Extracting Worst Case Perturbations for Robustness Analysis of Parameter-Dependent LTI Systems

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Abstract: In this paper, we deal with robust performance analysis problems of LTI systems depending on uncertain parameters. By following existing scaling-based approaches, we firstly derive computationally tractable parameter-independent LMI conditions to assess the robust performance, which are conservative in general. What makes the present approach novel is to take the dual of those LMs so that we can conclude the exactness of the analysis results. More precisely, we clarify that if the computed dual solution satisfies a certain rank condition, then we can ensure that the robust performance is never attained. In particular, we can extract the worst case perturbation that violates the underlying performance. Thus we provide viable tests for the exactness verification of LMI-based robust performance analysis.

Keywords: Robust performance, uncertain systems, linear matrix inequalities, duality theory, linear fractional transformation.

1. INTRODUCTION

This paper is concerned with the robustness analysis problems of linear time-invariant (LTI) systems depending on uncertain parameters (Barmish [1994]). These problems are naturally formulated as feasibility problems of linear matrix inequalities (LMIs) whose coefficient matrices are affected by the uncertain parameters. These LMIs, so called robust LMIs, arise when we deal with whole variety of robustness analysis and synthesis problems (see, ex., Scherer [2005, 2006]). Unfortunately, however, robust LMI problems are essentially intractable NP-hard problems. In view of these facts, main focus has been laid upon deriving sufficient LMI conditions that are less conservative and efficiently solved via LMI solvers.

Recently, stimulated by the theoretical advances on polynomial optimization via sum-of-squares decompositions (Lasserre [2001], Parrilo [2003]), novel contributions have been made to deal with robust LMIs in an asymptotically exact fashion (see, ex., Bliman [2004a], Henrion et al. [2004], Scherer [2005, 2006], Scherer and Hol [2006]). Among them, Scherer [2005, 2006] and Scherer and Hol [2006] showed a unified way for LMI relaxation, which enables us to obtain a hierarchy of LMIs with theoretical guarantee of asymptotic exactness. In addition, by taking the dual of these LMIs, viable tests for the exactness verification have been provided (Scherer [2005, 2006]).

In Ebihara et al. [2007], the authors pursued the direction related to but yet distinct from Scherer [2005, 2006], focusing on robustness analysis problems of continuous-time uncertain LTI systems. More precisely, the authors provided sound rank conditions for the exactness verification based on the particular block-moment matrix structure of the dual solution. This result is closely related to the LMI relaxation for polynomial matrix inequality (PMI) problems suggested in Henrion and Lasserre [2006], which is a genuine matrix counterpart of those in Lasserre [2001], Henrion and Lasserre [2005]. In comparison with the direct formulation as PMIs, one of the salient feature of the approach in Ebihara et al. [2007] is that it exploits the block-moment matrix structure of the dual solution so that the associated computational burden can keep moderate.

Our primary concern in this paper is to extend the results in Ebihara et al. [2007] so that we can deal with discrete-time system analysis in a unified fashion. To this end, we first analyze the generalized Lyapunov inequality (Scherer [2005]) for matrices depending on uncertain parameters. By following Ebihara et al. [2007], we convert this robust LMI into a numerically verifiable LMI via \((D,G)\)-scaling (Meinsma et al. [1997]) and take its dual for the exactness verification. Based on these preliminary results, we next clarify that, if the computed dual solution satisfies a certain rank condition, then the original robust LMI never holds. In particular, we can extract the worst case parameter perturbation that violates the robust LMI. We also show that these results can readily be extended to robust dissipative performance analysis by using the idea of Hamiltonian eigenvalue tests (Hagiwara [2005], Zhou and Doyle [1998]). Thus we can obtain consistent results to our preceding results for continuous-time system analysis (Ebihara et al. [2007]).

We use the following notations in this paper. The symbol \(S_n\) denotes the set of real symmetric matrices of the size \(n\). For a matrix \(A \in \mathbb{R}^{n \times n}\), we denote by \(\lambda(A)\) the set of its eigenvalues. For matrices \(A, B \in \mathbb{R}^{n \times n}\), the symbol \(\lambda(A, B)\) denotes the set of their generalized eigenvalues, i.e., the set of \(\lambda \in \mathbb{C}\) satisfying \(Ax = \lambda Bx\) for \(x \in \mathbb{C}^n \setminus \{0\}\). For \(A \in S_n\), we denote by \(\lambda_k(A)\) (\(k = 1, \ldots, n\))
its $k$-th eigenvalue. In addition, we use the notation $\text{In}(A) = (p, \nu, \zeta)$ to indicate that the number of the positive, negative and zero eigenvalues of $A$ are $p, \nu$ and $\zeta$, respectively. For a matrix $A \in \mathbb{R}^{n \times m}$, we denote its Moore-Penrose generalized inverse by $A^*$. For a matrix $A$ with partition $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $A_{11} \in \mathbb{R}^{n_1 \times m_1}$, $A_{22} \in \mathbb{R}^{n_2 \times m_2}$, we define $[A]_{m_1} := [A_{11} A_{12}]$ and $[A]_{m_2} := [A_{21} A_{22}]$. In particular, if $A$ is square and $n_i = m_i$ $(i = 1, 2)$, we define $(A)_{m_1} := A_{11}$ and $(A)_{m_2} := A_{22}$. Finally, for given $q, r, s \in \mathbb{R}$ satisfying $qr - s^2 < 0$, we define $D(q, r, s)$ and $\partial D(q, r, s)$ as follows: $D(q, r, s) := \left\{ \lambda \in \mathbb{C} : \begin{bmatrix} 1 & r \\ s & q \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & \lambda \end{bmatrix} < 0 \right\},$ $\partial D(q, r, s) := \left\{ \lambda \in \mathbb{C} : \begin{bmatrix} 1 & r \\ s & q \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & \lambda \end{bmatrix} = 0 \right\}.$

