New LMI Characterizations for Discrete-Time Descriptor Systems and Application to Multiobjective Control System Synthesis *

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Abstract: This paper presents new LMI characterizations for stability, $H_2$ and $H_\infty$ norms of discrete-time descriptor systems. Based on these characterizations, an iterative design procedure for multiobjective and structurally constrained feedback control are proposed. The first key idea of the iterative design procedure is embedding the previously designed feedback gain $\hat{K}$ in the descriptor representation of the closed-loop system. The second key idea of the iterative design procedure is linearizing the products terms of the actual controller parameter $K$ and the auxiliary variables by the assignment of variables instead of the ‘change of variables’ technique.

Keywords: multiobjective control, decentralized control, linear matrix inequalities, descriptor systems, dilation, Lyapunov variables.

1. INTRODUCTION

The aim of this paper is to propose LMI characterizations of discrete-time descriptor systems. This paper also proposes an iterative design procedure based on the proposed characterizations.

The dilated (or extended) LMI characterizations enable us to use parameter-dependent Lyapunov functions for robust system analysis and synthesis (de Oliveira et al., 1999; Peaucelle et al., 2000; Apkarian et al., 2001; de Oliveira et al., 2002) and independent Lyapunov functions for multiobjective control synthesis problems (Shimomura et al., 2001; Ebihara and Hagiwara, 2004). These results have promoted the great advance of the control theory. It should be also mentioned that Chen (2004) has pointed out that the dilated LMI characterizations are the LMI characterizations for adequate descriptor representation with redundant state variables.

On the other hand, there have been many attempts to improve the performance by iterative designs (Arzelier and Peaucelle, 2002; Ebihara et al., 2004; Shimomura and Fuji, 2005;Saeki, 2006). Although the iterative designs are effective, the procedures are rather specific for each combination of objective functions.

For continuous-time systems, the other dilated LMI characterizations for iterative design of multiobjective and structurally constrained feedback control are derived (Sebe, 2007). The key idea is dividing the system matrices into some pieces and reconstructing a descriptor system with the previously designed controller parameter. The proposed embedment of controller parameter ensures the improvement of achievable performance by iterative design. As the proposed procedure is based on the stability and performance characterizations of descriptor systems, any combination of the performance specifications and structural constraints on controllers can be dealt with. Contrary to the continuous-time case, the relation between the dilated LMI conditions for the stability and the stability (rigorously the admissibility) of descriptor systems is not so clear for the discrete-time systems. Therefore the iterative design procedure proposed by Sebe (2007) cannot be applied for discrete-time systems.

In this paper, new LMI characterizations for the stability, $H_2$ and $H_\infty$ norms of discrete-time descriptor systems are derived. Based on these characterizations, an iterative design procedure for multiobjective control and structurally constraint controller designs is proposed. Parallel to the continuous-time case, the proposed procedure embeds the previously designed feedback gain $\hat{K}$ in the descriptor representation of the closed-loop system. The proposed procedure linearizes the products terms of the actual controller parameter $K$ and the auxiliary variables by the assignment of variables instead of the ‘change of variables’ technique (Scherer et al., 1997; I. Masubuchi, 1998). The assignment of auxiliary variables for discrete-time systems is different from that for continuous-time systems. This paper also demonstrates the effectiveness of the proposed design procedure through numerical examples.

We use the following notations. $I$ and $O$ denote the identity and zero matrix, respectively. For a matrix $M$, $M^{-1}$ and $M^T$ are the inverse and transpose matrix of...
$M$, respectively. $\text{He}(M)$ is a shorthand notation for $M + M^T$. $\sigma(M)$ is the maximum singular value of $M$. In some partitioned symmetric matrices, the symbol ‘*’ denotes each of its symmetric block.

2. STABILITY CONDITION FOR DISCRETE-TIME DESCRIPTOR SYSTEMS

2.1 Preliminaries

Let us consider a discrete-time state space system

$$x(k + 1) = Ax(k)$$

(1)

where $x(k) \in \mathbb{R}^n$. A well-known stability condition for the system (1) is given as follows.

Proposition 1. The discrete-time system (1) is stable if and only if there exists a matrix $P$ such that

$$P - AP + PA > 0,$$

(2)

$$P = P^T > 0.$$  

(3)

Here are the dilated LMI conditions proposed by Oliveira et al. (1999) and Peaucelle et al. (2000).

