Robust Output Feedback Controller Scheme for a Class of Uncertain Nonlinear Systems

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Abstract: In this study, we present a new continuous output feedback type controller mechanism for the tracking problem of a class of uncertain nonlinear systems. The proposed strategy requires the uncertainties of the dynamical system to be first order differentiable and achieves semi-global asymptotic tracking when only the system outputs are measurable. The Controller design is based on a Lyapunov-type stability argument. Simulation studies on a two link planar robotic system are presented to illustrate the feasibility of the proposed strategy.

1. INTRODUCTION

The tracking control problem for uncertain nonlinear systems has been extensively studied for decades. This extensive research interest is not only due to the theoretically challenging nature of the problem but also due to practical needs, as the mathematical model for nearly all dynamical systems in control theory contain some uncertainty and feedback alone is not enough for preferential performance. Researchers have proposed many different types of controllers depending on the nature of the uncertainty. To give a few examples, when the parametric uncertainties are constant or slowly time-varying and the function containing the overall uncertainties can be linearly parametrizable, due to its continuous nature, adaptive control (Kristic et al., 1995; Sastry and Bodson, 1989) would be the preferred choice. Unfortunately in adaptive controllers each uncertain parameter has to be adapted separately, making the tuning process (due to the parameter update gains) moderately tedious. On the other hand when the uncertainties of the system are bounded by some known norm-based function, the theory of robust control (Qu, 1998) can be applied. From the implementation point of view, unlike adaptive controllers, the robust controllers have fewer gains to deal with. However, in most cases the convergence of the tracking error into an ultimate bound can be assured with the robust controllers and over shrinking this ultimate bound, for better performance, might cause undesirable system responses (like chattering). When the uncertainties of the dynamics are periodic, learning controllers (Arimoto et al., 1984; Messner et al., 1991; Dixon et al., 2002) can be used, the down part is, there are only limited number of systems with periodic dynamics (same is also true for the desired dynamics). Recently researches have proposed alternative methods to overcome the aforementioned disadvantages of adaptive and robust controllers. Motivated by the satisfactory performance of PI type controllers for many practical systems, a nonlinear Proportional-Integral (NPI) type controller was presented for a class of uncertain dynamical systems (Ortega et al., 2002). Though the proposed controller had some discontinuities, owing to its simple structure has fewer parameters to tune. Moreover, the effects of the controller gains on the system are predictable owing to the similarities of the controller structure to linear PI type counterpart. Therefore the implementation is quite easy. In 2004, Xian et al. presented a continuous tracking controller strategy for second order differentiable uncertain dynamics systems. The proposed controller strategy achieves asymptotic tracking and was backed up by a novel Lyapunov based analysis. However, our experience with the proposed method have shown that the control input signal, similar to variable-structure controller, has high order frequency components, which in practical implementations might trigger chattering like phenomenon.

The development of controllers that only require output measurements (i.e., output feedback (OFB)) has received considerable interest in literature due to the advantages of eliminating many sensors (e.g., reduced system complexity, cost, and noise). Global solutions to the OFB link position setpoint control problem have been presented by several researchers. For example, model-based global regulating OFB controllers were proposed in (Berguis and Nijmeijer, 1993; Burkov, 1993; Kelly, 1993). With the intent of overcoming the requirement of exact model knowledge, an OFB regulator was designed (Ortega et al., 1995); however, 1