2. PROBLEM FORMULATION AND GENERALIZED LYAPUNOV INEQUALITY

Let us consider the rational functions $M(\theta) : \mathbb{R}^L \to \mathbb{R}^{n \times n}$ and $J(\theta) : \mathbb{R}^L \to \mathbb{R}^{n \times n}$ represented by $M(\theta) = M_{11} + M_{12}(I_1 - \Delta_1(\theta)M_{21})^{-1}\Delta_1(\theta)M_{21},$ $J(\theta) = J_{11} + J_{12}(I_1 - \Delta_1(\theta)J_{21})^{-1}\Delta_1(\theta)J_{21},$ $\Delta_1(\theta) = \sum_{i=1}^{L} \theta_i E_{i,j},$ where the uncertain parameter $\theta_i$ varies over $\Theta_i := [-\delta, \delta]^L$. The matrices $E_{i,j} \in \mathbb{R}^{n \times n}$ $(i = 1, \ldots, L)$ are nonzero diagonal matrices whose diagonal entries are zero or one and satisfy $\sum_{i=1}^{L} E_{i,j} = I_1$. We assume that the LFTs in (1) are well-posed, i.e., $\det(I_1 - \Delta_1(\theta)M_{21}) \neq 0, \det(I_1 - \Delta_1(\theta)J_{21}) \neq 0, \forall \theta \in \Theta^L_\delta$.

Under these preparations, in this paper, we consider the next problem which is basically motivated from the robust $D$-stability analysis problem discussed in Barmish [1994].

**Problem 1.** For given $q, r, s \in \mathbb{R}$ with $qr - s^2 < 0$, determine whether the following condition holds or not: $\{ \lambda(M(\theta), J(\theta)) \cap \partial D(q, r, s) \} = \emptyset \quad \forall \theta \in \Theta^L_\delta.$ (2)

The next Lemma shows that Problem 1 can be reduced into a feasibility test of a robust LMI. This result readily follows from the generalized Lyapunov inequality for uncertainty-free matrices (Scherer [2005]).

**Lemma 1.** For given $q, r, s \in \mathbb{R}$ satisfying $qr - s^2 < 0$, the condition (2) holds if and only if there exists $P(\theta) : \mathbb{R}^L \to \mathbb{S}_n$ such that $\begin{bmatrix} r P(\theta) & s P(\theta) \\ s P(\theta) & q P(\theta) \end{bmatrix} - \begin{bmatrix} M(\theta)^T & M(\theta)^T \\ -J(\theta)^T & -J(\theta)^T \end{bmatrix} \preceq 0 \quad \forall \theta \in \Theta^L_\delta.$ (3)

The matrix-valued function $P(\theta)$ is often referred to as a parameter-dependent multiplier (Chen and Sugie [1996]). From the facts that: (i) $M(\theta)$ and $J(\theta)$ are continuous with respect to $\theta \in \Theta^L_\delta$; (ii) the set $\Theta^L_\delta$ is compact, (iii) the inequality (3) is strict, the multiplier $P(\theta)$ can be restricted to be a polynomial without loss of generality (Bilimann [2004b]). Nevertheless, the robust LMI (3) is still intractable mainly due to the following reasons:

1. The explicit degree of the polynomial $P(\theta)$ that enables us to achieve exact analysis is not known a priori, even though several results were obtained for robust $D$-stability analysis cases (Henrion et al. [2004], Zhang et al. [2003]).

2. Suppose we fix the multiplier $P(\theta)$ to be a polynomial of finite-degree. Then, in the single parameter case, we can reduce the robust LMI (3) into a parameter-independent LMI in an exact fashion via $(D, G)$-scaling (Meinsma et al. [1997]). However, this is not straightforwardly achieved in the multiple uncertain parameter cases, and those LMIs resulting from LMI relaxations are expected to be conservative in general.