Proposition 2. (de Oliveira et al., 1999) The discrete-time system (1) is stable if and only if there exist matrices $F$ and $G$ and $P$ such that

$$
\begin{bmatrix}
P - (GA)^T \\
-GA^T + G^T - P
\end{bmatrix} > 0,
$$

(4)

$$
P = P^T > 0.
$$

(5)

Proposition 3. (Peaucelle et al., 2000) The discrete-time system (1) is stable if and only if there exist matrices $F$, $G$, and $P$ such that

$$
\begin{bmatrix}
P - (FA + A^TF^T - (GA)^T \\
F^T - GA^T + G + G^T - P
\end{bmatrix} > 0,
$$

(6)

$$
P = P^T > 0.
$$

(7)

In these propositions, the matrix $P$ is the actual Lyapunov matrix for the system (1), and the matrices $G$ and $F$ are the auxiliary variables.

Let us consider a discrete-time descriptor system

$$
\tilde{E}\tilde{x}(k + 1) = \tilde{A}\tilde{x}(k)
$$

(8)

where $\tilde{x}(k) \in \mathbb{R}^n$. If the matrix $\tilde{E}$ is singular, the system might have impulsive modes. Therefore the admissibility of the system i.e., the regularity, impulse-free property and stability, should be considered. The rigorous definitions of these properties are found in Hsiung and Lee (1999).

Proposition 4. (Hsiung and Lee, 1999; Xu and Yang, 1999) The system (8) is admissible if and only if there exists a symmetric matrix $\tilde{X} \in \mathbb{R}^{n \times n}$ such that

$$
\tilde{E}^T \tilde{X} \tilde{E} - \tilde{A}^T \tilde{X} \tilde{A} > 0,
$$

(9)

$$
\tilde{E}^T \tilde{X} \tilde{E} \geq 0.
$$

(10)

2.2 Relation between the dilated LMI characterization and descriptor system

The relation between the stability conditions with dilated LMIs and the stability condition for descriptor system is now considered. Let us consider a descriptor system

$$
\begin{bmatrix}
I & O \\
O & O
\end{bmatrix} \tilde{x}(k + 1) = \begin{bmatrix}
O & I \\
A & -I
\end{bmatrix} \tilde{x}(k)
$$

(11)

where $\tilde{x}(k) = [x(k)^T \ x(k + 1)^T]^T$. This descriptor system is equivalent to the system (1). Let the Lyapunov matrix $\tilde{X} \in \mathbb{R}^{2n \times 2n}$ be

$$
\tilde{X} = \begin{bmatrix}
\tilde{X}_{11} & \tilde{X}_{12} \\
\tilde{X}_{12}^T & \tilde{X}_{22}
\end{bmatrix}
$$

(12)

where $\tilde{X}_{ij} \in \mathbb{R}^{n \times n}$ and $\tilde{X}_{ii} = \tilde{X}_{ii}^T$. From Proposition 4, the stability conditions of the descriptor system (11) are

$$
\begin{bmatrix}
\tilde{X}_{11} & O \\
O & O
\end{bmatrix} - \begin{bmatrix}
O & A^T \\
I & -I
\end{bmatrix} \tilde{X}_{12} \begin{bmatrix}
O & I \\
\tilde{X}_{12}^T & \tilde{X}_{22}
\end{bmatrix} \begin{bmatrix}
A & -I
\end{bmatrix} > 0,
$$

(13)

$$
\tilde{X}_{11} = \tilde{X}_{11}^T \geq 0.
$$

(14)

Furthermore, the strict positivity of the (1, 1) block in (13) implies

$$
\tilde{X}_{11} > A^T \tilde{X}_{22} A \geq 0.
$$

(15)

If we choose

$$
\tilde{X}_{11} = P > 0, \quad \tilde{X}_{12} = G, \quad \tilde{X}_{22} = O,
$$

(16)

then the conditions are reduced to those in Proposition 2. This implies that Proposition 2 is a sufficient condition for Proposition 4. Let us assign the matrices $F$, $G$ and $P$ as

$$
P = \tilde{X}_{11} > 0, \quad \begin{bmatrix}
F & G
\end{bmatrix} = \begin{bmatrix}
O & \frac{1}{2} A^T \\
I & -\frac{1}{2} I
\end{bmatrix} \begin{bmatrix}
\tilde{X}_{12} & \tilde{X}_{22}
\end{bmatrix},
$$

(17)

then (6) and (7) hold. Thus, Proposition 4 is a sufficient condition for Proposition 3. As Propositions 2 and 3 are equivalent, Proposition 4 with (11) is also equivalent to those propositions.