1 Global position tracking means that the controller must drive the link position error to zero for any finite, initial position and velocity tracking errors, with no conditions on the size of the initial tracking errors.
the stability result was semi-global\(^2\) asymptotic. On the other hand, a limitation that exists in almost all of the proposed OFB link position tracking controllers is the semi-global nature of the stability results. To give a few examples, a model-based observer was used to construct a semi-global exponential link position tracking controller (Lim et al., 1996). Variable structure OFB controllers were designed to compensate for parametric uncertainty (Canudas De Wit and Fixot, 1991; Canudas De Wit and Slotine 1991). Filter-based robust control schemes which produce semi-global, uniformly ultimately bounded (UUB) were designed in (Berghuis and Nijmeijer, 1994; Qu et al., 1995; Yuan and Stepanenko, 1991). Then adaptive OFB controllers were presented in (Burg et al., 1994; Burg et al., 1996; Kaneko and Horowitz, 1997; Zergeroglu et al., 1999) that yield semi-global asymptotic link position tracking in the presence of parametric uncertainty. In 1996, Loria developed a model-based controller which produces global uniform asymptotic tracking but the proposed method is only valid for a one degree-of-freedom (DOF) system. Then, a global OFB adaptive controller was designed for n-DOF robot manipulator (Zhang et al., 2000) Finally, a robust OFB tracking controller was proposed with a global, uniformly ultimately bounded link position tracking (Dixon et al., 2004).

In this work, we extend our previous result (Kuvulmaz and Zergeroglu, 2007) to output feedback case. When compared to (Ortega et al., 2002) our approach does not possess any singularities and when compared to (Xian et al., 2004) can compensate for the uncertainties for a larger class of nonlinear systems. However, our approach requires a high gain condition on the feedforward compensation gain which might be considered as a theoretical weakness.

The remaining of the paper is organized as follows: the model under consideration and the control problem are stated in Section 2. Section 3 contains the error system development, while controller design with the stability analysis to ensure asymptotic tracking and boundedness of the closed loop system are given in section 4. Simulations performed on a two-link, planar robotic mechanism are presented in Section 5 and lastly some concluding remarks with possible future research are presented in Section 6.

2 PROBLEM STATEMENT

For the ease of presentation\(^3\), we consider a second order single-input single-output dynamical system having the following form

\[ m(x) \ddot{x} + f(x, \dot{x}, \theta) = u(t) \]  \hspace{1cm} (1)

where \( x \) is the output and the notations \( \dot{x}, \ddot{x} \) are used to identify the first and second order derivatives of the output respectively. \( m(x) \neq 0 \) and \( f(x, \dot{x}, \theta) \) is first order differentiable nonlinear, uncertain function and \( u(t) \) is the control input.

Property 1: The inertia matrix can be upper and lower bounded by the following inequalities (Lewis et al., 1993)

\[ m_1 \leq m(x) \leq m_2 \]  \hspace{1cm} (2)

where \( m_1 \) and \( m_2 \) are positive constants.

Our control objective is to ensure that the state signal \( x(t) \) would track the given smooth reference trajectory \( x_f(t) \). To quantify this objective we defined the tracking error signal \( e(t) \) in the following form

\[ e_x = x_f - x \]  \hspace{1cm} (3)

In our analysis we will utilize the common assumption that the reference trajectory signal \( x_f(t) \) and its first three time derivatives are always bounded \( i.e. x_f(t), \ddot{x}_f(t), \dddot{x}_f(t), \ddot{x}_f(t) \in L_\infty \).

Velocity variables can not be measured because of that the related variables are obtained by filtering technique of position error. While position tracking error is the output of the system. The proposed filter dynamics is as follows

\[ r_f = -p - (k_i + 1)e \]  \hspace{1cm} (4)

\[ p = -p + e - e_x - (k_i + 1)r_f \]  \hspace{1cm} (5)

\[ \dot{e}_f = -e_f + r_f \]  \hspace{1cm} (6)

where the filter output \( r_f(t) \in \mathbb{R} \) will be used for the link velocity variable. \( p(t), e_f(t) \in \mathbb{R} \) are the auxiliary variables used to establish the velocity variable. \( k_i \) is the positive control gain which is defined as

\[ k_i = \frac{1}{m_1} (k_x + 1) \]  \hspace{1cm} (7)

where \( k_x \) is the positive nonlinear damping gain. In order to form the open loop position error system, dynamics of the filter output would be obtained. Taking the time derivative of (4) and inserting for \( p \) from (5) we get the dynamics of the filter output

\[ r_f = -e_x + \dot{e} - p + e - e_f - (k_i + 1)r_f \]  \hspace{1cm} (8)