In view of these current state of the art, all we can do amounts to resorting to LMI conditions that are conservative in general. To obtain exactness certificates even under these difficult situation, in this paper, we follow the idea in Ebihara et al. [2007] and consider the dual LMI problem. This enables us to derive viable tests for the exactness verification as we see in the next two sections. We note that the discussion in Ebihara et al. [2007] is restricted to the case where $J(\theta) = I$ and $(q, r, s) = (0, 0, 1)$ in Problem 1.

3. ROBUST $D$-NONSINGULARITY ANALYSIS

3.1 Single Uncertain Parameter Case

Let us first consider the single parameter case. In this case, we can state the next result.

**Theorem 1.** For given odd number $N$ and $\delta > 0$, let us consider the following LMI-LME conditions with respect to $H \in \mathbb{S}_{(N+3)\times n} (n := n + 1)$: $\begin{bmatrix} H_0 & H_1 & \cdots & H_{(N+1)/2} \\ H_1^T & \ddots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \cdots \\ H_{(N+1)/2}^T & \cdots & \cdots & H_N \end{bmatrix} \succeq 0,$ where $\delta^2(H_{(N+1)/2}) - (H_{(N+1)/2})_{n+1} \geq 0,$ $\begin{bmatrix} 0_{n,2n} \\ -I_{2n} \\ \vdots \\ -I_{2n} \\ 0_{n,2n} \end{bmatrix}^T = 0, \quad W^T H_0 W = 0,$ $\begin{bmatrix} r I_{n} s I_{n} q I_{n} \end{bmatrix} (H_j)_{2n}^T + \begin{bmatrix} r I_{n} s I_{n} q I_{n} \end{bmatrix} (H_j)_{2n} = 0 \quad (j = 0, \cdots, N)$

Then, the following two assertions hold:

(i) If (4) is infeasible, then the condition (2) holds.

(ii) Suppose (4) is feasible and has a solution $H$. Let us denote the full-rank factorization of $H$ by

$H = \begin{bmatrix} H_0 & \cdots & H_0 \\ H_{(N+1)/2} & \cdots & H_{(N+1)/2} \end{bmatrix} (H_j)_{2n} \in \mathbb{R}^{2m \times m},$ (5)

and define $\begin{bmatrix} (H_0)_{2n}^T & \cdots & (H_{(N+1)/2})_{2n}^T \end{bmatrix} \in \mathbb{R}^{(N+1)n/2 \times m}, H := \begin{bmatrix} (H_1)_{2n}^T & \cdots & (H_{(N+1)/2})_{2n}^T \end{bmatrix} \in \mathbb{R}^{(N+1)n/2 \times m}.$
Then, if
\[
\text{rank}(\mathcal{H}) = \text{rank}(\mathcal{H}), 
\]
the condition (2) never holds. More precisely, if we define
\[
\Omega := \mathcal{H}^T \mathcal{H},
\]
then this matrix \( \Omega \) satisfies \( \Omega \in \mathbf{S}_m \) and \( \lambda(\Omega) \subset \Theta_3 \). In addition, we have \( \lambda(M(\lambda_\kappa(\Omega)), J(\lambda_\kappa(\Omega))) \cap \partial D(q,r,s) \neq \emptyset \) for all \( k = 1, \ldots, m \).

Due to limited space, we omit the proof of this theorem. It should be noted that, in the case where we have only one uncertain parameter, we can apply those results in [Ebihara and Hagihara, 2005; Meinsma et al., 1997] so that we can obtain an LMI that ensures the existence of \( N \)-th degree polynomial \( P_N(\theta) \) satisfying (3) in an exact fashion. The LMI (4) corresponds to the dual of this LMI, which follows immediately from the convex duality theory (Balakrishnan and Vandenberghe, 2003). From this procedure, we see that the LMI (4) is infeasible if and only if (3) is feasible as \( P(\theta) = P_{31}(\theta) \). Thus, the assertion (i) in Theorem 1 readily follows.

The importance of the theorem lies in the assertion (ii), which provides a viable tests for the exactness verification. Namely, if the dual LMI (4) is feasible and if the computed dual solution satisfies the rank condition (6), then we can conclude that (2) never holds. In addition, the worst case parameter perturbation \( \theta^* \) such that \( \{\lambda(M(\theta^*_w)), J(\theta^*_w)) \cap \partial D(q,r,s) \neq \emptyset \) can be obtained as eigenvalues of the matrix \( \Omega \in \mathbf{S}_m \) which can be readily computed by constructing \( \Omega \) from the dual solution. This result surely goes beyond the standard primal LMI approach that allows us to conclude the assertion (i) only. We note that the key to derive the result (ii) lies on the particular block-Hankel matrix structure of the dual solution \( \mathcal{H} \).

**Remark 1.** The size of the LMI (3) and the rank condition for the exactness verification (6) of course depend on \( N \), the degree of the employed multiplier. By increasing \( N \), we can show that the condition (6) becomes more likely to be satisfied in the sense that if there exists a dual solution satisfying (3) and (6) for \( N = N_1 \), then there always exists a dual solution satisfying (3) and (6) for \( N \geq N_1 \).