The stability condition proposed by Peaucelle et al. (2000) has advantages:

- There does not exist product terms of the Lyapunov matrix $P$ and the system matrix $A$. Thus the conditions with parameter dependent Lyapunov functions are readily applicable for the robust stability analysis and state feedback synthesis of systems with polytopic uncertainties.
- There are no indefinite quadratic terms of the system matrix $A$. This fact makes the controller design problem as LMIs.
- There are large number of decision variables, which may reduce the conservativeness in controller design.

From these reasons, the dilated LMI characterizations are powerful tools for controller designs of robust control, multiobjective control and structural constraint control.

Unfortunately, these dilated LMI characterizations are the stability conditions for the descriptor system (11), and are not for general descriptor systems. Thus, the conditions can not be applied to the iterative design procedure proposed by Sebe (2007).

2.3 Main results

In this subsection, the generalized stability condition for descriptor systems is given. The $H_2$ and $H_{\infty}$ norm conditions for descriptor systems are also given. Let a given discrete-time descriptor system be

$$
\tilde{E}\tilde{x}(k + 1) = \tilde{A}\tilde{x}(k) + \tilde{B}\tilde{w}(k),
$$

(18a)

$$
\tilde{z}(k) = \tilde{C}\tilde{x}(k) + \tilde{D}\tilde{w}(k)
$$

(18b)
where \( x(k) \in \mathbb{R}^d, w(k) \in \mathbb{R}^m, z(k) \in \mathbb{R}^p \). The transfer function from \( w(k) \) to \( z(k) \) is defined by
\[
G(z) = \tilde{C}(z\tilde{E} - \tilde{A})^{-1}\tilde{B} + \tilde{D}.
\] (19)
Assume that
\[
\text{rank } \tilde{E} = r. \tag{20}
\]
There always exist two non-singular matrices \( \tilde{U} \) and \( \tilde{V} \) such that
\[
\tilde{U}\tilde{E}\tilde{V} = \text{diag } \{E, O\} \tag{21}
\]
where \( E \in \mathbb{R}^{r \times r} \) is non-singular. Note that these two matrices \( \tilde{U} \) and \( \tilde{V} \) can be found by the singular value decomposition, which is numerically stable. Let us partition the matrices \( \tilde{U} \) and \( \tilde{V} \) as
\[
\tilde{U} = \begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{bmatrix}, \quad \tilde{V} = \begin{bmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix} \tag{22}
\]
where \( \tilde{U}_1 \in \mathbb{R}^{r \times l}, \tilde{U}_2 \in \mathbb{R}^{(l-r) \times l}, \tilde{V}_1 \in \mathbb{R}^{r \times r}, \tilde{V}_2 \in \mathbb{R}^{r \times (l-r)} \). Then, the following theorem holds.

**Theorem 5.** The given descriptor system (18) is admissible if and only if there exist matrices \( P \in \mathbb{R}^{r \times r} \) and \( \tilde{F} \in \mathbb{R}^{(l-r) \times (l-r)} \) such that
\[
(P (\tilde{U}_1\tilde{E}) - (\tilde{U}_1\tilde{A})^T P (\tilde{U}_1\tilde{A})) + \text{He}(\tilde{F}\tilde{U}_2\tilde{A} \tilde{B}) + \begin{bmatrix} \gamma_2^2 I \\ \gamma_3^2 \end{bmatrix} > 0, \tag{23}
\]
\[
P = P^T > O. \tag{24}
\]

**Remark 6.** If the matrices \( \tilde{E} \) and \( \tilde{A} \) are chosen as (11), Theorem 5 coincides with Proposition 3.

**Theorem 7.** The given descriptor system (18) is admissible if and only if there exist matrices \( P \in \mathbb{R}^{(l+r) \times (l+r)} \) and \( \tilde{D} \in \mathbb{R}^{(l-r) \times (l-r)} \) such that
\[
(P \tilde{U}_1\tilde{E})^T P (\tilde{U}_1\tilde{E}) - (\tilde{U}_1\tilde{A})^T P (\tilde{U}_1\tilde{A}) + \text{He}(\tilde{F}\tilde{U}_2\tilde{A} \tilde{B}) + \begin{bmatrix} \gamma_2^2 I \\ \gamma_3^2 \end{bmatrix} > 0, \tag{25}
\]
\[
P = P^T > O. \tag{26}
\]

**Remark 7.** In this section, let us apply the idea in Sebe (2007) to the discrete-time \( H_2/H_{\infty} \) static feedback design. Note that fixed order dynamic controllers can be similarly designed with augmented matrices given in Iwasaki and Skelton (1994). Furthermore, the design procedure proposed here can also deal with multiobjective control problems with any combinations of objective functions and structural constraints on controllers.