Inserting the value of \( p \) from (4) in equation (8),

\[ r_f = -(k_i + 1)e + e_x - e_f - r_f \]  \hspace{1cm} (9)

where the filtered error signal \( \eta(t) \in \mathbb{R} \) is defined as

\[ \eta = \dot{e} + e_e + r_f \]  \hspace{1cm} (10)

If (10) is rearranged, we get the error dynamics as

\[ \dot{\eta} = e - r_f + \eta \]  \hspace{1cm} (11)

In addition, we define an auxiliary term named integral effect injection term \( \xi \) as

\[ 14126 \]
\[ \xi(t) = e(t) + \int_0^t (e(\tau) + r_f(\tau)) \, d\tau + e(0) \]  

\(12\)

It is obvious that the equation (12) can also be expressed as

\[ \xi(t) = \int_0^t \eta(\tau) \, d\tau \]  

\(13\)

3. ERROR SYSTEM DEVELOPMENT

Taking the time derivative of (10), multiplying both sides of the resultant equation by \(m(x)\) and inserting for \(\dot{x}(t), \dot{r}_f, e(f)\) from (1), (9) and (11) respectively, we have

\[ m \ddot{\eta} = -k_i m \eta + N - m (2 r_f + e_f) - u \]  

\(14\)

After adding and subtracting \(\frac{1}{2} m \dot{\eta}\) term to the both sides of the (14), we get

\[ m \ddot{\eta} = -k_i m \eta + N - m (2 r_f + e_f) - u + \frac{1}{2} m \dot{\eta} - \frac{1}{2} m \eta \]  

\(15\)

where the auxiliary signal is defined as

\[ \eta = m \dot{x} + f(x, \dot{x}, \theta) \]  

\(16\)

At this point we define the desired version of the auxiliary signal \(N\), \(N_d\) such that

\[ N_d := \begin{cases} N & \text{if } x_d = x \\text{or } \dot{x}_d = \dot{x} \\ 0 & \text{else} \end{cases} \]  

\(17\)

Note that due to assumption that the reference trajectory term \(x_d\) is third order differentiable, the newly defined “desired” version of the auxiliary term can be proven to be at least first order differentiable (i.e. \(N_d(t), \dot{N}_d(t) \in L_\infty\)). Adding and subtracting \(N_d\) to the right hand side of (17), we obtain

\[ m \ddot{\eta} = -k_i m \eta + \tilde{N} + N_d - u - \frac{1}{2} m \dot{\eta} \]  

\(18\)

where the function \(\tilde{N}\) is defined as

\[ \tilde{N} := N - N_d - m (2 r_f + e_f) + \frac{1}{2} m \dot{\eta} \]  

\(19\)

Remark 1: Since the auxiliary function \(N\) defined in (16) is continuously differentiable, we can show that \(\tilde{N}\) can be upper bounded in the following manner:

\[ \|k\| \leq \rho(j(t)) \|f\| \]  

\(20\)

where \(\|\|\) denotes the standard Euclidean norm, \(j(t)\) is the vector function as

\[ j := [e, e_f, r_f, \eta] \]  

\(21\)

and \(\rho(\cdot)\) is a positive defined non-decreasing bounding function.

Based on the subsequent stability analysis we propose the following nonlinear PI control law

\[ u(t) = -(k_i + 1) r_f + (k_i + 1) e + \beta \tanh(e + e_f) \]  

\[ + k_i \int_0^t (e_f + r_f)(\tau) \, d\tau \]  

\(22\)

where \(k_i, k_i\) and \(\beta\) are positive control gains. Substituting (22) into (18) the closed loop dynamics for the filtered tracking error is obtained as

\[ m \ddot{\eta} = -\frac{1}{2} m \dot{\eta} + \tilde{N} + N_d - k_i m \eta + (k_i + 1) r_f \]

\[ - k_i \int_0^t \eta(\tau) \, d\tau - \beta \tanh(e + e_f) - e \]  

\(23\)

4. ANALYSIS

Before going into the stability analysis, we state the following Lemma which will be useful by helping us prove the Lyapunov candidate function is lower bounded.