**Remark 2.** The exactness test (6) should be compared with those reported in the literature (Scherer, 2005, 2006). To this end, for simplicity, let us consider the robust \( D \)-stability analysis case where \( J(\theta) = I \). In this case, we can confirm that the existence of the worst case parameter is also ensured if the following condition holds:
\[
\exists \theta \in \Theta_3 \text{ such that } \mathcal{H}_k = \theta^k \mathcal{H}_0 \ (k = 1, \ldots, N + 1).
\]
It is also true that if \( \text{rank}(\mathcal{H}) = 1 \), the above condition is automatically satisfied. Namely, the exactness verification test (7) goes beyond the common rank-one exactness principle \( \text{rank}(\mathcal{H}) = 1 \).

On the other hand, from the form of the equality constraints in (4), we can readily prove that if (7) holds, then (6) holds. Here, we note that, even though the discussion in Scherer [2005, 2006] does not clearly mention the block-Hankel matrix structure of the dual solution, we can confirm that the exactness verification test to (4), in the spirit of Scherer [2005, 2006], can be given as (7). It follows that, at least in the case where we deal with robust \( D \)-stability analysis problems, the suggested exactness test (6) is more general than those in Scherer [2005, 2006].

### 3.2 Multiple Uncertain Parameter Case

Let us next consider the case where we have \( L \)-multiple uncertain parameters in Problem 1. To tackle this problem, we consider an affine multiplier of the form \( P(\theta) = P_0 + \sum_{i=1}^{L} \theta_i P_i \) in (3). Then, by following a close argument to [Ebihara et al., 2007], we are led to the next result.

**Theorem 2.** Let us consider the following LMI-LME conditions with respect to \( \mathcal{H} \in \mathbf{S}_m \):

\[
\mathcal{H} = \begin{bmatrix}
\mathcal{H}_{00} & \mathcal{H}_{01} & \cdots & \mathcal{H}_{0L} \\
\mathcal{H}_{10} & \mathcal{H}_{11} & \cdots & \mathcal{H}_{1L} \\
\vdots & \vdots & \ddots & \vdots \\
\mathcal{H}_{0L} & \mathcal{H}_{1L} & \cdots & \mathcal{H}_{LL-1,L} \\
\mathcal{H}_{L0} & \mathcal{H}_{L1} & \cdots & \mathcal{H}_{LL} \\
\end{bmatrix} \succeq 0,
\]

\[
\delta^2 \mathcal{H}_{ii} - \mathcal{H}_{ii} \geq 0 \ (i = 1, \ldots, L),
\]

\[
\begin{bmatrix}
\frac{r_{02n}}{I_{2L}} & \frac{r_{02n}}{V_i} \\
\frac{s_{1n}}{I_{2L}} & \frac{s_{1n}}{V_i} \\
\end{bmatrix}^T \begin{bmatrix}
H_{00} - \mathcal{H}_{ii} \\
\mathcal{H}_{ii} \\
\mathcal{H}_{ii} - \mathcal{H}_{ii} \\
\mathcal{H}_{ii} - \mathcal{H}_{ii} \\
\end{bmatrix}
\begin{bmatrix}
\frac{r_{02n}}{I_{2L}} & \frac{r_{02n}}{V_i} \\
\frac{s_{1n}}{I_{2L}} & \frac{s_{1n}}{V_i} \\
\end{bmatrix} = 0, \quad \mathbf{W}^T \mathcal{H}_{00} \mathbf{W} = 0,
\]

where
\[
\mathbf{W} := \begin{bmatrix}
E_{11} & M_{21} & 0 & E_{11} & M_{22} & 0 \\
0 & E_{21} & J_{21} & 0 & E_{21} & J_{22} \\
\end{bmatrix}^T \\
(i = 1, \ldots, L).
\]

Then, the following two assertions hold:

(i) If (8) is infeasible, then condition (2) holds.

(ii) Suppose (8) is feasible and has a solution \( \mathcal{H} \). Then, if
\[
\text{rank}(\mathcal{H}_{00}^n) = \text{rank}(\mathcal{H}),
\]
the condition (2) never holds. More precisely, if we denote the full-rank factorization of \( \mathcal{H} \) by
\[
\mathcal{H} = \begin{bmatrix}
H_0 \\
\vdots \\
H_L \\
\end{bmatrix} \begin{bmatrix}
H_0 \\
\vdots \\
H_L \\
\end{bmatrix}^T,
\]

and define \( \Omega_i := (H_{0i})^T H_{0i} \ (i = 1, \ldots, L) \), then these matrices satisfy \( \Omega_i \in \mathbf{S}_m \), \( \lambda(\Omega_i) \subset \Theta_3 \) \ (i = 1, \ldots, L) and share all eigenvectors \( u_k \in \mathbb{C}^m \ (k = 1, \ldots, m) \) in common. In addition, if we denote by \( \lambda_k(\Omega_i) \) \ (i = 1, \ldots, L) the eigenvalue of \( \Omega_i \) corresponding to the common eigenvector \( u_k \), we have \( \lambda(M(\lambda_{w,k}), J(\lambda_{w,k})) \cap \partial D(q,r,s) \neq \emptyset \) for all \( k = 1, \ldots, m \) where \( \lambda_{w,k} = \{ \lambda_1(\Omega_1), \ldots, \lambda_L(\Omega_L) \} \).