Let us consider a generalized plant
\[
x(k + 1) = Ax(k) + B_1w_1(k) + B_2w_2(k),
\]
\[
z_1(k) = C_0x(k) + D_{00}w_0(k) + D_{01}w_1(k) + D_{02}u(k),
\]
\[
z_2(k) = C_1x(k) + D_{10}w_0(k) + D_{11}w_1(k) + D_{12}u(k),
\]
\[
y(k) = C_2x(k) + D_{20}w_0(k) + D_{21}w_1(k) + D_{22}u(k). \tag{32}
\]
Let \( T_1 (i = 0, 1) \) denote the transfer functions from \( w_i \) to \( z_i \). Assume \( \gamma_\infty \) is a given scalar. Then, the problem is to find a static feedback \( u = Ky \) which minimizes \( \gamma_\infty = \| T_0 \|_2 \) under the constraint \( \| T_1 \|_\infty < \gamma_\infty \).

As this paper aims to propose an iterative design procedure, let us assume that a previously designed feedback gain \( K \) is given. With this designed \( K \) and the feedback \( K \) to be designed, the explicit descriptor representation of the closed-loop system, which is used for controller design, is proposed as follows:
\[
\tilde{E}\tilde{x}(k + 1) = \tilde{A}\tilde{x}(k) + \tilde{B}\tilde{w}(k), \tag{33a}
\]
\[
\tilde{z}(k) = \tilde{C}\tilde{x}(k) + \tilde{D}\tilde{w}(k) \tag{33b}
\]
where \( \tilde{E} = \text{block diag} (I, O) \),
\[
\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & B_{10} & B_{11} \\ A_{21} & A_{22} & B_{20} & B_{21} \\ C_{11} & C_{12} & D_{00} & D_{01} \\ C_{11} & C_{12} & D_{10} & D_{11} \end{bmatrix} = \begin{bmatrix} A & B_0 & B_1 \\ O & I & O \\ C_0 & D_{00} & D_{01} \\ C_1 & O & D_{10} & D_{11} \end{bmatrix}
\]
+ \begin{bmatrix} O \\ D_{02} \\ D_{12} \end{bmatrix} \begin{bmatrix} O \\ D_{10} \end{bmatrix} (K - \tilde{K}) \}
\]
\[
\times \begin{bmatrix} C_2 \\ D_{20} \\ D_{21} \end{bmatrix}, \tag{33c}
\]
\[
B_2 = B_1B_2R_1, \quad \xi = B_Ru, \tag{33d}
\]
\[
\tilde{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad \tilde{w} = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}, \quad \tilde{z} = \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}. \tag{33e}
\]

The idea of the decomposition of \( B_2 \) in (33d) is first proposed in Saeki (2006) and is also discussed in Sebe (2007). Mostly the recommended decomposition is
\[
B_1 = U_{B2}^\top Z_{B2}, \quad B_R = Z_{B2}V_{B2}^\top \tag{34}
\]
where $U_{B2} \Sigma_{B2} V_{B2}^T$ is the singular value decomposition of $B_2$.

Applying Theorems 7 and 9 to the descriptor system (33), the $H_2/H_\infty$ control problem can be formulated as an optimization problem below.