Lemma 1: Consider the auxiliary function \(\beta_i(t)\) defined as

\[ \beta_i := - \int_0^t w_i \, d\tau + \delta \]  

\(24\)

with

\[ w_i := \eta (N_d - \beta \tanh(e + e_f)) \]  

\(25\)

where the auxiliary constant term \(\delta\), explicitly given in the following form

\[ \delta := \beta \left( \ln \cosh(e(0) + e_f(0)) + 1 \right) \]

\[ - (e(0) + e_f(0)) N_d(0) \]  

\(26\)

When the constant scalar control gain \(\beta\) of (26) is selected to satisfy

\[ \beta \geq k \left[ N_d \right] \]  

\(27\)

with the high but bounded design gain \(k\) defined as

\[ k = \max \left\{ 1 + \epsilon, \frac{1}{\|\tanh(w)\|} \right\} \]  

\(28\)

where \(\epsilon > 0\). Then \(\beta_i(t)\) will always be lower bounded by zero (i.e. \(\beta_i \geq 0\)) or equivalently

\[ \int_0^t w_i(\tau) \, d\tau \leq \delta \]  

\(29\)

Proof: For presentation easiness, a simple transformation is given in the following form by using (6) and (10)

\[ \eta = w + w \]  

\(30\)

\[ w = e + e_f \]  

\(31\)

To prove the Lemma, we take only the part inside the integral in (24) and substitute (30) to obtain
\[ \int w(\tau) \, d\tau = \int \left[ w(\tau)(N_j(\tau) - \beta \tanh(w(\tau))) \right] d\tau \]
\[ + \int \frac{d w(\tau)}{d\tau} N_j(\tau) \, d\tau \]
\[ - \int \beta \frac{d w(\tau)}{d\tau} \tanh(w(\tau)) \, d\tau \, . \]

Evaluating the third term and integrating the second term on the right hand side by parts, we have
\[ \int w(\tau) \, d\tau = \int \left[ w(\tau) \left( N_j(\tau) - \frac{dN_j(\tau)}{d\tau} - \beta \tanh(w(\tau)) \right) \right] d\tau \]
\[ + \left[ w(t) \left( N_j(t) - \beta \left( \ln(\cosh(w(t))) + 1 \right) \right) \right] \]
\[ + \left[ \beta \left( \ln(\cosh(w(0))) + 1 \right) - w(0) N_j(0) \right] \, , \]
where the \( \beta \) term has been added and subtracted from the right hand side. Now we can upper bound (33) in the following way
\[ \int w(\tau) \, d\tau \leq \int \left[ w(\tau) \left( N_j(\tau) + \frac{dN_j(\tau)}{d\tau} - \beta \tanh(w(\tau)) \right) \right] d\tau \]
\[ + \left[ w(t) \left( N_j(t) - \beta \left( \ln(\cosh(w(t)) + 1) \right) \right) \right] \]
\[ + \left[ \beta \left( \ln(\cosh(w(0))) + 1 \right) - w(0) N_j(0) \right] \, . \]

Where the \( w(t) \tanh(w(\tau)) \geq 0 \) has been utilized. Notice that in (34), the integral term (the first line) on the right hand side of the inequality will exactly be zero when \( w(t) = 0 \) and will have negative values for all other values of \( w(t) \) when the controller gain \( \beta \) is selected to satisfy (27). Similarly the same selection of \( \beta \) will also ensure the negative semi definiteness of second term on the right hand side of (34) (as \( k \left( \ln(\cosh(w)) + 1 \right) - w \geq 0 \) for \( k \geq (1 + e^2) \) ). And from the definition of \( \delta_j \) given in (26), it is straightforward to show that (29) holds.

