T_2(k)}^2) \]
\[ + J_c(r_c) + J_d(r_d) \]
\[ \geq J_c(r_c) + J_d(r_d) \]
\[ = J_{cT}(x_0^*, \hat{u}_*, \hat{w}, w_d, v, r_c, r_d) \]
because \( x_0^* = \bar{x}_0, \ w^* = 0, \ w_d^* = 0, \ v^* = 0 \) and \( e(t) = 0 \) for \( t \in [0, T]. \) Moreover, using this optimal input \( \hat{u}_* = \hat{u}_*, \) the inequality
\[ J_{cT}(x_0, \hat{u}_*, \hat{w}, w_d, v, r_c, r_d) = \sum_{k=0}^{N} \mathbf{E}_{R_c}(\|\hat{w}(k) - T_{2}^{-1}(k)R_d(k)e(k)\|_{T_2(k)}^2) \]
\[ + J_c(r_c) + J_d(r_d) \]
\[ \leq J_c(r_c) + J_d(r_d) \]
\[ = J_{cT}(x_0^*, \hat{u}_*, \hat{w}, w_d, v, r_c, r_d) \]
holds. By these inequalities, the following theorem, which gives the solution of \( H_\infty \) tracking problem by output feedback,
hold.

**Proposition 4.1.** Consider the system (1) and the performance index (3), and suppose A1 and A2. Then the \( H_\infty \) Tracking Problem is solvable by Output Feedback if and only if there exist \( X(t) > 0 \) and \( Q(t) > 0 \) satisfying the conditions \( X(0^-) < \gamma^2R^{-1}, X(T) = O \) and \( Q(0^-) = \gamma^2P_0^{-1} \) such that the Riccati equations (4)(5), (15)(16) and the condition (17) hold over \([0, T].\) A saddle point strategy is given by
\[ x_0^* = \gamma^2R^{-1} - X(0^-)^{-1}\theta(0), \]
\[ w^* = \frac{1}{\gamma}B'_1(X + \theta), \ w_d^* = 0, \ v^* = 0, \]
\[ u_c^*(k) = -T_{2}^{-1}(k)B'_d \]
\[ \times [X(k\tau)(A_d\bar{x}_d(k\tau) + B_{3d}r_d(k)) + \theta_c(k\tau^+)] \]
\[ + L_2[y(k) - C_2\bar{x}_c(k\tau)] \]
where \( \theta(t), t \in [0, T] \) satisfies (6) and \( \theta_c(s), s \in [t, t + h] \) satisfies (7), \( \bar{x}_d(t) \) is the ‘causal’ part of (12) at time \( t. \) \( \bar{x}_c \) is the expected value of \( \bar{x} \) over \( \hat{R}_s \) and \( \hat{R}_h, \) but the actual value \( \bar{x}_d(t) \) is determined based on the information \( g(k), R_s(t+h) \) and \( R_{h}+h \) with \( 0 \leq h \leq N. \) Moreover, the value of the game is
\[ J_{cT}(x_0^*, \hat{u}_*, w^*, w_d', v, r_c, r_d) = \sum_{k=0}^{N} \mathbf{E}_{R_c}(\|T_{2}^{-1}(k)B'_dQe(k)\|_{T_2(k)}^2) \]
\[ + J_c(r_c) + J_d(r_d) \]
(18)
where \( \theta_1(t) = \theta(t) - \theta_c(t), \bar{x}_d(t) = \bar{x}(t) - \bar{x}_c(t), t \in [0, T], \)
\( J_c(r_c) \) and \( J_d(r_d) \) are given by (9) and (10).

**(Proof)**

**Sufficiency** We have already described the sufficient condition for the solvability of the tracking problem. The optimal input \( u_c^*(k) \), \( k \in [0, N] \) can be adopted using only the causal parts \( \theta_c(t), \bar{x}_d(t) \) determined on-line based on the information \( R_{s+h}, R_{h}+h \) and \( g(k), k \in [0, N] \).

**Necessity.** On this game problem, the reference signals \( r_c(t) \) and \( r_d(t) \) are arbitrary. Therefore, by considering the case of \( r_c(t) \equiv 0 \) and \( r_d(t) \equiv 0, \) one can easily deduce the necessity for the solvability of our game problem. (Refer to [5][6] and etc.) (QED.)