**Problem 12.** Find positive definite matrices $P_2$, $P_\infty$, $Q$, and matrices $K$, $F$, $G$, $H$ such that

\[
\begin{align*}
\text{minimize } & \gamma_2 \\
\text{subject to } & \gamma_2^2 \geq \text{trace } Q, \\
& \begin{bmatrix} M_1 & 0 \\ \tilde{C}_{01} & \tilde{C}_{02} \end{bmatrix} > 0, \\
& \begin{bmatrix} M_2 & 0 \\ \tilde{C}_{02} \tilde{D}_{00} \end{bmatrix} > 0, \\
& \begin{bmatrix} M_3 & 0 \\ \tilde{C}_{11} & \tilde{C}_{12} \tilde{D}_{11} \end{bmatrix} > 0, \\
& M_1 = \begin{bmatrix} P_2 & 0 \\ 0 & P_\infty \end{bmatrix} - \begin{bmatrix} \tilde{A}_{11}^T & \tilde{A}_{12}^T \\ \tilde{A}_{12} & \tilde{A}_{22}^T \end{bmatrix} P_2 \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{12} & \tilde{A}_{22} \end{bmatrix} + \text{He} \left\{ \tilde{F} \begin{bmatrix} \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \right\}, \\
& M_2 = \begin{bmatrix} P_\infty & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \tilde{A}_{11}^T & \tilde{A}_{12}^T \\ \tilde{A}_{12} & \tilde{A}_{22}^T \end{bmatrix} P_\infty \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{12} & \tilde{A}_{22} \end{bmatrix} + \text{He} \left\{ \tilde{G} \begin{bmatrix} \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \right\}, \\
& M_3 = \begin{bmatrix} P_\infty & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \tilde{A}_{11}^T & \tilde{A}_{12}^T \\ \tilde{A}_{12} & \tilde{A}_{22}^T \end{bmatrix} P_\infty \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{12} & \tilde{A}_{22} \end{bmatrix} + \text{He} \left\{ \tilde{H} \begin{bmatrix} \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \right\}.
\end{align*}
\]

Similar to the conventional dilated LMI characterizations, there are no product terms of the Lyapunov variables $P$, and the controller parameter $K$ in the above conditions. On the other hand, the product terms of the auxiliary variables $F$, $G$, $H$ and the controller parameter $K$ exist. Thus, linearization is required to make the problem tractable.

Here, the conceptual iterative design procedure is summarized. The actual linearization will be mentioned in the next subsection.

**Algorithm 1.** Let $K^{(i)}$ be the feedback gain which is designed at the $i$-th design iteration.

(i) Find a static feedback $K$ which satisfies $\|T_i\|_\infty < \gamma_\infty$. Set $K^{(0)} = K$ and $i = 1$.

(ii) Set $K = K^{(i-1)}$ in (33). Solve Problem 12 with the linearized conditions of (36), (37) and (38). (We will mention the actual linearization in the next subsection.) Set $K^{(i)} = K$ where $K$ is the solution.

(iii) If a stopping criterion is satisfied, exit. Otherwise, set $i = i + 1$ and go to Step (ii).

With this iterative design procedure, let us define some variables. Let $\gamma_2^{(i)}$ be the guaranteed upper bound, i.e., the optimized value $\gamma_2$ in Problem 12, and $\gamma_2^{(i)}$ be the actually achieved $H_2$ norm of the closed-loop system with the designed $K^{(i)}$. Evidently, $\gamma_2^{(i)} \leq \gamma_2^{(i)}$. The matrices $P_2^{(i)}$ and $P_\infty^{(i)}$ denote the Lyapunov variables which evaluate the $H_2$ and $H_\infty$ norms of the closed-loop system with $K^{(i)}$.

3.2 Linearization by assignment of variables

In many controller design methods based on LMI characterizations, the ‘change of variables’ technique is used to linearize the product terms of the Lyapunov or auxiliary variables and the controller parameter at the expense of the unification of the Lyapunov or auxiliary variables. The unification induces conservative results. Instead of the ‘change of variables’ technique, the assignment of variables is proposed to linearize the product terms by Sebe (2007) for the continuous-time systems. Similar to the continuous-time case, linearization with the Lyapunov variables $P_2^{(i-1)}$ and $P_\infty^{(i-1)}$ is proposed in this subsection.

As mentioned before, there exist the product terms of the auxiliary variables $F$, $G$, $H$ and the controller parameter $K$. Obviously, the problem becomes linear, if the auxiliary variables are fixed. Thus we now propose to linearize the conditions (36), (37) and (38) by fixing these auxiliary variables. Explicit assignments are given below:

\[
F = -\tilde{A}_{11}^T \tilde{A}_{21}^T P_2^{(i-1)} B_L, \quad G = -\tilde{A}_{12}^T B_L^T P_2^{(i-1)} B_L, \quad H = -\tilde{A}_{12}^T B_L^T P_\infty^{(i-1)} B_L.
\]

Then, the next theorem holds.

**Theorem 13.** Assume the conditions (36), (37) and (38) be linearized by the variable assignments (42). Then, the inequality $\gamma_2^{(i)} \leq \gamma_2^{(i-1)}$ holds.