We also omit the proof for this theorem. Again, the assertion (ii) is important in this theorem, which indicates that if the dual LMI (4) is feasible and the computed dual solution satisfies the rank condition (9), then we can conclude that (2) never holds. In addition, the worst case parameter perturbations can readily be extracted as eigenvalues of \( \Omega_i \in \mathbf{S}_m \) \ (i = 1, \ldots, L) corresponding to the common eigenvector \( u_k \) \ (k = 1, \ldots, m).

**Remark 3.** Even though we have restricted our attention to the affine multiplier in Theorem 2, it is possible to employ higher-degree polynomial multipliers and derive corresponding rank conditions for the exactness verification. However, extensive numerical experiments indicate that, for most of problem instances, we can obtain exact
results via the affine multiplier. Thus, we do not pursue the direction of higher-degree polynomial multipliers in this paper.

3.3 Numerical Example

Let us consider the discrete-time system described by

\[ x(t + 1) = A(\theta)x(t), \quad A(\theta) = A_0 + \theta_1 E_1 + \theta_2 E_2. \]  

(11)

Here, the matrices \( A_0, E_1 \) and \( E_2 \) are given in Example 3 of Ramos [2001] and \( \theta_1 \in [-0.0615, 0.8822] \), \( \theta_2 \in [-0.0793, 0.7977] \). The problem we posed here is to analyze the robust stability of this discrete-time system.

Since the parameter variation is asymmetrical with respect to the origin, our method cannot be applied directly to this problem. To get around this difficulty, we first determine the nominal parameter \( \theta_0 = [0.41035, 0.35920]^T \) by taking the center of each parameter’s variation. This allows us to represent the parameter variation as \( \theta_i \in \{ \theta_{\text{u}}, \theta_{\text{l}}, \theta_{\text{u} + \theta_{\text{l}}} \} \) for the \( i = 1, 2 \) where \( \theta_{\text{u}} = [0.47185, 0.43850] \). It follows that we can describe \( A(\theta) \) as an LFT form (1) where

\[ M_{11} = A_0 + \theta_{\text{u}} E_1 + \theta_{\text{u} + \theta_{\text{l}}} E_2, \quad M_{12} = [I \ I], \quad M_{22} = 0, \]

\[ M_2 = \begin{bmatrix} \theta_{\text{u}} E_1 & 0 \\ 0 & \theta_{\text{l}} E_2 \end{bmatrix}, \quad \Delta_2(\theta) = \begin{bmatrix} \delta_{11, \text{u}} & 0 \\ 0 & \delta_{11, \text{l}} \end{bmatrix}, \quad (\hat{\theta} \in \Theta_\delta^2 = [-1,1]^2). \]

Through this equivalent problem reformulation, we solved the dual LMI (8) in Theorem 2. It turns out that (8) is infeasible and thus we can readily conclude this system is robustly stable.

We next seek for the robust stability margin \( \delta_{\text{max}} \), which is defined by the maximal value such that the system remains stable for all \( \theta \in \Theta_\delta^2 = [-\delta, \delta]^2 \). To this end, we carried out a bisection search over \( \delta \) by regarding the parameter range as \( \Theta_\delta^2 \). At the minimal value of \( \delta \) such that (8) is feasible, we examined whether the rank condition (9) holds. It turns out that rank\((\mathcal{H}_{900})\) = rank\(\mathcal{H}\) = 2 and thus the suggested rank condition is satisfied. The worst case parameter that destabilizes this system was obtained as \( \theta_{\text{u}} = [-0.0616 - 0.0794]^T \). We can confirm that \( \lambda(A(\theta_0)) = [0.4066 \pm 0.9136, -0.1473 \pm 0.3345] \), whose absolute values are 1.0000, 1.0000, 0.3655 and 0.3655, respectively.

4. ROBUST DISSIPATION PERFORMANCE ANALYSIS

Based on the preceding detailed analysis on the generalized Lyapunov inequality, we next move on to the robust dissipation performance analysis of uncertain LTI systems. For simplicity, we focus our attention on the discrete-time robust \( H_\infty \) performance analysis problem described below.

**Problem 2.** Let us given rational functions \( A(\theta) : R^L \to R^{p \times n}, B(\theta) : R^L \to R^{p \times p_1}, C(\theta) : R^L \to R^{n \times n}, D(\theta) : R^L \to R^{p_2 \times p_1} \) with no poles over \( \Theta_\delta^L \). We assume that \( A(\theta) \) is Schur stable for all \( \theta \in \Theta_\delta^L \). With these matrices, let us consider the discrete-time LTI system described by

\[ P(z, \theta) = \frac{A(\theta)B(\theta)}{C(\theta)D(\theta)} \]  

(12)

Then, for given \( \gamma > 0 \), determine whether

\[ \|P(z, \theta)\| < \gamma \quad \forall \theta \in \Theta_\delta^L \]  

(13)

holds or not.