We are now ready to present the following Theorem

**Theorem 1:** The control law of (22) ensures that all the signals in the closed loop system of (23) will remain bounded and the semi-global asymptotic convergence and stability of the error signal \( e(t) \) is guaranteed in the sense that
\[
\lim_{t \to \infty} e(t) = 0
\]
provided that the control gain \( \beta \) is adjusted according to (27) and damping gain \( k_s \) is selected to satisfy
\[
k_s \rho^2 \left( \frac{\sqrt{\delta_j^2}}{\lambda_1} \right) \left| \tilde{y}(0) \right| \]
where \( \lambda_1 \) and \( \lambda_2 \) are defined as
\[
\lambda_1 = \frac{1}{2} \min \left\{ m_1, 1, k_s, 0 \right\}
\]
\[
\lambda_2 = \frac{1}{2} \max \left\{ m_2, 1, k_s, \delta_j \right\}
\]
and \( s(t) \) is the vector function defined as
\[
s = \left[ \sqrt{z} \quad \sqrt{\xi} \right]^T
\]

**Proof:** To prove the Theorem we define a non negative function of the form
\[
V(t) = \frac{1}{2} m \eta^2 + Z + \frac{1}{2} k_s \xi^2
\]
with \( Z(t) \) is defined to have the form
\[
Z := \beta_p + \beta_i
\]
where \( \beta_p \) is selected as
\[
\beta_p = \frac{1}{2} \tilde{e}^2 + \frac{1}{2} \tilde{e}_j^2 + \frac{1}{2} \tilde{r}_j^2
\]
and \( \beta_i \) term is the lower bounded function defined in (24). We can upper and lower bound (40) by using \( \lambda \_1 \) and \( \lambda \_2 \) terms in the following way
\[
\lambda \| \| \leq V \leq \lambda \| \|
\]
Taking the time derivative of (40) substituting (7) and (23) we obtain
\[
\dot{V} = -\tilde{e}^2 - \tilde{e}_j^2 - \tilde{r}_j^2 - k_s \eta^2 + \eta \tilde{N}
\]
And from (20), we can state the following upper bound for the time derivative of function \( V \) given in (40)
\[
\dot{V} \leq -\| \| + \| \| \rho(j) \| \| - k_s \eta^2
\]
Adding and subtracting \( \rho^2(j) \| \| / 4 k_s \) term to the right hand side of (45) yields
\[
\dot{V} \leq -\| \| + \rho^2(j) \| \| / 4 k_s
\]
In the above equation the terms in the brackets is square of \( \sqrt{k_s \eta - \rho(j) \| \| / 2 \sqrt{k_s}} \) and due to the negative sign on its front, is always negative, this enables us to further upper bounded (46) to have the following form
\[
\dot{V} \leq -\left[ 1 - \rho^2(j) / 4 k_s \right] \| \|
\]
When the damping gain \( k_s \) is selected sufficiently large we can obtain
\[
\dot{V} \leq -\| \| , \quad k_s \rho^2 \| \| / 4
\]
for some \( \psi \geq 0 \). Equation (48) can also be defined as
\[
\dot{V} \leq -\| \| , \quad k_s \rho^2 \| \| / 4
\]
\[ V \leq -\psi \| \eta \|^2, \quad k_\eta \frac{1}{4} \rho^2 \left( \frac{V(t)}{\lambda_1} \right) \]

With the help of (43) we can rearrange (49) in the following way
\[ \dot{V} \leq -\psi \| \eta \|^2, \quad k_{\eta} \frac{1}{4} \rho^2 \left( \frac{\| V(0) \|}{\lambda_1} \right) \]

Notice that, \( V(0) \) is the maximum value of \( V(t) \). So we can adjust \( k_{\eta} \) according to \( V(0) \) such that
\[ \dot{V} \leq -\psi \| \eta \|^2, \quad k_{\eta} \frac{1}{4} \rho^2 \left( \frac{\| \eta \|}{\lambda_1} \right) \]

and finally when the damping gain is selected as follows we can upper bound the time derivative of \( V(t) \) according to initial conditions
\[ \dot{V} \leq -\psi \| \eta \|^2, \quad k_{\eta} \frac{1}{4} \rho^2 \left( \frac{\| \eta \|}{\lambda_1} \right) \]