**Proof.** Let $K = \tilde{K} = K^{(i-1)}$, $P_2 = \tilde{P}_2 = P_2^{(i-1)}$, $P_\infty = P_\infty^{(i-1)}$, and substitute (42) into the constraints (36), (37), (38), then the inequalities become

\[
\begin{align*}
\text{Problem 12. } & \quad \begin{bmatrix} P_2 - (\delta)^T P_2 (A + B_2 \tilde{K} C_2) & O & 0 \\ O & B_L^T \tilde{P}_2 B_L & O \\ C_0 + D_{02} \tilde{K} C_2 & O & I \end{bmatrix} > 0, \\
& \begin{bmatrix} B_L^T \tilde{P}_2 B_L & O & 0 \\ O & Q - (\delta)^T (B_0 + B_2 \tilde{K} D_{20}) & O \\ O & D_{00} + D_{02} \tilde{K} D_{20} & I \end{bmatrix} > 0, \\
& \begin{bmatrix} N_3 & 0 \\ \begin{bmatrix} C_1 + D_{12} \tilde{K} C_2 & O \\ D_{12} + D_{12} \tilde{K} D_{21} \end{bmatrix} & I \end{bmatrix} > 0, \\
N_3 &= \begin{bmatrix} \tilde{P}_\infty & O \\ O & B_L^T \tilde{P}_2 B_L & O \\ O & O & \gamma_2^{(i)} I \end{bmatrix} - [\delta]^T \tilde{P}_\infty (A + B_2 \tilde{K} C_2) O B_L + B_2 \tilde{K} D_{21}.
\end{align*}
\]

The inequalities (43) and (44) are the $H_2$ constraints and the inequality (45) is the $H_\infty$ constraint of the closed-loop system with $\tilde{K}$, respectively. As the matrices $\tilde{P}_2$ and $\tilde{P}_\infty$ are the Lyapunov matrices which evaluate the $H_2$ and $H_\infty$ norms, the optimal value is $\gamma_2^{(i-1)}$. □

3.3 Remarks on implementation

Similar to the continuous-time case in Sebe (2007), we can modify the term $B_L^T P_2^{(i-1)} B_L$ in the assignments (42). The
Two examples demonstrate the efficiency of the proposed method. All the examples are carried out by Robust Control Toolbox in MATLAB (Release 2007a) on a PC (Pentium 4, 3.2GHz with 2GB RAM). For the two examples, the decomposition of $B_2$ is $B_2 = B_3$ and $B_R = I$, and the stopping criterion is $|\gamma_{a}^{(i-1)} - \gamma_{a}^{(i)}| < 1 \times 10^{-5}$.

Example 14. ($H_2/H_\infty$ control). Let us consider the $H_2/H_\infty$ control problem in Oliveira et al. (2002). The state space data of the generalized plant are

$$\begin{bmatrix} A & B_0 & B_1 & B_2 \\ C_0 & D_{00} & D_{01} & D_{02} \\ C_1 & D_{10} & D_{11} & D_{12} \\ C_2 & D_{20} & D_{21} & D_{22} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0.5 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -0.5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$ 

The problem is to find a strictly proper output feedback controller which minimizes $\|T_0\|_2$ under the $H_\infty$ constraint $\|T_{i\xi}\|_\infty < \gamma_\infty$ ($i = 1, 2, 3$), where $T_{i\xi}$ are the diagonal elements of $T_1$ and $\gamma_\infty$ is a given value.

Three design methods are used to design controllers for comparison, the common Lyapunov variable design by Scherer et al. (1997), the dilated LMI characterization by Oliveira et al. (2002) and the proposed method.

First, let us compare the three design methods from the viewpoint of the feasible $H_\infty$ norm constraints. Without $H_2$ norm optimization, the problem becomes the simultaneous $H_\infty$ optimization problem to find a controller which minimizes the worst $H_\infty$ norm of $T_{i\xi}$. Table 1 shows the minimum feasible $\gamma_\infty$ for each design method. Please note that worst $H_\infty$ norm achieved by the optimal (non-robust) $H_2$ controller is $21.757 (= \gamma_2$). The $H_2$ optimal controller is optimal for $H_2/H_\infty$ control problem with $\gamma_\infty \geq \gamma$. In other words, it is meaningless to specify $H_\infty$ constraint with $\gamma_\infty \geq \gamma$. The minimum feasible $\gamma_\infty$ for Scherer’s method is larger than $\gamma$, and that for Oliveira’s is slightly smaller than $\gamma$. This fact implies that these two methods do not provide efficient design results for this example. For the proposed procedure, the controller designed by Oliveira’s method is selected as an initial controller. Then, after 8 iterations, the minimum feasible $\gamma_\infty$ for the proposed method is 10.865 and is much smaller than $\gamma$. The state space data of the designed controller for $\gamma_\infty = 10.865$ is

$$\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} = \begin{bmatrix} -1.2192 & -3.9646 & 0.2890 & -0.1017 \\ 0.9316 & -0.9926 & 0.0632 & 2.4815 \\ 0.0040 & 0.0387 & -0.5029 & -0.9873 \\ -1.6088 & -3.9381 & 1.2711 & 0 \end{bmatrix}.$$ 