**Remark 4.1.** \( J_c(r_c) \) and \( J_d(r_d) \), which mean the tracking errors including the preview information vector \( \theta \), are equal to zero, if \( r_c = 0, r_d = 0 \) and \( \theta(t) = 0 \) at all \( t \in [0, T]. \) Namely, in the case of neither inputting any reference signals nor considering any preview information, these tracking error terms are reduced to zero.

The saddle-point strategies are given by arbitrary \( \{r_c\} \) and \( \{r_d\} \) and the jump parts (16) of the Riccati equation do not depend on the coefficient matrices \( B_{3d}, D_{13d}, D_{13d} \) of the reference signals \( \{r_c\} \) and \( \{r_d\} \). Therefore we can deduced the controller gains \( L_1 \) and \( L_2 \) by considering...
the case where \( \{r_c\} \) and \( \{r_d\} \) are identically zero. (See ([5][6]) for details.) Now let \( Z(t) = \gamma^2 Q^{-1}(t) \) with \( Z(k\tau) = \gamma^2 Q^{-1}(k\tau) \) and \( Z(k\tau^+) = \gamma^2 Q^{-1}(k\tau) \) We consider the following Riccati equation with jump parts.

\[
\dot{Z} = \bar{A}Z + Z\bar{A}^T + B_1B_1^T t \neq k\tau
\]  
\[
Z(k\tau^+) = A_d Z(k\tau)A_d^T
\]
\[
- (R_{22}T_{22}R_{22})'(k) + (F_{12}Z F_{12})'(k)
\]
\[
V_z(k) > aI \text{ for some } a > 0
\]
\[
(19)
\]
\[
(20)
\]
\[
(21)
\]
where
\[
\begin{align*}
R_2(k) &= B_d^T X(k\tau) A_d, \\
T_{12}(k) &= \gamma^2 I_m - T_{22}^{-1/2} R_2(k) Z(k\tau) R_{22}^T T_{22}^{-1/2}(k), \\
T_{22}(k) &= \bar{D}_{21}\bar{D}_{21}^T + C_d Z(k\tau) C_d^T, \\
R_{12}(k) &= T_{22}^{-1/2} R_2(k) Z(k\tau) A_d^T, \\
R_{22}(k) &= C_d Z(k\tau) A_d^T, \\
S_{22}(k) &= C_d Z(k\tau) R_{22}^T T_{22}^{-1/2}(k), \\
V_z(k) &= [T_{12}^T + S_{22}^T T_{22}^{-1} S_{22}] (k), \\
F_{12}(k) &= [V_{Z}^{-1}(R_{12} - S_{22} T_{22}^{-1} R_{22})](k), \\
F_{22}(k) &= -[T_{22}^{-1} (R_{22} + S_{22} F_{12})](k), \\
\dot{\bar{D}}_{21} &= [D_{21} \bar{A}_d].
\end{align*}
\]

Finally, utilizing this Riccati equation, we get the following theorem.

**Theorem 4.1.** Consider the system (1) and the performance index (3), and let \( \gamma > 0 \) be a given scalar. Suppose A1 and A2. Then each of the H\(_\infty\) Tracking Problems is solvable by Output Feedback if and only if there exist matrices \( K \) and \( K_d \) such that (A + BZK, A + BDKd) is exponentially stable.

The sufficient and necessary conditions for the solvability of the H\(_\infty\) tracking problems in the finite horizon case are the same as the ones for the solvability of the standard H\(_\infty\) control problems and so we can obtain the convergence and stability conditions for the H\(_\infty\) tracking problems in the infinite horizon case.

Now we consider the following conditions for the system (1).

**J1:** ([A, A_d], [B_1, O], [C_1, C_1d]) is stabilizable and detectable.

**J2:** ([A, A_d], [O, B_2d], [O, C_2d]) is stabilizable and detectable.

**Theorem 5.1.** Consider the system (1) and the performance index (3) with \( T \to \infty \), and let \( \gamma > 0 \) be a given scalar. Suppose A1 and A2. Also suppose that \( R \) is sufficiently small. Then each of the H\(_\infty\) Tracking Problems is solvable by Output Feedback if and only if there exist \( \tau \)-periodic stabilizing solutions \( X(t) > O \) and \( Z(t) > O \) satisfying (19)-(21) over \([0, T]\) such that \( X(0) < \gamma^2 R^{-1} \), \( X(T) = O \) and \( Z(0) = P_0 \). Then the output feedback controller for each case of the H\(_\infty\) tracking problems is given by (22) and (23) respectively Now \( \theta \) and \( \vartheta \) are given by