In Ebihara et al. [2007], the continuous-time counterpart of this problem was investigated. In particular, by using the idea of the Hamiltonian eigenvalue tests (Hagiwara [2005], Zhou and Doyle [1998]), the problem is first reduced into a \( \mathcal{D} \)-nonsingularity analysis problem. Then, based on the similar results to Theorems 1 and 2, effective analysis methods with exactness verification have been proposed.

We follow this strategy to deal with the discrete-time robust \( H_\infty \) performance analysis problem. To this end, we introduce the next lemma.

**Lemma 2.** (Hagiwara [2005]) Let us consider the discrete-time LTI system \( P(z) = [A, B, C, D] \) with \( A \) being Schur stable. Then, \( \|P(z)\| < \gamma \) holds if and only if the following three conditions hold:

(i) The matrix \( D \) satisfies

\[ R_x := D^T D - \gamma I \prec 0. \]  

(14)

(ii) For one \( z_0 \) taken from \( \mathcal{D}(1, -1, 0) \) at one’s discretion,

\[ F_x - F_{\infty} = (n, n, 0) \]  

(15)

holds where

\[ F_x := [-B R_x^{-1} B^T A - B R_x^{-1} D C], \]

\[ F_{\infty} := [z_0^I I \ 0]. \]

(iii) The generalized eigenvalue condition

\[ \{ \lambda(M_x, J_x) \cap \mathcal{D}(1, -1, 0) \} = \emptyset \]  

(16)

holds where

\[ J_x := [A - C^T D R_x^{-1} B^T C - C^T D R_x^{-1} D^T C], \]

\[ M_x := -B R_x^{-1} B^T A - B R_x^{-1} D^T C. \]

When dealing with uncertainty-free systems, it is straightforward to verify the conditions (i), (ii) and (iii). However, if the system matrices are affected by the uncertain parameters as in Problem 2, those matrices \( R_x, F_x, J_x \) and \( M_x \) depend on the parameter \( \theta \) as in \( R_x(\theta), F_x(\theta), J_x(\theta) \) and \( M_x(\theta) \) and thus it is far from obvious to check the corresponding conditions (i), (ii) and (iii). In addition, the conditions (i) and (ii) particularly appear for discrete-time system analysis and this fact makes the problem more complicated in comparison with the continuous-time case.

To deal with Problem 2 by means of Lemma 2, we first note that \( D(\theta), F_x(\theta), J_x(\theta) \) and \( M_x(\theta) \) admit LFT representation of the form

\[ D(\theta) = D_{11} + D_{12} \theta C_{D_{22}} - D_{\theta}(\theta) D_{22}^{-1} D_{\theta}(\theta) D_{21}, \]

\[ F_x(\theta) = F_{11} + F_{12} \theta C_{F_{22}} - D_{\theta}(\theta) F_{22}^{-1} D_{\theta}(\theta) F_{21}, \]

\[ J_x(\theta) = J_{11} + J_{12} \theta C_{J_{22}} - D_{\theta}(\theta) J_{22}^{-1} D_{\theta}(\theta) J_{21}, \]

\[ M_x(\theta) = M_{11} + M_{12} \theta C_{M_{22}} - D_{\theta}(\theta) M_{22}^{-1} D_{\theta}(\theta) M_{21}, \]  

(17)

where \( \Delta_{\theta}(\theta) = \sum_{i=1}^{L} \theta_i E_{0,i}, \Delta_{\theta}(\theta) = \sum_{i=1}^{L} \theta_i E_{r,i} \) and \( \sum_{i=1}^{L} E_{0,i} = I_{p_1}, \sum_{i=1}^{L} E_{r,i} = I_{p_1} \). These LFT representations are always possible since \( A(\theta), B(\theta), C(\theta) \) and \( D(\theta) \) are rational. Then, it is apparent from Lemma 2 that \( \|P(z, \theta)\| < \gamma \) holds if and only if the following three conditions hold:

\[ D(\theta)^T D(\theta) - \gamma^2 I \succ 0 \quad \forall \theta \in \Theta_\delta^L \]  

(18)

\[ (F_x(\theta) - F_{\infty}) = (n, n, 0) \ 
\forall \theta \in \Theta_\delta^L, \]

(19)

\[ \{ \lambda(M_x(\theta), J_x(\theta)) \cap \mathcal{D}(1, -1, 0) \} = \emptyset \ 
\forall \theta \in \Theta_\delta^L. \]  

(20)