Next, let us examine the guaranteed and achieved $H_2$ norms. For the proposed method, the simultaneous $H_\infty$ controller above is selected as an initial controller. Then, Figure 1 shows the relation between the $H_\infty$ constraint $\gamma_\infty$ and the guaranteed and achieved $H_2$ norms. As the guaranteed and achieved $H_2$ norms by the proposed method are same, only the guaranteed norm is shown. It is easy to see that the proposed method provides efficient design results. For example, only the proposed method can solve the problem with $\gamma_\infty = 15$, and the state space data of the designed controller is

$$\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} = \begin{bmatrix} -0.7692 & 3.1366 & 0.1370 & 0.1377 \\ -0.9702 & -0.6705 & -0.0309 & 2.0843 \\ -0.0909 & -1.1001 & -0.5000 & 0.9205 \\ 1.2565 & -3.4486 & 0.8083 & 0 \end{bmatrix}.$$ 

which achieves $\|T_0\|_2 = 32.434$. This controller is obtained after 16 iterations.
Table 1. Feasible $H_\infty$ constraint and actual $H_\infty$ norm (Example 14).

<table>
<thead>
<tr>
<th>Method</th>
<th>feasible $H_\infty$ constraint</th>
<th>actual $H_\infty$ norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scherer et al.</td>
<td>29.081</td>
<td>16.854</td>
</tr>
<tr>
<td>Oliveira et al.</td>
<td>20.958</td>
<td>15.153</td>
</tr>
<tr>
<td>proposed</td>
<td>10.865</td>
<td>10.865</td>
</tr>
<tr>
<td>$H_2$ optimal</td>
<td>−</td>
<td>21.757</td>
</tr>
</tbody>
</table>

Example 15. (Decentralized $H_2$ control). The proposed method is applied to a decentralized $H_2$ controller design. The generalized plant is also borrowed from Oliveira et al. (2002) and its state space data are

$$A = \begin{bmatrix} 0.8189 & 0.0863 & 0.0900 & 0.0813 \\ 0.2524 & 1.0033 & 0.0313 & 0.2004 \\ -0.0545 & 0.0102 & 0.7901 & -0.2580 \\ -0.1918 & -0.1034 & 0.1602 & 0.8604 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0.0953 \\ 0.0145 \\ 0.0862 \\ -0.0011 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.0045 & 0.0044 \\ 0.1001 & 0.0100 \\ 0.0003 & -0.0136 \\ -0.0051 & 0.0936 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$D_{02} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C_2 = I, D_{00} = O.$$  

The problem is to find a decentralized state feedback which minimizes $\|T_0\|_2$. Let the initial decentralized state feedback $K^{(0)}$ be

$$K^{(0)} = \begin{bmatrix} -2.5841 & -5.0228 & 0 & 0 \\ 0 & 0 & 1.9632 & -8.6826 \end{bmatrix},$$

which is designed by Oliveira et al. (2002). After 11 iterations, we obtained a decentralized state feedback

$$K^{(11)} = \begin{bmatrix} -0.4723 & -0.3001 & 0 \\ -0.0011 & 0 & -0.3782 & -0.1848 \end{bmatrix},$$

which attains $\gamma^{(11)} = 0.2728$. Figure 2 shows the convergence of the achieved $H_2$ norm.

Ebihara et al. (2004) propose an alternating projection method for structural constraint controller designs. Although their method is effective, their method does not ensure the monotonic decrease of the performance index. Table 2 shows the computational expense of Ebihara’s and the proposed methods. The proposed method is much effective from the viewpoint of computational expense. Furthermore, Ebihara’s method requires bisection method to obtain the optimal controller and it takes 62 iterations for a given $\gamma_2 = 0.2735$.

5. CONCLUSIONS

This paper introduces new dilated LMI characterizations for discrete-time descriptor systems. Based on the characterizations an iterative design procedure is proposed for multiobjective control and structurally constrained controller designs for discrete-time systems.