\[
\begin{cases}
\dot{\theta}(t) = -\bar{A}(t)\theta(t) + \bar{B}(t)r_c(t), t \neq k\tau \\
\theta(k\tau^+) = \bar{A}(k\tau)\theta(k\tau^+) + \bar{B}(k\tau)r_c(k)
\end{cases}
\]

and

\[
\begin{cases}
\dot{\vartheta}(s) = -\bar{A}(s)\vartheta(s) + \bar{B}(s)r_c(s), s \neq k\tau \\
\theta_c(t) = \bar{A}(c)\theta_c(c) + \bar{B}(c)r_c(c), \quad (k\tau^+ \leq t \leq k\tau + \tau_f)
\end{cases}
\]

where, for \( k\tau^+ \leq t \leq (k + 1)\tau \), \( \tau_f = t + h\tau \) and \( k_f = k + h \).
6. A NUMERICAL EXAMPLE

We consider the following numerical example, (cf., [1], [8])

\[ \dot{x} = A \dot{x} + B_1 w_s + B_2 \tilde{u}, \quad \tilde{u} \in \mathbb{R}^2, w_s \in \mathbb{R}^1, \]
\[ g(k) = C_2 x(k\tau) + D_{21} w_d(k) + v(k) \]
\[ A = \begin{bmatrix} 0 & 1 \\ -1 & 0.4 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \]
\[ C_2 = [0 1 0], \quad D_{21} = [0 1] \]

where the control input \( \tilde{u} \) is realized through a zero-order hold i.e., \( \tilde{u}(t) = u(k), \quad k\tau < t < (k + 1)\tau \), and \( \tau \) is a sampling period. The dynamics of this system can be represented by the following linear system with impulsive effects (or linear jump system). ([5] [6])

\[ \dot{x} = Ax + B_1 w, \quad t \neq k\tau, \quad x(0) = x_0 \]
\[ x(k\tau^+) = A_d x(k\tau) + B_d u(k), \quad x \in \mathbb{R}^3, w \in \mathbb{R}^1 \]

where

\[ A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0.4 & 1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \]
\[ B_1 = \text{col}(1,-1,0) \] and \( B_d = \text{col}(0,0,1) \). Motivated by the above jump system representation of the sampled-data system, we consider the following jump system with a feedforward term of the reference signal at the jump part and introduce an objective function.

\[ \dot{x} = Ax + B_1 w, \quad t \neq k\tau, \quad x(0) = x_0 \]
\[ x(k\tau^+) = A_d x(k\tau) + B_d u(k) + B_{3d} r_d(k) \]
\[ z_d(k) = C_{1d} x(k\tau) + D_{12d} u(k) + D_{13d} r_d(k) \]
\[ y(k) = C_2 x(k\tau) + D_{21} w_d(k) + v(k) \]

where

\[ B_{3d} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \quad C_{1d} = \begin{bmatrix} 0.35 & 1.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]
\[ D_{12d} = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \quad D_{13d} = \begin{bmatrix} -1.0 \\ 0 \end{bmatrix} \]

\[ J_{dt}(x_0, u, w, w_d, v, r_d) = -\gamma^2 x_0^T R^{-1} x_0 \]
\[ -\gamma^2 [\|w\|^2 + \|w_d\|^2 + \|v\|^2] \]
\[ + \sum_{k=0}^{N} E_{R_k} \{[C_{1d} x(k) - r_d(k)]^2 + 0.5^2 \|u(k)\|^2 \} \]

where \( N\tau < T < (N + 1)\tau \) and \( T \) is assumed to be very large. By the term \( B_{3d} r_d(k) \), the tracking performance can be expected to be improve as similar to [1] [8].

Let \( \gamma = 20, \tau = 0.05 \) and we design a output feedback law by which the function \( J_{dt} \) is minimized. We apply the results of \( H_\infty \) tracking for \( r_d(k) = 2 \sin(2k) \) with various step lengths of preview, and show the simulation results. It is shown that increasing the preview steps form \( h = 0 \) to \( h = 3, 6, 9, 12 \), improves the tracking performance. In fact, the square values \( \|C_{1d} x(k) + D_{13d} r_d(k)\|^2 \) of the tracking errors are shown in Fig. 1 and it is clear the tracking error decreases as increasing the preview steps by this figure. For the preview length \( h = 24 \), the tracking performance becomes better, and, for \( h = 36 \), finally, the tracking error almost tends to zero except for a little vibration.

**REFERENCES**