From the structure of (40) and (52) it is clear that \( V \geq 0, \dot{V} \leq 0 \) \( V \in L_\infty \) and due to the structure of \( V \) all the signals contained in \( V \), are also bounded, that is \( \varepsilon(t), \eta(t), j(t) \in L_\infty \). From (11) and (21) we can conclude that \( \varepsilon(t), j(t) \in L_\infty \). Thus the control input signal of (22) is bounded. It follows from (23) that \( \dot{\eta} \in L_\infty \) that is all signals in the closed loop error system are bounded. Finally, from the structure of (52) we can conclude that \( j \), therefore \( e \in L_2 \). With the above information and direct application of Barbalat’s Lemma (Sadegh and Horowitz, 1990) we can wrap up that the tracking error term, \( e(t) \) will approach to zero as time passes to infinity, as proposed in (35).

5. SIMULATION RESULTS

To illustrate the performance of the proposed control scheme, we have performed simulations on a 2-link, revolute, direct-drive robot manipulator with the following dynamics (DDM Operations Manual, 1992)

\[
\begin{bmatrix}
\tau_1 \\
\tau_2
\end{bmatrix} =
\begin{bmatrix}
p_1 + 2p_3 \cos(x_2) & p_2 + p_3 \cos(x_2) \\
p_2 + p_3 \cos(x_2) & p_2
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} +
\begin{bmatrix}
-p_3 \sin(x_2) \dot{x}_2 & -p_3 \sin(x_2) \dot{x}_2 + \dot{x}_2 \\
p_3 \sin(x_2) \dot{x}_1 & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2
\end{bmatrix}
\]

where \( x, \dot{x}, \ddot{x} \) are link position, velocity and acceleration vectors respectively. The unknown but constant parameters representing the robot parameters are taken as \( p_1 = 3.473 \) [kg.m²], \( p_2 = 0.193 \) [kg.m²], \( p_3 = 0.242 \) [kg.m²], \( f_{\|} = 5.3 \) [Nm.s], \( f_{\|} = 1.1 \) [Nm.s] during the simulation studies. The robot’s reference trajectory is selected as

\[
\begin{bmatrix}
x_{d1}(t) \\
x_{d2}(t)
\end{bmatrix} =
\begin{bmatrix}
0.7 \sin(t)(1 - \exp(-0.3t^3)) \\
1.2 \sin(t)(1 - \exp(-0.3t^3))
\end{bmatrix}_{rad}
\]

For both joints the initial position values are selected to be 0.5236 rad (30 degrees). After the tuning process a fair performance for the controller was achieved when the controller gains were selected as follows
\[ k_{\eta} = 40, \quad \rho = 5, \quad k_1 = 500, \quad k_2 = 200, \quad \beta = 100 \]

The simulation results are shown in Figures 1-2. Figure 1 presents the link position tracking errors while Figure 2 presents the control torques applied to each link motor during the simulations. As can be observed from the simulation results the control gain, \( \beta \), is not high compared to the other controller gains. During the simulation studies we have also observed that cranking up \( \beta \) does not affect on the controller performance much. Therefore, it is our belief that the high gain condition given in (27) is only a theoretical drawback.

6. CONCLUSION

In this paper, we have presented a new output feedback controller strategy for the tracking control of a class of uncertain nonlinear systems. Despite the parametric uncertainties in the system dynamics, both constant and/or time-varying, the proposed controller guarantees semi-global asymptotic tracking, and only requires the parametric uncertainties to be first order differentiable. Due to the continuous nature of the controller the proposed method can also be used in backstepping type controller designs. Moreover since the controller proposed can be formulated as a N-PI type controller, implementation and gain tuning are straightforward compared to other uncertainty compensating controllers in the literature. However the proposed controller requires a theoretical high gain condition on the compensation gain, \( \beta \), given in (27). Future studies will concentrate on reducing this high gain condition.

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Fig. 2. Control Torque Inputs