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REFERENCES


**Appendix A. OUTLINES OF PROOFS**

**A.1 Proof of Theorem 5**

**Lemma 16.** (Kuijper, 1994) The given descriptor system (18) is regular and impulse-free, if and only if

$$\text{Im}\dot{E} + \text{Im} \hat{A}(\text{Ker}\hat{E}) = \mathbb{R}^n.$$ (A.1)

**Proof of Theorem 5.** Necessity. From Proposition 4, there exists a symmetric matrix $\hat{X}$ which satisfies

$$\tilde{E}^T \hat{X} \tilde{E} - \hat{A}^T \hat{X} \hat{A} > 0,$$ (A.2)

$$\tilde{E}^T \hat{X} \tilde{E} > 0.$$ (A.3)

if the given descriptor system (18) is admissible. Let us define $\tilde{Y}$ as

$$\tilde{Y} = \tilde{U}^{-T} \tilde{X} \tilde{U}^{-1} = \begin{bmatrix} \tilde{Y}_{11} & \tilde{Y}_{12} \\ \tilde{Y}_{12}^T & \tilde{Y}_{22} \end{bmatrix}.$$ (A.4)

With this $\tilde{Y}$ and multiplying $\tilde{V}^T$ and $\tilde{V}$, (A.2) and (A.3) become

$$\begin{bmatrix} E & O \end{bmatrix}^T \tilde{Y} \begin{bmatrix} E & O \end{bmatrix} - (\tilde{U} \tilde{A} \tilde{V})^T \tilde{Y} (\tilde{U} \tilde{A} \tilde{V}) > 0,$$ (A.5)

$$\tilde{E}^T \tilde{Y}_{11} E > 0.$$ (A.6)

Similar to (15), the strict positivity of (A.6) can be shown. The inequality (A.5) can be rewritten as

$$\begin{bmatrix} E^T & O \end{bmatrix} \tilde{Y}_{11} \begin{bmatrix} E & O \end{bmatrix} - (\tilde{U}_1 \tilde{A} \tilde{V})^T \tilde{Y}_{11} (\tilde{U}_1 \tilde{A} \tilde{V})$$

$$- (\tilde{U} \tilde{A} \tilde{V})^T \begin{bmatrix} O & \tilde{Y}_{12} \\ \tilde{Y}_{12} & \tilde{Y}_{22} \end{bmatrix} (\tilde{U} \tilde{A} \tilde{V}) > 0.$$ (A.7)

Then, the assignments

$$P = \tilde{Y}_{11}, \quad F = (\tilde{U} \tilde{A})^T \begin{bmatrix} \tilde{Y}_{12} \\ \frac{1}{2} \tilde{Y}_{22} \end{bmatrix}$$ (A.8)

accomplish the necessity.

Sufficiency. Multiplying $\tilde{V}^T$ and $\tilde{V}$, the condition (23) becomes

$$(\tilde{U}_1 \tilde{E} \tilde{V})^T P (\tilde{U}_1 \tilde{E} \tilde{V}) - (\tilde{U}_1 \tilde{A} \tilde{V})^T P (\tilde{U}_1 \tilde{A} \tilde{V})$$

$$\text{He} \{\tilde{V}^T \tilde{F} \tilde{U}_2 \tilde{A} \tilde{V} \} > 0.$$ (A.9)

Let

$$\tilde{U} \tilde{A} \tilde{V} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}.$$ (A.10)

From Lemma 16, if the system is not impulse-free, there exist a vector $p(\neq o)$ which satisfies $\tilde{A}_{22} p = o$. Multiplying $[o^T \ p^T]^T$ and its transpose to (A.9), the left hand side of the inequality becomes 0. This contradicts the negativity of (A.9). Thus the system is regular and impulse-free, and $\tilde{A}_{22}$ is non-singular. Applying the elimination lemma, the inequality (A.9) is equivalent to

$$(A_{11} - A_{12} A_{22}^T A_{21})^T P (A_{11} - A_{12} A_{22}^T A_{21}) - E^T P E < 0.$$ (A.11)

This shows the stability of exponential modes of the system. □

**A.2 Outlines of the other proofs**

Once the conditions for the admissibility of discrete-time descriptor systems are obtained, the conditions for $H_2$ and $H_\infty$ norm of the systems can be derived directly from the results in Stykel (2006) and Hsiung and Lee (1999).