Since we can apply Theorems 1 and 2 for the analysis of (20), it remains to show how we deal with (18) and (19).
In the sequel, we naturally assume that the nominal performance condition $\|P(z, 0)\|_{\infty} < \gamma$ holds. It should be emphasized that this in particular implies

$$D_{11}^T D_{11} - \gamma^2 I \prec 0,$$

(21)

$$\text{In } (F_{11} - F_{20}) = (n, n, 0).$$

(22)

Then, from continuity arguments, we can rewrite (18) and (19) equivalently as follows:

$$\det(D(\theta)^T D(\theta) - \gamma^2 I) \neq 0 \quad \forall \theta \in \Theta^L_{D},$$

(23)

$$\det(F_{11}(\theta) - F_{20}) \neq 0 \quad \forall \theta \in \Theta^L_{F}.$$  

(24)

It is also true that the above condition is equivalent to

$$(F_{11}(\theta) - F_{20})^*(F_{11}(\theta) - F_{20}) > 0 \quad \forall \theta \in \Theta^L_{F}.$$  

(25)

With these equivalent reformulations, we can deal with (18) and (19) very easily in the single parameter case.

4.1 Single Uncertain Parameter Case

Let us consider the single parameter case. In this case, we can verify (18) and (19) exactly by computing eigenvalues of a fixed matrix as shown in the next Lemmas.

**Lemma 3.** The condition (23) holds if and only if

$$\left\{ \lambda \left( \begin{bmatrix} -\gamma^2 I & 0 & 0 \\ D_{11}^T & -I & 0 \\ 0 & 0 & -I \\ D_{12}^T & -I & 0 \\ 0 & 0 & -I \\ D_{21}^T & 0 & 0 \\ D_{22}^T & 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 0 & D_{11}^T \\ 0 & 0 & D_{12}^T \\ 0 & 0 & D_{21}^T \\ 0 & 0 & D_{22}^T \end{bmatrix} \right\} \cap \left\{ (-\infty, -\frac{1}{\delta}] \cup [\frac{1}{\gamma}, \infty) \right\} = \emptyset. \quad (26)$$

**Lemma 4.** The condition (24) holds if and only if

$$\left\{ \lambda \left( \begin{bmatrix} F_{11} - F_{20} \\ F_{12} \\ I \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -F_{21} & -F_{22} \end{bmatrix} \right\} \cap \left\{ (-\infty, -\frac{1}{\delta}] \cup [\frac{1}{\gamma}, \infty) \right\} = \emptyset. \quad (27)$$

With these lemmas and Theorem 1, we can obtain the next result.

**Theorem 3.** Let us consider Problem 2 with $L = 1$.

(i) If either of the conditions (26) and (27) fails, we can conclude that (13) never holds.

(ii) If both of the conditions (26) and (27) are satisfied and the dual LMI (4) corresponding to (20) is feasible, then we can conclude that (13) holds.

(iii) If the dual LMI (4) corresponding to (20) is feasible and the computed dual solution satisfies the rank condition (6), we can conclude that (13) never holds.

4.2 Multiple Uncertain Parameter Case

Problem 2 becomes much more intractable in the case of multiple uncertain parameters. Obviously, we can apply Theorem 2 to assess the condition (20). Thus the rest of this subsection is devoted to the technical details how we deal with the conditions (18) and (19).

As before, it is hard to deal with (18) and (25) exactly since they depend on multiple uncertain parameters. To get around this difficulty, as in Subsection 3.2, we apply the modified $(D, G)$-scaling (Ebihara et al. [2007]) to these parameter-dependent inequalities and obtain numerically tractable but conservative in general LMI conditions. By considering corresponding dual LMIs, it turns out that we can obtain consistent results for Theorem 2. The results are summarized in the next two lemmas.

**Lemma 5.** Let us consider the following LMI with respect to $G \in S_{(L+1)(p_1+q)} \setminus \{0\}$:

$$G = \begin{bmatrix} G_{00} & G_{01} & \cdots & G_{0L} \\ G_{10} & G_{11} & \cdots & G_{1L} \\ \vdots & \vdots & \ddots & \vdots \\ G_{L0} & G_{L1} & \cdots & G_{LL} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} 0_{p_1, r} \\ -I \end{bmatrix}^T \begin{bmatrix} S_{1} \\ -S_{1} \end{bmatrix} = 0,$$

where

$$\delta^2 G_{00} - G_{ii} \succeq 0 \quad (i = 1, \ldots, L), \quad \delta^2 F_{11}^T - F_{20}^T \succeq 0.$$

(28)

Here, we defined $S_i := [E_{q,1} F_{21} E_{q,1} F_{22}]^T (i = 1, \ldots, L).$

(i) If (28) is infeasible, then the condition (18) holds.

(ii) Suppose (28) is feasible and has a solution $G$. Then, (29) is the condition (18) never holds. More precisely, if we denote the full-rank factorization of $G$ by

$$G = \begin{bmatrix} G_0 & \cdots & G_L \end{bmatrix}^T, \quad G_j \in R^{(p_1+q) \times m} (j = 0, \ldots, L),$$

and define $\Omega_i := G_i^0 G_i (i = 1, \ldots, L)$, then these matrices satisfy $\Omega_i \in S_m$, $\lambda(\Omega_i) \subset \Theta^L_F (i = 1, \ldots, L)$ and share all eigenvectors $u_k \in C^m (k = 1, \ldots, m)$ in common.

In addition, if we denote by $\lambda_k(\Omega_i) (i = 1, \ldots, L)$ the eigenvalue of $\Omega_i$, corresponding to the common eigenvector $u_k$, we have $D(\Omega_i)^T D(\Omega_i) - \gamma^2 I \prec 0$ for all $i = 1, \ldots, m$ where $\Omega_i = \{ \lambda_k(\Omega_i) : k = 1, \ldots, m \}$.

**Lemma 6.** Let us consider the following LMI with respect to $G \in S_{(L+1)(2n+r)} \setminus \{0\}$:

$$G = \begin{bmatrix} G_{00} & G_{01} & \cdots & G_{0L} \\ \vdots & \vdots & \ddots & \vdots \\ G_{L0} & G_{L1} & \cdots & G_{LL} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} 0_{2n+r, r} \\ -L \end{bmatrix}^T \begin{bmatrix} T_1 \\ -T_1 \end{bmatrix} = 0,$$

(30)

$$\delta^2 G_{00} - G_{ii} \succeq 0 \quad (i = 1, \ldots, L), \quad \begin{bmatrix} T_1^T - F_{20}^T \\ F_{11}^T - F_{21}^T \end{bmatrix} \begin{bmatrix} G_{00} \\ T_1^T - F_{20}^T \end{bmatrix} = 0.$$

Here, we defined $T_i := [E_{v,1} F_{21} E_{v,1} F_{22}]^T (i = 1, \ldots, L).$

(i) If (30) is infeasible, then the condition (25) holds.

(ii) Suppose (30) is feasible and has a solution $G$. Then, if $\text{rank}(G_{00}) = \text{rank}(G)$, then the condition (25) never holds. More precisely, if we denote the full-rank factorization of $G$ by

$$G = \begin{bmatrix} G_0 & \cdots & G_L \end{bmatrix}^T, \quad G_j \in R^{(2n+r) \times m} (j = 0, \ldots, L),$$

and define $\Omega_i := G_i^0 G_i (i = 1, \ldots, L)$, then these matrices satisfy $\Omega_i \in S_m$, $\lambda(\Omega_i) \subset \Theta^L_F (i = 1, \ldots, L)$ and share all eigenvectors $u_k \in C^m (k = 1, \ldots, m)$ in common.
eigenvalue of $\Omega_i$ corresponding to the common eigenvector $u_k$, we have $\ln(F_\gamma(\theta_{w,k}) - F_{z_0}) \neq (n,n,0)$ for all $k = 1, \ldots, m$ where $\theta_{w,k} = [\lambda_k(\Omega_1) \cdots \lambda_k(\Omega_L)]^T$.

We see that these results are surely consistent with Theorem 2. With these results and Theorem 2, we are led to the next result.

**Theorem 4.** Let us consider Problem 2.

(i) If all of the dual LMI$s$ (28), (30) and (8) corresponding respectively to (18), (19) and (20) are infeasible, then (13) holds.

(ii) If the dual LMI (28) corresponding to (18) is feasible and the computed dual solution satisfies the rank condition (29), then (13) never holds.

(iii) If the dual LMI (30) corresponding to (19) is feasible and the computed dual solution satisfies the rank condition (31), then (13) never holds.

(iv) If the dual LMI (8) corresponding to (20) is feasible and the computed dual solution satisfies the rank condition (9), then (13) never holds.

We note that, if we resort to the standard primal-LMI-based approaches, all we can conclude is the assertion (i) in Theorem 4 (this corresponds to the case where all of the primal LMI$s$ of (28), (30) and (8) are feasible). By investigating the dual LMI$s$ and considering the structure of the dual solution, we have succeeded in deriving exactness verification tests as in (ii), (iii) and (iv). More precisely, if one of the rank conditions (29), (31) and (9) is satisfied, we can readily extract the worst case parameter perturbations.

5. CONCLUSION

In this paper, we considered robust performance analysis problems of LTI systems depending on uncertain parameters. We extended our dual LMI approach in Ebihara et al. [2007] so that we can deal with discrete-time system analysis in a unified fashion. This has been achieved by the detailed analysis on the generalized Lyapunov inequalities depending rationally upon the uncertain parameters. In stark contrast with the standard primal-LMI-based approaches, the suggested dual LMI approach would be effective to extract the worst case parameter perturbations and to conclude the exactness of the computed results. From numerical experiments, we confirmed that the suggested method is surely effective to achieve exact analysis.

At the same time, this paper showed that the robust dissipation performance analysis based on the Hamiltonian eigenvalue tests becomes rather complicated particularly for discrete-time systems. Alternative approach by means of KYP-lemma should be promising and this topic is currently under investigation.

REFERENCES


